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Weakly and weak* p-convergent operators

Saak Gabriyelyan[a]

[a]*Department of Mathematics, Ben-Gurion University of the Negev, Israel*

Abstract. Let $p \in [1,\infty]$. Being motivated by weakly p-convergent and weak* p-convergent operators between Banach spaces introduced by Fourie and Zeekoei, we introduce and study the classes of weakly p-convergent and weak* p-convergent operators between arbitrary locally convex spaces. Relationships between these classes of operators are given, and we show that they have ideal properties. Numerous characterizations of weakly p-convergent and weak* p-convergent operators are given.

1. Introduction

Unifying the notion of unconditionally converging operators and the notion of completely continuous operators, Castillo and Sánchez selected in [3] the class of p-convergent operators. An operator $T : X \to Y$ between Banach spaces X and Y is called *p-convergent* if it transforms weakly p-summable sequences into norm null sequences (all relevant definitions are given in Section 2). Using this notion they introduced and study Banach spaces with the Dunford–Pettis property of order p (DPP_p for short) for every $p \in [1,\infty]$. A Banach space X is said to have the DPP_p if every weakly compact operator from X into a Banach space Y is p-convergent.

The influential article of Castillo and Sánchez [3] inspired an intensive study of p-versions of numerous geometrical properties of Banach spaces and new classes of operators of p-convergent type. The following two classes of operators between Banach spaces were introduced and studied by Fourie and Zeekoei in [6] and [7], respectively, where the Banach dual of a Banach space X is denoted by X^*.

Definition 1.1. Let $p \in [1,\infty]$, and let X and Y be Banach spaces. An operator $T : X \to Y$ is called

(i) *weakly p-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weakly null sequence $\{\eta_n\}_{n\in\omega}$ in Y^* and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in X;

(ii) *weak* p-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weak* null sequence $\{\eta_n\}_{n\in\omega}$ in Y^* and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in X. □

It should be mentioned that if $p=\infty$, weakly p-convergent operators are known as *weak Dunford–Pettis* operators (see [1, p. 349]) and weak* p-convergent operators are known as *weak* Dunford–Pettis* operators (see [5]).

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Email address: saak@bgu.ac.il (Saak Gabriyelyan)

Numerous characterizations and applications of weakly p-convergent and weak* p-convergent operators between Banach spaces and in particular between Banach lattices were obtained in [6–8]. These results motivate us to consider these classes of operators in the general case of locally convex spaces.

Definition 1.2. Let $p \in [1,\infty]$, and let E and L be separated topological vector spaces. A linear map $T : E \to L$ is called

(i) *weakly p-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weakly null sequence $\{\eta_n\}_{n\in\omega}$ in L'_β and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E;

(ii) *weak* p-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weak* null sequence $\{\eta_n\}_{n\in\omega}$ in L' and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E. □

Relationships between the classes of p-convergent, weakly p-convergent and weak* p-convergent operators are given in Proposition 3.1, and Proposition 3.2 provides a sufficient condition on the range space L under which all these three classes of operators coincide.

In [2] Bourgain and Diestel introduced the class of limited operators between Banach spaces. More general classes of limited completely continuous and limited p-convergent operators were defined and studied by Salimi and Moshtaghioun [19] and Fourie and Zeekoei [7], respectively. We generalize these classes by introducing the classes of (q',q)-limited p-convergent and (q',q)-(V^*) p-convergent operators from a locally convex space E to a locally convex space L, where $p,q,q' \in [1,\infty]$ and $q' \leq q$. In Proposition 3.6 we show that these new classes have ideal properties. If the space L contains an isomorphic copy of ℓ_∞, in Theorems 3.8 and 3.10 we show that all weakly p-convergent operators are (q',q)-limited p-convergent (resp., (q',q)-(V^*) p-convergent) if and only if so are all weak* p-convergent operators if and only if so is the identity operator $\mathrm{id}_E : E \to E$.

The main results of the article are Theorems 3.12 and 3.14 in which we give numerous characterizations of weak* p-convergent and weakly p-convergent operators between locally convex spaces.

2. Preliminaries results

We start with some necessary definitions and notations used in the article. Set $\omega := \{0,1,2,\dots\}$. All topological spaces are assumed to be Tychonoff (= completely regular and T_1). The closure of a subset A of a topological space X is denoted by $\overline{A}$, $\overline{A}^X$ or $\mathrm{cl}_X(A)$. A topological space X is defined to be *selectively sequentially pseudocompact* if for any sequence $\{U_n\}_{n\in\omega}$ of open sets of X there exists a sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} U_n$ containing a convergent subsequence. A function $f : X \to Y$ between topological spaces X and Y is called *sequentially continuous* if for any convergent sequence $\{x_n\}_{n\in\omega} \subseteq X$, the sequence $\{f(x_n)\}_{n\in\omega}$ converges in Y and $\lim_n f(x_n) = f(\lim_n x_n)$. We denote by $C(X)$ the vector space of all continuous $\mathbb{F}$-valued functions on X. A subset A of a topological space X is called

- *relatively compact* if its closure $\bar{A}$ is compact;
- (*relatively*) *sequentially compact* if each sequence in A has a subsequence converging to a point of A (resp., of X);
- *functionally bounded in X* if every $f \in C(X)$ is bounded on A.

The space $C(X)$ endowed with the pointwise topology is denoted by $C_p(X)$.

Let E be a locally convex space. We assume that E is over the field $\mathbb{F}$ of real or complex numbers. We denote by $\mathcal{N}_0(E)$ (resp., $\mathcal{N}_0^c(E)$) the family of all (resp., closed absolutely convex) neighborhoods of zero of E. The family of all bounded subsets of E is denoted by $\mathsf{Bo}(E)$. The topological dual space of E is denoted by E'. The value of $\chi \in E'$ on $x \in E$ is denoted by $\langle \chi, x\rangle$ or $\chi(x)$. A sequence $\{x_n\}_{n\in\omega}$ in E is said to be *Cauchy* if for every $U \in \mathcal{N}_0(E)$ there is $N \in \omega$ such that $x_n - x_m \in U$ for all $n,m \geq N$. It is easy to see that a sequence $\{x_n\}_{n\in\omega}$ in E is Cauchy if and only if $x_{n_k} - x_{n_{k+1}} \to 0$ for every (strictly) increasing sequence (n_k) in ω. We denote by E_w and E_β the space E endowed with the weak topology $\sigma(E,E')$ and with the strong topology $\beta(E,E')$, respectively. The topological dual space E' of E endowed with weak* topology $\sigma(E',E)$ or with the strong topology $\beta(E',E)$ is denoted by E'_{w^*} or E'_β, respectively. The closure of a subset A in the weak topology is denoted by $\overline{A}^{w}$ or $\overline{A}^{\sigma(E,E')}$, and $\overline{B}^{w^*}$ (or $\overline{B}^{\sigma(E',E)}$) denotes the closure of $B \subseteq E'$ in the weak* topology. The *polar* of a subset A of E is denoted by

$$A^\circ := \{\chi \in E' : \|\chi\|_A \leq 1\}, \quad \text{where} \quad \|\chi\|_A = \sup\big\{|\chi(x)| : x \in A \cup \{0\}\big\}.$$

A subset B of E' is *equicontinuous* if $B \subseteq U^\circ$ for some $U \in \mathcal{N}_0(E)$. The family of all continuous linear maps (= operators) from an lcs H to an lcs L is denoted by $\mathcal{L}(H, L)$.

Let $p \in [1, \infty]$. Then p^* is defined to be the unique element of $[1, \infty]$ which satisfies $\frac{1}{p} + \frac{1}{p^*} = 1$. For $p \in [1, \infty)$, the space ℓ_{p^*} is the dual space of ℓ_p. We denote by $\{e_n\}_{n\in\omega}$ the canonical basis of ℓ_p, if $1 \leq p < \infty$, or the canonical basis of c_0, if $p = \infty$. The canonical basis of ℓ_{p^*} is denoted by $\{e_n^*\}_{n\in\omega}$. Denote by ℓ_p^0 and c_0^0 the linear span of $\{e_n\}_{n\in\omega}$ in ℓ_p or c_0 endowed with the induced norm topology, respectively.

A subset A of a locally convex space E is called

- *precompact* if for every $U \in \mathcal{N}_0(E)$ there is a finite set $F \subseteq E$ such that $A \subseteq F + U$;
- *sequentially precompact* if every sequence in A has a Cauchy subsequence;
- *weakly* (*sequentially*) *compact* if A is (sequentially) compact in E_w;
- *relatively weakly compact* if its weak closure $\overline{A}^{\sigma(E,E')}$ is compact in E_w;
- *relatively weakly sequentially compact* if each sequence in A has a subsequence weakly converging to a point of E;
- *weakly sequentially precompact* if each sequence in A has a weakly Cauchy subsequence.

Note that each sequentially precompact subset of E is precompact, but the converse is not true in general, see Lemma 2.2 of [11].

Let $p \in [1, \infty]$. A sequence $\{x_n\}_{n\in\omega}$ in a locally convex space E is called

- *weakly p-summable* if for every $\chi \in E'$, it follows

  $$(\langle \chi, x_n\rangle)_{n\in\omega} \in \ell_p \text{ if } p < \infty, \text{ and } (\langle \chi, x_n\rangle)_{n\in\omega} \in c_0 \text{ if } p = \infty;$$

- *weakly p-convergent to $x \in E$* if $\{x_n - x\}_{n\in\omega}$ is weakly p-summable;
- *weakly p-Cauchy* if for each pair of strictly increasing sequences $(k_n), (j_n) \subseteq \omega$, the sequence $(x_{k_n} - x_{j_n})_{n\in\omega}$ is weakly p-summable.

The family of all weakly p-summable sequences of E is denoted by $\ell_p^w(E)$ or $c_0^w(E)$ if $p = \infty$.

A sequence $\{\chi_n\}_{n\in\omega}$ in E' is called *weak* p-summable* (resp., *weak* p-convergent to* $\chi \in E'$ or *weak* p-Cauchy*) if it is weakly p-summable (resp., weakly p-convergent to $\chi \in E'$ or weakly p-Cauchy) in E'_{w^*}.

Generalizing the classical notions of limited subsets, p-limited subsets, p-(V^*) subsets and coarse p-limited subsets of a Banach space X and p-(V) subsets of the Banach dual X^* introduced in [2], [16], [4], [13] and [17], respectively, we defined in [11, 12] the following notions. Let $1 \leq p \leq q \leq \infty$, and let E be a locally convex space E. Then:

- a non-empty subset A of E is a (p, q)-(V^*) *set* (resp., a (p, q)-*limited set*) if

  $$\Big(\sup_{a\in A} |\langle \chi_n, a\rangle|\Big) \in \ell_q \ \text{ if } q < \infty, \quad \text{or} \quad \Big(\sup_{a\in A} |\langle \chi_n, a\rangle|\Big) \in c_0 \ \text{ if } q = \infty,$$

  for every weakly (resp., weak*) p-summable sequence $\{\chi_n\}_{n\in\omega}$ in E'_β. (p, ∞)-(V^*) sets and $(1, \infty)$-(V^*) sets will be called simply p-(V^*) *sets* and (V^*) *sets*, respectively. Analogously, (p, p)-limited sets and (∞, ∞)-limited sets will be called *p-limited sets* and *limited sets*, respectively.
- a non-empty subset A of E is a *coarse p-limited set* if for every $T \in \mathcal{L}(E, \ell_p)$ (or $T \in \mathcal{L}(E, c_0)$ if $p = \infty$), the set $T(A)$ is relatively compact.
- a non-empty subset B of E' is a (p, q)-(V) *set* if

  $$\Big(\sup_{\chi\in B} |\langle \chi, x_n\rangle|\Big) \in \ell_q \ \text{ if } q < \infty, \quad \text{or} \quad \Big(\sup_{\chi\in B} |\langle \chi, x_n\rangle|\Big) \in c_0 \ \text{ if } q = \infty,$$

  for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E. (p, ∞)-(V) sets and $(1, \infty)$-(V) sets will be called simply p-(V) *sets* and (V) *sets*, respectively.

Recall that a locally convex space E

- is *sequentially complete* if each Cauchy sequence in E converges;
- (*quasi*)*barrelled* if every $\sigma(E',E)$-bounded (resp., $\beta(E',E)$-bounded) subset of E' is equicontinuous;
- c_0-(*quasi*)*barrelled* if every $\sigma(E',E)$-null (resp., $\beta(E',E)$-null) sequence is equicontinuous.

The following weak barrelledness conditions introduced and studied in [11] will play a considerable role in the article. Let $p \in [1,\infty]$. A locally convex space E is called *p-barrelled* (resp., *p-quasibarrelled*) if every weakly p-summable sequence in E'_{w^*} (resp., in E'_β) is equicontinuous. It is clear that E is ∞-barrelled if and only if it is c_0-barrelled.

We shall consider also the following linear map introduced in [11]

$$S_p : \mathcal{L}(E,\ell_p) \to \ell_p^w(E'_{w^*}) \quad \big(\text{or } S_\infty : \mathcal{L}(E,c_0) \to c_0^w(E'_{w^*}) \text{ if } p=\infty\big)$$

defined by $S_p(T) := \big(T^*(e_n^*)\big)_{n\in\omega}$.

The following p-versions of weakly compact-type properties are defined in [11] generalizing the corresponding notions in the class of Banach spaces introduced in [3] and [14]. Let $p \in [1,\infty]$. A subset A of a locally convex space E is called

- (*relatively*) *weakly sequentially p-compact* if every sequence in A has a weakly p-convergent subsequence with limit in A (resp., in E);
- *weakly sequentially p-precompact* if every sequence from A has a weakly p-Cauchy subsequence.

It is clear that each relatively weakly sequentially p-compact subset of E is weakly sequentially p-precompact.

Let E and L be locally convex spaces. An operator $T \in \mathcal{L}(E,L)$ is called *weakly sequentially compact* (resp., *weakly sequentially p-compact, weakly sequentially p-precompact* or *coarse p-limited*) if there is $U \in \mathcal{N}_0(E)$ such that $T(U)$ is a relatively weakly sequentially compact (resp., relatively weakly sequentially p-compact, weakly sequentially p-precompact or coarse p-limited) subset of L. Generalizing the notion of p-convergent operators between Banach spaces and following [11], an operator $T \in \mathcal{L}(E,L)$ is called *p-convergent* if T sends weakly p-summable sequences of E to null sequences of L.

3. Main results

The following assertion gives the first relationships between different p-convergent types of operators. Recall that a locally convex space E is called *Grothendieck* or has the *Grothendieck property* if the identity map $\mathrm{id}_{E'} : E'_{w^*} \to \big(E'_\beta\big)_w$ is sequentially continuous.

Proposition 3.1. *Let $p \in [1,\infty]$, E and L be locally convex spaces, and let $T : E \to L$ be a linear map. Then:*

(i) *if T is finite-dimensional and continuous, then T is p-convergent, coarse p-limited and weak* p-convergent;*
(ii) *if T is weak* p-convergent, then it is weakly p-convergent; the converse it true if L has the Grothendieck property;*
(iii) *if L is ∞-quasibarrelled and T is p-convergent, then T is weakly p-convergent;*
(iv) *if L is c_0-barrelled and T is p-convergent, then T is weak* p-convergent.*

Proof. (i) follows from the corresponding definitions and (iv) of Proposition 4.2 of [12] (which states that every finite subset of E is coarse p-limited).

(ii) follows from the fact that every weakly null sequence in L'_β is also weak* null and the definition of the Grothendieck property.

(iii), (iv): Let $\{\eta_n\}_{n\in\omega} \subseteq L'_\beta$ be a weakly (resp., weak*) null-sequence, and let $\{x_n\}_{n\in\omega} \subseteq E$ be a weakly p-summable sequence. As L is ∞-quasibarrelled (resp., c_0-barrelled), the sequence $\{\eta_n\}_{n\in\omega}$ is equicontinuous. Now, fix an arbitrary $\varepsilon > 0$. Choose $U \in \mathcal{N}_0(L)$ such that

$$|\langle \eta_n, y\rangle| < \varepsilon \quad \text{for every } n \in \omega \text{ and each } y \in U. \tag{1}$$

Since T is p-convergent, $T(x_n) \to 0$ in L, and hence there is $N_\varepsilon \in \omega$ such that

$$T(x_n) \in U \quad \text{for every } n \geq N_\varepsilon. \tag{2}$$

Then (1) and (2) imply

$$|\langle \eta_n, T(x_n)\rangle| < \varepsilon \quad \text{for every } n \geq N_\varepsilon.$$

Therefore $\langle \eta_n, T(x_n)\rangle \to 0$ as $n \to \infty$. Thus T is weakly (resp., weak*) p-convergent. □

Below we consider the case when the classes of all weakly p-convergent, weak* p-convergent and p-convergent operators coincide. Observe that the conditions of the proposition are satisfied if L is a separable reflexive Fréchet space or a reflexive Banach space. In particular, this proposition generalizes Corollary 2.4 and Proposition 2.5 of [7].

Proposition 3.2. *Let $p \in [1, \infty]$, E be a locally convex space, and let L be an ∞-quasibarrelled Grothendieck space such that U° is weak* selectively sequentially pseudocompact for every $U \in \mathcal{N}_0(L)$. Then for an operator $T : E \to L$, the following assertions are equivalent:*

- (i) *T is weakly p-convergent;*
- (ii) *T is weak* p-convergent;*
- (iii) *T is p-convergent.*

Proof. The equivalence (i)⇔(ii) follow from (ii) of Proposition 3.1, and the implication (iii)⇒(i) follows from (iii) of Proposition 3.1.

(ii)⇒(iii) Suppose for a contradiction that there is a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E such that $T(x_n) \not\to 0$. Without loss of generality we can assume that there is $V \in \mathcal{N}_0^c(L)$ such that $T(x_n) \notin V$ for every $n \in \omega$. By the Hahn–Banach separation theorem, for every $n \in \omega$ there is $\eta_n \in V^\circ$ such that $|\langle \eta_n, T(x_n)\rangle| > 1$. For every $n \in \omega$, set

$$U_n := \{\chi \in V^\circ : |\langle \chi, T(x_n)\rangle| > 1\}.$$

Then U_n is a weak* open neighborhood of η_n in V°. Since V° is selectively sequentially pseudocompact in the weak* topology, for every $n \in \omega$ there exists $\chi_n \in U_n$ such that the sequence $\{\chi_n\}_{n\in\omega}$ contains a subsequence $\{\chi_{n_k}\}_{k\in\omega}$ which weak* converges to some functional $\chi \in V^\circ$. Taking into account that the subsequence $\{x_{n_k}\}_{k\in\omega}$ is also weakly p-summable and the operator T is weak* p-convergent the inclusion $\chi_{n_k} \in U_{n_k}$ implies

$$1 < \langle \chi_{n_k}, T(x_{n_k})\rangle = \langle \chi_{n_k} - \chi, T(x_{n_k})\rangle + \langle \chi, T(x_{n_k})\rangle \to 0 \ \text{ as } k \to \infty,$$

a contradiction. □

In (iv) of Proposition 3.1 the condition of being a c_0-barrelled space is not necessary in general even for operators, but this condition cannot be completely omitted even for metrizable spaces, and also the condition in (iii) of being ∞-quasibarrelled is essential as the following example shows.

Example 3.3. Let $p \in [1, \infty]$.

- (i) There are metrizable non-c_0-barrelled spaces E and L such that each operator $T : E \to L$ is finite-dimensional and hence it is p-convergent and weak* p-convergent.
- (ii) There is a metrizable non-c_0-barrelled space E such that the identity map id_E is p-convergent and weakly p-convergent, but it is not weak* p-convergent.
- (iii) There are a non-∞-quasibarrelled space E and an ∞-convergent operator $T : E \to E$ such that T is not weakly ∞-convergent.

Proof. (i) Let $E = (c_0)_p$ be the Banach space c_0 endowed with the topology induced from $\mathbb{F}^\omega$, and let $L := c_0^0$. Then E and L are metrizable and non-barrelled. Since metrizable locally convex spaces are barrelled if and only if they are c_0-barrelled (see Proposition 12.2.3 of [15]), it remains to note that, by (ii) of Example 5.4 of [11], each $T \in \mathcal{L}(E, L)$ is finite-dimensional.

(ii) Let $E = L = C_p(\mathbf{s})$, where $\mathbf{s} = \{x_n\}_{n\in\omega} \cup \{x_\infty\}$ is a convergent sequence. Then E is a metrizable space and hence quasibarrelled. By Proposition 12.2.3 of [15] and the Buchwalter—Schmets theorem, E is not c_0-barrelled. Since E carries its weak topology, id_E is trivially p-convergent for every $p \in [1,\infty]$. Therefore, by (iii) of Proposition 3.1, id_E is weakly p-convergent. To show that id_E is not weak* p-convergent, for every $n \in \omega$, let $\eta_n := \delta_{x_n} - \delta_{x_\infty}$ and $f_n := 1_{\{x_n\}}$, where δ_x is the Dirac measure at the point x and 1_A is the characteristic function of a subset $A \subseteq \mathbf{s}$. It is well known that $C_p(\mathbf{s})' = L(\mathbf{s})$ algebraically, where $L(\mathbf{s})$ is the free locally convex space over $\mathbf{s}$. Now it is clear that the sequence $\{\eta_n\}_{n\in\omega}$ is weak* null in L' and $(f_n) \in \ell_p^w(E)$ (or $\in c_0^w(E)$ if $p = \infty$). However, since $\langle \eta_n, \mathrm{id}_E(f_n)\rangle = f_n(x_n) - f_n(x_\infty) = 1 \not\to 0$ we obtain that id_E is not weak* p-convergent.

(iii) Let $\mathbf{s} = \{x_n\}_{n\in\omega} \cup \{x_\infty\}$ be a convergent sequence, and let $E = L(\mathbf{s})$ be the free locally convex space over $\mathbf{s}$. By Example 5.5 of [11], the space E is not 1-quasibarrelled and hence it is not ∞-quasibarrelled. Let $T = \mathrm{id}_E : E \to E$ be the identity operator. Since, by Theorem 1.2 of [10], E has the Schur property the operator T is trivially p-convergent. We show that T is not weakly ∞-convergent. To this end, as in the proof of (ii), we consider two sequences $\{\eta_n := \delta_{x_n} - \delta_{x_\infty}\}_{n\in\omega} \subseteq E$ and $\{f_n\}_{n\in\omega} \subseteq E' = C(\mathbf{s})$. It is clear that $\{\eta_n\}_{n\in\omega}$ is weakly null in E. Since, by Proposition 3.4 of [9], the space E'_β is the Banach space $C(\mathbf{s})$, it is easy to see that the sequence $\{f_n = 1_{\{x_n\}}\}_{n\in\omega}$ is weakly null (= weakly ∞-summable). Taking into account that

$$\langle f_n, T(\eta_n)\rangle = \langle f_n, \delta_{x_n} - \delta_{x_\infty}\rangle = f_n(x_n) - f_n(x_\infty) = 1 \not\to 0,$$

it follows that T is not weakly ∞-convergent. □

Below we generalize the classes of limited, limited completely continuous and limited p-convergent operators between Banach spaces.

Definition 3.4. Let $p, q, q' \in [1,\infty]$, $q' \le q$, and let E and L be locally convex spaces. A linear map $T : E \to L$ is called

- *weakly (p,q)-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weakly q-summable sequence $\{\eta_n\}_{n\in\omega}$ in L'_β and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E;
- *weak* (p,q)-convergent* if $\lim_{n\to\infty}\langle \eta_n, T(x_n)\rangle = 0$ for every weak* q-summable sequence $\{\eta_n\}_{n\in\omega}$ in L' and each weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E;
- *(p,q)-limited* if $T(U)$ is a (p,q)-limited subset of L for some $U \in \mathcal{N}_0(E)$; if $p = q$ or $p = q = \infty$ we shall say that T is *p-limited* or *limited*, respectively;
- (p,q)-(V^*) if $T(U)$ is a (p,q)-(V^*) subset of L for some $U \in \mathcal{N}_0(E)$; if $q = \infty$ or $p = 1$ and $q = \infty$ we shall say that T is p-(V^*) or (V^*), respectively;
- *(q',q)-limited p-convergent* if $T(x_n) \to 0$ for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E which is a (q',q)-limited subset of E; if $q' = q$ or $q' = q = \infty$ we shall say that T is *q-limited p-convergent* or *limited p-convergent*, respectively;
- *(q',q)-(V^*) p-convergent* if $T(x_n) \to 0$ for every weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E which is a (q',q)-(V^*) subset of E; if $q = \infty$ or $q' = q = \infty$ we shall say that T is *q'-(V^*) p-convergent* or *(V^*) p-convergent*, respectively. □

It is clear that weakly p-convergent operators are exactly weakly (p,∞)-convergent operators, and weak* p-convergent operators are exactly weak* (p,∞)-convergent operators.

Lemma 3.5. *Let $p, q, q' \in [1,\infty]$, $q' \le q$, and let E and L be locally convex spaces. If $T \in \mathcal{L}(E, L)$ is finite-dimensional, then T is (p,q)-limited, (p,q)-(V^*), (q',q)-limited p-convergent and (q',q)-(V^*) p-convergent.*

Proof. Since T is finite-dimensional, there are a finite subset F of L and $U \in \mathcal{N}_0(E)$ such that $T(U) \subseteq \overline{\mathrm{acx}}(F)$. Therefore, by Lemma 3.1 of [12], $T(U)$ is a (p,q)-limited set and hence a (p,q)-(V^*) set. Whence T is (p,q)-limited and (p,q)-(V^*). Since, by (i) of Proposition 3.1, T is p-convergent it is trivially (q',q)-limited p-convergent and (q',q)-(V^*) p-convergent. □

The next proposition stands ideal properties of the classes of operators introduced in Definition 3.4.

Proposition 3.6. *Let $p,q,q' \in [1,\infty]$, $q' \le q$, $\lambda,\mu \in \mathbb{F}$, the spaces E_0, E, L_0 and L be locally convex, and let $Q \in \mathcal{L}(E_0,E)$, $T,S \in \mathcal{L}(E,L)$ and $R \in \mathcal{L}(L,L_0)$. Then:*

(i) *if T and S are weakly (p,q)-convergent, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$;*
(ii) *if T and S are weak* (p,q)-convergent, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$;*
(iii) *if T and S are (p,q)-limited operators, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$;*
(iv) *if T and S are (p,q)-(V^*) operators, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$;*
(v) *if T and S are (q',q)-limited p-convergent, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$;*
(vi) *if T and S are (q',q)-(V^*) p-convergent, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$;*
(vii) *if T and S are coarse p-limited, then so are $R \circ T \circ Q$ and $\lambda T + \mu S$.*

Proof. Since the case $\lambda T + \mu S$ is trivial, we consider the case $R \circ T \circ Q$.

(i) and (ii): Let $\{\eta_n\}_{n\in\omega}$ be a weakly (resp., weak*) q-summable sequence in $(L_0)'_\beta$, and let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E_0. Then, by (iii) of Lemma 4.6 of [11], the sequence $\{Q(x_n)\}_{n\in\omega}$ is weakly p-summable in E. Since R^* is weak* and strongly continuous by Theorems 8.10.5 and 8.11.3 of [18], respectively, Lemma 4.6 of of [11] implies that the sequence $\{R^*(\eta_n)\}_{n\in\omega}$ is weakly (resp., weak*) q-summable in L'_β. Now the definition of weakly (resp., weak* (p,q)-convergent operators implies

$$\lim_{n\to\infty}\langle \eta_n, R\circ T\circ Q(x_n)\rangle = \lim_{n\to\infty}\langle R^*(\eta_n), T\big(Q(x_n)\big)\rangle = 0.$$

Thus $R \circ T \circ Q$ is weakly (resp., weak*) (p,q)-convergent, as desired.

(iii) immediately follows from (iv) of Lemma 3.1 of [12].

(iv) immediately follows from (iv) of Lemma 7.2 of [11].

(v) and (vi): Let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E_0 which is a (q',q)-limited (resp., (q',q)-(V^*)) subset of E. Then, by Lemma 4.6 of of [11] and (iv) of Lemma 3.1 of [12] (resp., (iv) of Lemma 7.2 of [11]), the sequence $\{Q(x_n)\}_{n\in\omega}$ is a weakly p-summable sequence in E which is a (q',q)-limited (resp., (q',q)-(V^*)) subset of E. Since T is (q',q)-limited (resp., (q',q)-(V^*)) p-convergent, it follows that $T(Q(x_n)) \to 0$ in the space L. The continuity of R implies that $R \circ T \circ Q(x_n) \to 0$ in L_0. Thus $R \circ T \circ Q$ is (q',q)-limited (resp., (q',q)-(V^*)) p-convergent.

(vii) immediately follows from (iii) of Lemma 4.1 of [12] (which states that continuous images of coarse p-limited sets are coarse p-limited). □

Corollary 3.7. *Let $p,q,q' \in [1,\infty]$, $q' \le q$, and let E and L be locally convex spaces. If the identity operator $\mathsf{id}_E : E \to E$ is weakly (p,q)-convergent (resp., weak* (p,q)-convergent, (q',q)-limited, (q',q)-(V^*), (q',q)-limited p-convergent, (q',q)-(V^*) p-convergent or coarse p-limited), then so is every $T \in \mathcal{L}(E,L)$.*

Proof. The assertion follows from the equality $T = T \circ \mathsf{id}_E \circ \mathsf{id}_E$ and Proposition 3.6. □

Corollary 3.7 motivates the study of locally convex spaces E for which the identity operator $\mathsf{id}_E : E \to E$ has one of the properties from the corollary. If the space L contains an isomorphic copy of ℓ_∞ we can partially reverse Corollary 3.7 as follows.

Theorem 3.8. *Let $p,q,q' \in [1,\infty]$, $q' \le q$, and let E and L be locally convex spaces. If L contains an isomorphic copy of ℓ_∞, then the following assertions are equivalent:*

(i) *each weakly p-convergent operator from E into L is (q',q)-limited p-convergent;*

(ii) *each weak* p-convergent operator from E into L is (q', q)-limited p-convergent;*
(iii) *the identity operator $\mathrm{id}_E : E \to E$ is (q', q)-limited p-convergent.*

Proof. (i)⇒(ii) follows from (ii) of Proposition 3.1.

(ii)⇒(iii) Suppose for a contradiction that id_E is not (q', q)-limited p-convergent. Then there is a (q', q)-limited weakly p-summable sequence $S = \{x_n\}_{n\in\omega}$ in E which does not converge to zero in E. Without loss of generality we can assume that $S \cap U = \emptyset$ for some closed absolutely convex neighborhood U of zero in E. Since S, being also a weakly null-sequence, is a bounded subset of E, there is $a > 1$ such that $S \subseteq aU$. For every $n \in \omega$, by the Hahn–Banach separation theorem, choose $\chi_n \in U^\circ$ such that $|\langle \chi_n, x_n\rangle| > 1$. Therefore

$$1 < |\langle \chi_n, x_n\rangle| \le a \quad \text{for every } \ n \in \omega. \tag{3}$$

Since $\{\chi_n\}_{n\in\omega} \subseteq U^\circ$, the sequence $\{\chi_n\}_{n\in\omega}$ is equicontinuous. Therefore, by Lemma 14.13 of [11], the linear map

$$Q : E \to \ell_\infty, \quad Q(x) := \big(\langle \chi_n, x\rangle\big)_{n\in\omega},$$

is continuous.

By assumption, there is an embedding $R : \ell_\infty \to L$. Consider the operator $T := R \circ Q : E \to L$. We claim that T is weak* p-convergent. Indeed, let $(y_n)_{n\in\omega} \in \ell_p^w(E)$ (or $(y_n)_{n\in\omega} \in c_0^w(E)$ if $p = \infty$) and let $\{\eta_n\}_{n\in\omega}$ be a weak* null-sequence in $(\ell_\infty)'$. Then $\{R^*(\eta_n)\}_{n\in\omega}$ is also a weak* null-sequence in $(\ell_\infty)'$. Therefore, by the Grothendieck property of ℓ_∞, $\{R^*(\eta_n)\}_{n\in\omega}$ is weakly null in the Banach dual space $(\ell_\infty)'_\beta$. Since $\{y_n\}_{n\in\omega}$ is weakly null, it follows that $\{Q(y_n)\}_{n\in\omega} \subseteq \ell_\infty$ is also weakly null. Therefore, by the Dunford–Pettis property of ℓ_∞, we obtain

$$\lim_{n\to\infty} \langle \eta_n, T(y_n)\rangle = \lim_{n\to\infty} \langle R^*(\eta_n), Q(y_n)\rangle = 0.$$

Thus T is a weak* p-convergent operator.

To get a desired contradiction it remains to prove that T is not (q', q)-limited p-convergent by showing that $T(x_k) \not\to 0$. To this end, choose $W \in \mathcal{N}_0(L)$ such that $W \cap R(\ell_\infty) \subseteq R(B_{\ell_\infty})$. The inequalities (3) and the bijectivity of R imply

$$T(x_k) = R \circ Q(x_k) = R\big(\langle \chi_n, x_k\rangle\big) \notin R(B_{\ell_\infty}) \quad \text{for every } \ k \in \omega.$$

Since the range of T is contained in $R(\ell_\infty)$ and R is an embedding the choice of W implies that $T(x_k) \notin W$ for all $k \in \omega$. Thus T is not (q', q)-limited p-convergent.

(iii)⇒(i) follows from Corollary 3.7. □

Corollary 3.7 and Theorem 3.8 immediately imply

Corollary 3.9. *Let $p, q, q' \in [1, \infty]$, $q' \le q$, and let E be a locally convex space. Then the identity operator $\mathrm{id}_E : E \to E$ is (q', q)-limited p-convergent if and only if so is any operator $T : E \to \ell_\infty$.*

Below we obtain an analogous characterization of locally convex spaces E for which the identity map id_E is (q', q)-(V^*) p-convergent. We omit its proof because it can be obtained from the proof of Theorem 3.8 just replacing "(q', q)-limited" by "(q', q)-(V^*)".

Theorem 3.10. *Let $p, q, q' \in [1, \infty]$, $q' \le q$, and let E and L be locally convex spaces. If L contains an isomorphic copy of ℓ_∞, then the following assertions are equivalent:*

(i) *each weakly p-convergent operator from E into L is (q', q)-(V^*) p-convergent;*
(ii) *each weak* p-convergent operator from E into L is (q', q)-(V^*) p-convergent;*
(iii) *the identity operator $\mathrm{id}_E : E \to E$ is (q', q)-(V^*) p-convergent.*

Corollary 3.7 and Theorem 3.10 immediately imply

Corollary 3.11. *Let $p, q, q' \in [1, \infty]$, $q' \le q$, and let E be a locally convex space. Then the identity operator $\mathrm{id}_E : E \to E$ is (q', q)-(V^*) p-convergent if and only if so is any operator $T : E \to \ell_\infty$.*

Let $p \in [1, \infty]$. We shall say that a locally convex space E is (*weakly*) *sequentially locally p-complete* if the closed absolutely convex hull of a weakly p-summable sequence is weakly sequentially p-compact (resp., weakly sequentially p-precompact). It is clear that if $p = \infty$ and E is weakly angelic (for example, E is a strict (LF)-space), then E is sequentially locally ∞-complete if and only if it is locally complete.

Now we characterize weak* p-convergent operators.

Theorem 3.12. *Let $p \in [1, \infty]$, E and L be locally convex spaces, and let $T : E \to L$ be an operator. Consider the following assertions:*

(i) *T is weak* p-convergent;*
(ii) *T transforms weakly sequentially p-precompact subsets of E to limited subsets of L;*
(iii) *T transforms (relatively) weakly sequentially p-compact subsets of E to limited subsets of L;*
(iv) *T transforms weakly p-summable sequence of E to limited subsets of L;*
(v) *$S \circ T$ is p-convergent for each $S \in \mathcal{L}(L, Z)$ and any locally convex (or the same, Banach) space Z such that U° is weak* selectively sequentially pseudocompact for every $U \in \mathcal{N}_0(Z)$;*
(vi) *$S \circ T$ is p-convergent for each $S \in \mathcal{L}(L, c_0)$;*
(vii) *for any normed space X and each weakly sequentially p-precompact operator $R : X \to E$, the operator $T \circ R$ is limited;*
(viii) *for any normed space X and each weakly sequentially p-precompact operator $R : X \to E$, the adjoint $R^* \circ T^* : L'_{w^*} \to X'_\beta$ is ∞-convergent;*
(ix) *if $R \in \mathcal{L}(\ell_1^0, E)$ is weakly sequentially p-precompact, then $T \circ R$ is limited;*
(x) *for every normed space Z and each weakly sequentially p-compact operator S from Z to E, the composition $T \circ S$ is a limited linear map;*
(xi) *for any operator $S \in \mathcal{L}(\ell_{p^*}, E)$, the linear map $T \circ S$ is limited.*

Then:

(A) (i)⇔(ii)⇔(iii)⇔(iv);
(B) (i)⇒(v)⇒(vi), *and if L is c_0-barrelled, then* (vi)⇒(i);
(C) (ii)⇒(vii)⇔(viii)⇒(ix), *and if E is weakly sequentially locally p-complete, then* (ix)⇒(i);
(D) *if $1 < p < \infty$, then* (iii)⇒(x)⇒(xi), *and if E is sequentially complete, then* (xi)⇒(i).

Proof. (i)⇒(ii) Let A be a weakly sequentially p-precompact subset of E, and suppose for a contradiction that $T(A)$ is not limited. Therefore there are a weak* null sequence $\{\eta_n\}_{n\in\omega}$ in L', a sequence $\{x_n\}_{n\in\omega}$ in A and $\varepsilon > 0$ such that $|\langle \eta_n, T(x_n)\rangle| > \varepsilon$ for every $n \in \omega$. Since A is weakly sequentially p-precompact, without loss of generality we assume that $\{x_n\}_{n\in\omega}$ is weakly p-Cauchy.

For $n_0 = 0$, since $\{\eta_n\}_{n\in\omega}$ is weak* null we can choose $n_1 > n_0$ such that $|\langle \eta_{n_1}, T(x_{n_0})\rangle| < \frac{\varepsilon}{2}$. Proceeding by induction on k, we can choose $n_{k+1} > n_k$ such that $|\langle \eta_{n_{k+1}}, T(x_{n_k})\rangle| < \frac{\varepsilon}{2}$. Since the sequence $\{x_{n_{k+1}} - x_{n_k}\}_{k\in\omega}$ is weakly p-summable and T is weak* p-convergent, we obtain

$$\left\langle \eta_{n_{k+1}}, T\Big(x_{n_{k+1}} - x_{n_k}\Big)\right\rangle \to 0.$$

On the other hand,

$$\left|\left\langle \eta_{n_{k+1}}, T\Big(x_{n_{k+1}} - x_{n_k}\Big)\right\rangle\right| \ge \left|\left\langle \eta_{n_{k+1}}, T\Big(x_{n_{k+1}}\Big)\right\rangle\right| - \left|\left\langle \eta_{n_{k+1}}, T\Big(x_{n_k}\Big)\right\rangle\right| > \tfrac{\varepsilon}{2},$$

a contradiction.

(ii)⇒(iii) and (iii)⇒(iv) are trivial.

(iv)⇒(i) follows from the definition of limited sets.

(i)⇒(v) Assume that T is weak* p-convergent, and suppose for a contradiction that $S \circ T$ is not p-convergent for some $S \in \mathcal{L}(L, Z)$ and some locally convex (resp., Banach) space Z such that U° is weak* selectively sequentially pseudocompact for every $U \in \mathcal{N}_0(Z)$. Then there are a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and a closed absolutely convex neighborhood $V \subseteq Z$ of zero such that $ST(x_n) \notin V$ for every $n \in \omega$. By the Hahn–Banach separation theorem, for every $n \in \omega$ there is $\eta_n \in V^\circ$ such that $|\langle \eta_n, ST(x_n)\rangle| > 1$. For every $n \in \omega$, set

$$U_n := \{\chi \in V^\circ : |\langle \chi, ST(x_n)\rangle| > 1\}.$$

Then U_n is a weak* open neighborhood of η_n in V°. Since V° is selectively sequentially pseudocompact in the weak* topology, for every $n \in \omega$ there exists $\chi_n \in U_n$ such that the sequence $\{\chi_n\}_{n\in\omega}$ contains a subsequence $\{\chi_{n_k}\}_{k\in\omega}$ which weak* converges to some functional $\chi \in V^\circ$. By Theorem 8.10.5 of [18], the adjoint operator S^* is weak* continuous and hence $S^*(\chi_{n_k}) \to S^*(\chi)$ in the weak* topology. Since T is weak-weak sequentially continuous, we have $ST(x_{n_k}) \to 0$ in the weak topology. Taking into account that T is weak* p-convergent, we obtain

$$|\langle \chi_{n_k}, ST(x_{n_k})\rangle| \le |\langle S^*(\chi_{n_k} - \chi), T(x_{n_k})\rangle| + |\langle \chi, ST(x_{n_k})\rangle| \to 0 \text{ as } k \to \infty.$$

Since, by the choice of χ_{n_k}, $|\langle \chi_{n_k}, ST(x_{n_k})\rangle| > 1$ we obtain a desired contradiction.

(v)⇒(vi) follows from the well known fact that $B^\circ_{c_0}$ is even a weak* metrizable compact space.

(vi)⇒(i) Assume that L is c_0-barrelled. To show that T is weak* p-convergent, fix a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and a weak* null sequence $\{\chi_n\}_{n\in\omega}$ in L'. Since L is c_0-barrelled, (ii) of Proposition 4.17 of [11] implies that there is $S \in \mathcal{L}(L, c_0)$ such that $S(y) = \big(\langle \chi_n, y\rangle\big)_{n\in\omega}$ for every $y \in L$. By assumption ST is p-convergent. Therefore $ST(x_n) \to 0$ in c_0. Taking into account that $\{e^*_n\}_{n\in\omega}$ is bounded in the Banach space $(c_0)' = \ell_1$, we obtain

$$\Big|\langle \chi_k, T(x_k)\rangle\Big| = \Big|\langle e^*_k, (\langle \chi_n, T(x_k)\rangle)_{n\in\omega}\rangle\Big| = \Big|\langle e^*_k, ST(x_k)\rangle\Big| \le \|e^*_k\|_{\ell_1} \cdot \|ST(x_k)\|_{c_0} = \|ST(x_k)\|_{c_0} \to 0.$$

This shows that T is weak* p-convergent.

(ii)⇒(vii) Assume that T transforms weakly sequentially p-precompact subsets of E to limited subsets of L, and let $R : X \to E$ be a weakly sequentially p-precompact operator. Then the set $R(B_X)$ is weakly sequentially p-precompact, and hence $TR(B_X)$ is a limited subset of L. Thus the operator $T \circ R$ is limited.

(vii)⇔(viii) follows from the equivalence (i)⇔(ii) of Theorem 5.5 of [12] applied to $T \circ R$ and $p = \infty$.

(vii)⇒(ix) is obvious.

(ix)⇒(i) Assume additionally that E is weakly sequentially locally p-complete. Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, by Proposition 14.9 of [11], the linear map $R : \ell^0_1 \to E$ defined by

$$R(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \dots, a_n \in \mathbb{F})$$

is continuous. It is clear that $R\big(B_{\ell^0_1}\big) \subseteq \overline{\mathrm{acx}}(S)$. Since E is weakly sequentially locally p-complete it follows that $\overline{\mathrm{acx}}(S)$ is weakly sequentially p-precompact. Therefore R is a weakly sequentially p-precompact operator and hence, by (ix), TR is limited. Whence for every weak* null sequence $\{\eta_n\}_{n\in\omega}$ in L' we obtain

$$|\langle \eta_n, T(x_n)\rangle| = |\langle \eta_n, TR(e_n)\rangle| \le \sup_{x\in B_{\ell^0_1}} |\langle \eta_n, TR(x)\rangle| \to 0 \text{ as } n \to \infty.$$

Thus T is weak* p-convergent.

(iii)⇒(x) Let Z be a normed space, and let $S : Z \to E$ be a weakly sequentially p-compact operator. Then $S(B_Z)$ is a relatively weakly sequentially p-compact subset of E. By (iii), the set $TS(B_Z)$ is limited. Thus the linear map $T \circ S$ is limited.

(x)⇒(xi) By Proposition 1.4 of [3] (or by (ii) of Corollary 13.11 of [11]), the identity operator $\mathrm{id}_{\ell_{p^*}}$ of ℓ_{p^*} is weakly sequentially p-compact. Hence each operator $S = S \circ \mathrm{id}_{\ell_{p^*}} \in \mathcal{L}(\ell_{p^*}, E)$ is weakly sequentially p-compact. Thus, by (x), $T \circ S$ is limited for every $S \in \mathcal{L}(\ell_{p^*}, E)$.

(xi)⇒(i) Assume that E is sequentially complete. Let $\{\chi_n\}_{n\in\omega}$ be a weak* null sequence in L', and let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. By Proposition 4.14 of [11], there is $S \in \mathcal{L}(\ell_{p^*}, E)$ such that $S(e_n^*) = x_n$ for every $n \in \omega$ (where $\{e_n^*\}_{n\in\omega}$ is the canonical unit basis of ℓ_{p^*}). By (xi), $T \circ S$ is limited. Therefore

$$\left|\langle \chi_n, T(x_n)\rangle\right| = \left|\langle \chi_n, TS(e_n^*)\rangle\right| \le \sup_{x\in B_{\ell_{p^*}}} |\langle \chi_n, TS(x)\rangle| \to 0 \quad \text{as } n \to \infty.$$

Thus the linear map T is weak* p-convergent. □

Remark 3.13. The condition on L to be c_0-barrelled in the implication (vi)⇒(i) in (B) of Theorem 3.12 is essential. Indeed, let $p \in [1,\infty]$, $E = L = C_p(\mathbf{s})$ and let $T = \mathrm{id} : E \to L$ be the identity map. Then, by (ii) of Example 3.3, L is a metrizable non-c_0-barrelled space and T is not weak* p-convergent. On the other hand, it is easy to see that each $S \in \mathcal{L}(L, c_0)$ is finite-dimensional (see Lemma 17.18 of [11]), and therefore $S \circ T$ is also finite-dimensional and hence p-convergent (see (i) of Proposition 3.1). Thus the implication (vi)⇒(i) in (B) of Theorem 3.12 does not hold. □

Below we characterize weakly p-convergent operators (for the definition of the map S_∞ see Section 2).

Theorem 3.14. *Let $p \in [1,\infty]$, E and L be locally convex spaces, and let $T : E \to L$ be an operator. Consider the following assertions:*

(i) *T is weakly p-convergent;*
(ii) *T transforms weakly sequentially p-precompact subsets of E to ∞-(V^*) subsets of L;*
(iii) *T transforms weakly sequentially p-compact subsets of E to ∞-(V^*) subsets of L;*
(iv) *T transforms weakly p-summable sequences of E to ∞-(V^*) subsets of L;*
(v) *$R \circ T$ is p-convergent for any Banach space Z and each $R \in \mathcal{L}(L, Z)$ with weakly sequentially precompact adjoint $R^* : Z'_\beta \to L'_\beta$;*
(vi) *$R \circ T$ is p-convergent for each $R \in \mathcal{L}(L, c_0)$ with weakly sequentially precompact adjoint $R^* : \ell_1 \to L'_\beta$;*
(vii) *$R \circ T$ is p-convergent for each operator $R \in \mathcal{L}(L, c_0)$ such that $S_\infty(R) = (\chi_n)$ is weakly null in L'_β;*
(viii) *for any normed space X and each weakly sequentially p-precompact operator $R : X \to E$, the operator $T \circ R$ is ∞-(V^*);*
(ix) *for every normed space X and each weakly sequentially p-precompact operator S from X to E, the map $S^* \circ T^*$ is ∞-convergent;*
(x) *if $R \in \mathcal{L}(\ell_1^0, E)$ is weakly sequentially p-precompact, then $T \circ R$ is ∞-(V^*);*
(xi) *for every normed space Z and each weakly sequentially p-compact operator S from Z to E, the composition $T \circ S$ is a ∞-(V^*) linear map;*
(xii) *for any operator $S \in \mathcal{L}(\ell_{p^*}, E)$, the linear map $T \circ S$ is ∞-(V^*).*

Then:

(A) (i)⇔(ii)⇔(iii)⇔(iv);
(B) (i)⇒(v)⇒(vi), *and if L is c_0-barrelled and L'_β is weakly sequentially locally ∞-complete, then* (vi)⇒(i);
(C) (i)⇒(vii), *and if L is c_0-barrelled, then* (vii)⇒(i);
(D) (ii)⇒(viii)⇔(ix)⇒(x), *and if E is weakly sequentially locally p-complete, then* (x)⇒(i);
(E) *if $1 < p < \infty$, then* (i)⇒(viii)⇒(xi)⇒(xii), *and if E is sequentially complete, then* (xii)⇒(i).

Proof. (i)⇒(ii) Let A be a weakly sequentially p-precompact subset of E, and suppose for a contradiction that $T(A)$ is not an ∞-(V^*) set. Therefore there is a weakly null sequence $\{\eta_n\}_{n\in\omega}$ in L'_β, a sequence $\{x_n\}_{n\in\omega}$ in A and $\varepsilon > 0$ such that $|\langle \eta_n, T(x_n)\rangle| > \varepsilon$ for every $n \in \omega$. Since A is weakly sequentially p-precompact, without loss of generality we assume that $\{x_n\}_{n\in\omega}$ is weakly p-Cauchy.

For $n_0 = 0$, since $\{\eta_n\}_{n\in\omega}$ is also weak* null we can choose $n_1 > n_0$ such that $|\langle \eta_{n_1}, T(x_{n_0})\rangle| < \frac{\varepsilon}{2}$. Proceeding by induction on k, we can choose $n_{k+1} > n_k$ such that $|\langle \eta_{n_{k+1}}, T(x_{n_k})\rangle| < \frac{\varepsilon}{2}$. Since the sequence $\{x_{n_{k+1}} - x_{n_k}\}_{k\in\omega}$ is weakly p-summable and T is weakly p-convergent, we obtain

$$\left\langle \eta_{n_{k+1}}, T\left(x_{n_{k+1}} - x_{n_k}\right)\right\rangle \to 0.$$

On the other hand, for every $k \in \omega$, we have

$$\left|\left\langle \eta_{n_{k+1}}, T\left(x_{n_{k+1}} - x_{n_k}\right)\right\rangle\right| \geq \left|\left\langle \eta_{n_{k+1}}, T\left(x_{n_{k+1}}\right)\right\rangle\right| - \left|\left\langle \eta_{n_{k+1}}, T\left(x_{n_k}\right)\right\rangle\right| > \frac{\varepsilon}{2},$$

a contradiction.

(ii)⇒(iii) and (iii)⇒(iv) are trivial.

(iv)⇒(i) follows from the definition of ∞-(V^*).

(i)⇒(v) Assume that T is weakly p-convergent, and suppose for a contradiction that $R \circ T$ is not p-convergent for some Banach space Z and $R \in \mathcal{L}(L, Z)$ with weakly sequentially precompact adjoint $R^* : Z'_\beta \to L'_\beta$. Then there are a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and $\delta > 0$ such that $\|RT(x_n)\| > \delta$ for every $n \in \omega$. By the Hahn–Banach separation theorem, for every $n \in \omega$ there is $\eta_n \in (\delta B_Z)^\circ$ such that $|\langle R^*(\eta_n), T(x_n)\rangle| = |\langle \eta_n, RT(x_n)\rangle| > 1$. Since R^* is weakly sequentially precompact, passing to subsequences if needed we can assume that the sequence $\{R^*(\eta_n)\}_{n\in\omega} \subseteq L'_\beta$ is weakly Cauchy.

For $n_0 = 0$, choose $n_1 > n_0$ such that $|\langle R^*(\eta_{n_0}), T(x_{n_1})\rangle| < \frac{1}{2}$ (this is possible since $\{x_n\}_{n\in\omega}$ and hence also $\{T(x_n)\}_{n\in\omega}$ are weakly null). By induction on $k \in \omega$, for every $k > 0$ choose $n_{k+1} > n_k$ such that $|\langle R^*(\eta_{n_k}), T(x_{n_{k+1}})\rangle| < \frac{1}{2}$. As $\{R^*(\eta_n)\}_{n\in\omega} \subseteq L'_\beta$ is weakly Cauchy, the sequence $\{R^*(\eta_{n_k}) - R^*(\eta_{n_{k+1}})\}_{k\in\omega}$ is weakly null in L'_β. Taking into account that $\{x_{n_k}\}_{n\in\omega}$ is also weakly p-summable and T is weakly p-convergent we obtain

$$0 \leftarrow \left|\left\langle R^*(\eta_{n_k} - \eta_{n_{k+1}}), T(x_{n_{k+1}})\right\rangle\right| \geq \left|\left\langle R^*(\eta_{n_{k+1}}), T(x_{n_{k+1}})\right\rangle\right| - \left|\left\langle R^*(\eta_{n_k}), T(x_{n_{k+1}})\right\rangle\right| > \frac{1}{2},$$

a contradiction.

(v)⇒(vi) is trivial.

(vi)⇒(i) Assume that L is c_0-barrelled and L'_β is weakly sequentially locally ∞-complete. To show that T is weakly p-convergent, fix a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and a weakly null sequence $\{\chi_n\}_{n\in\omega}$ in L'_β. Since L is c_0-barrelled, (ii) of Proposition 4.17 of [11] implies that there is $R \in \mathcal{L}(L, c_0)$ such that $R(y) = \left(\langle \chi_n, y\rangle\right)_{n\in\omega}$ for every $y \in L$. Since $\{\chi_n\}_{n\in\omega}$ is weakly null in L'_β and L'_β is weakly sequentially locally ∞-complete, it follows that $\overline{\mathrm{acx}}\left(\{\chi_n\}_{n\in\omega}\right)$ is weakly sequentially precompact. Taking into account that $R^*(e_n^*) = \chi_n$ for every $n \in \omega$ (where as usual $\{e_n^*\}_{n\in\omega}$ is the canonical unit basis of ℓ_1), we obtain $R^*(B_{\ell_1}) \subseteq \overline{\mathrm{acx}}\left(\{\chi_n\}_{n\in\omega}\right)$. Therefore R^* is weakly sequentially precompact, and hence, by (vi), RT is p-convergent. Hence $RT(x_n) \to 0$ in c_0. Therefore

$$\left|\langle \chi_k, T(x_k)\rangle\right| = \left|\langle e_k^*, (\langle \chi_n, T(x_k)\rangle)_{n\in\omega}\rangle\right| = \left|\langle e_k^*, RT(x_k)\rangle\right| \leq \|e_k^*\|_{\ell_1} \cdot \|RT(x_k)\|_{c_0} = \|RT(x_k)\|_{c_0} \to 0.$$

Thus T is weakly p-convergent.

(i)⇒(vii) Assume that T is weakly p-convergent, and suppose for a contradiction that $R \circ T$ is not p-convergent for some operator $R \in \mathcal{L}(L, c_0)$ such that the sequence $S_\infty(R) = (\chi_n)$ is weakly null in L'_β. Then there are a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and $\varepsilon > 0$ such that $\|RT(x_n)\|_{c_0} \geq \varepsilon$ for every $n \in \omega$. Recall that $R(y) = \left(\langle \chi_n, y\rangle\right)_n \in c_0$ for every $y \in L$. Then for every $n \in \omega$, we have (where as usual $\{e_i^*\}_{i\in\omega}$ is the canonical unit basis of $\ell_1 = (c_0)'$)

$$\varepsilon \leq \|RT(x_n)\|_{c_0} = \sup_{i\in\omega} |\langle e_i^*, RT(x_n)\rangle| = \sup_{i\in\omega} |\langle R^*(e_i^*), T(x_n)\rangle| = \sup_{i\in\omega} |\langle \chi_i, T(x_n)\rangle|. \tag{4}$$

For $n_0 = 0$, choose $i_0 \in \omega$ such that $|\langle \chi_{i_0}, T(x_{n_0})\rangle| \geq \frac{\varepsilon}{2}$. Since T is weak-weak sequentially continuous and because the sequence $\{x_n\}_{n\in\omega}$ is also weakly null, it follows that $T(x_n) \to 0$ in the weak topology of L. Therefore we can choose $n_1 > n_0$ such that

$$|\langle \chi_i, T(x_{n_1})\rangle| < \frac{\varepsilon}{2} \quad \text{for every } i \leq i_0. \tag{5}$$

By (4), there is $i_1 \in \omega$ such that $|\langle \chi_{i_1}, T(x_{n_1})\rangle| \geq \frac{\varepsilon}{2}$. Taking into account (5) we obtain that $i_1 > i_0$. Since $T(x_n) \to 0$ in the weak topology of L, there exists $n_2 > n_1$ such that

$$|\langle \chi_i, T(x_{n_2})\rangle| < \tfrac{\varepsilon}{2} \quad \text{for every } i \leq i_1. \tag{6}$$

By (4), there is $i_2 \in \omega$ such that $|\langle \chi_{i_2}, T(x_{n_2})\rangle| \geq \frac{\varepsilon}{2}$. By (6), we obtain that $i_2 > i_1$. Continuing this process we find two sequences $\{\chi_{i_k}\}_{k\in\omega}$ and $\{T(x_{n_k})\}_{n\in\omega}$ such that $\{i_k\}_{k\in\omega}$ and $\{n_k\}_{k\in\omega}$ are strictly increasing and $|\langle \chi_{i_k}, T(x_{n_k})\rangle| \geq \frac{\varepsilon}{2}$ for every $k \in \omega$. Clearly, the sequence $\{\chi_{i_k}\}_{k\in\omega}$ is weakly null in L'_β and the sequence $\{x_{n_k}\}_{n\in\omega}$ is weakly p-summable in E. Then the weak p-convergence of T and the choice of these two sequences imply

$$\tfrac{\varepsilon}{2} \leq \lim_{k\to\infty} |\langle \chi_{i_k}, T(x_{n_k})\rangle| = 0$$

which is impossible.

(vii)⇒(i) To show that T is weakly p-convergent, fix a weakly p-summable sequence $\{x_n\}_{n\in\omega}$ in E and a weakly null sequence $\{\chi_n\}_{n\in\omega}$ in L'_β. Since L is c_0-barrelled, (ii) of Proposition 4.17 of [11] implies that there is $R \in \mathcal{L}(L, c_0)$ such that $R(y) = \big(\langle \chi_n, y\rangle\big)_{n\in\omega}$ for every $y \in L$. Since $\{\chi_n\}_{n\in\omega}$ is weakly null in L'_β and $S_\infty(R) = (\chi_n)$, (vii) implies that RT is p-convergent. Hence $RT(x_n) \to 0$ in c_0. If $\{e^*_n\}_{n\in\omega}$ is the canonical unit basis of $(c_0)' = \ell_1$, we obtain

$$\Big|\langle \chi_k, T(x_k)\rangle\Big| = \Big|\langle e^*_k, (\langle \chi_n, T(x_k)\rangle)_{n\in\omega}\rangle\Big| = \Big|\langle e^*_k, RT(x_k)\rangle\Big| \leq \|e^*_k\|_{\ell_1} \cdot \|RT(x_k)\|_{c_0} = \|RT(x_k)\|_{c_0} \to 0.$$

Thus T is weakly p-convergent.

(ii)⇒(viii) Assume that T transforms weakly sequentially p-precompact subsets of E to ∞-(V^*) subsets of L, and let $R : X \to E$ be a weakly sequentially p-precompact operator. Then the set $R(B_X)$ is weakly sequentially p-precompact, and hence $TR(B_X)$ is an ∞-(V^*) subset of L. Thus the operator $T \circ R$ is ∞-(V^*).

(viii)⇔(ix) follows from the equivalence (i)⇔(ii) in Theorem 14.1 of [11] applied to $T \circ R$ and $p = \infty$.

(viii)⇒(x) is obvious.

(x)⇒(ii) Assume that E is weakly sequentially locally p-complete. Let $S = \{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. Then, by Proposition 14.9 of [11], the linear map $R : \ell_1^0 \to E$ defined by

$$R(a_0e_0 + \cdots + a_ne_n) := a_0x_0 + \cdots + a_nx_n \quad (n \in \omega,\ a_0, \dots, a_n \in \mathbb{F})$$

is continuous. It is clear that $R\big(B_{\ell_1^0}\big) \subseteq \overline{\mathrm{acx}}(S)$. Since E is weakly sequentially locally p-complete it follows that $\overline{\mathrm{acx}}(S)$ is weakly sequentially p-precompact. Therefore R is a weakly sequentially p-precompact operator and hence, by (x), TR is ∞-(V^*). Whence for every weakly null sequence $\{\eta_n\}_{n\in\omega}$ in L'_β we obtain

$$|\langle \eta_n, T(x_n)\rangle| = |\langle \eta_n, TR(e_n)\rangle| \leq \sup_{x\in B_{\ell_1^0}} |\langle \eta_n, TR(x)\rangle| \to 0 \text{ as } n \to \infty.$$

Thus T is weakly p-convergent.

(viii)⇒(xi) is trivial.

(xi)⇒(xii) Let $1 < p < \infty$.the identity operator $\mathrm{id}_{\ell_{p^*}}$ of ℓ_{p^*} is weakly sequentially p-compact. Hence, each operator $S = S \circ \mathrm{id}_{\ell_{p^*}} \in \mathcal{L}(\ell_{p^*}, E)$ is weakly sequentially p-compact. Thus, by (xi), $T \circ S$ is an ∞-(V^*) operator for every $S \in \mathcal{L}(\ell_{p^*}, E)$.

(xii)⇒(i) Let $1 < p < \infty$ and assume that E is sequentially complete. Let $\{\chi_n\}_{n\in\omega}$ be a weakly null sequence in L'_β, and let $\{x_n\}_{n\in\omega}$ be a weakly p-summable sequence in E. By Proposition 4.14 of [11], there is $S \in \mathcal{L}(\ell_{p^*}, E)$ such that $S(e^*_n) = x_n$ for every $n \in \omega$ (where $\{e^*_n\}_{n\in\omega}$ is the canonical unit basis of ℓ_{p^*}). By (xii), $T \circ S$ is a ∞-(V^*) map. Therefore

$$\Big|\langle \chi_n, T(x_n)\rangle\Big| = \Big|\langle \chi_n, TS(e^*_n)\rangle\Big| \leq \sup_{x\in B_{\ell_{p^*}}} |\langle \chi_n, TS(x)\rangle| \to 0 \quad \text{as } n \to \infty.$$

Thus the linear map T is weakly p-convergent. □

Remark 3.15. The condition on E to be sequentially complete in (D) of Theorem 3.12 and in (E) of Theorem 3.14 is essential. Indeed, let $1 < p < \infty$, $E = \ell^0_{p^*}$, $L = \ell_q$ with $p^* \leq q < \infty$, and let $T = \mathrm{id} : E \to L$ be the identity inclusion. Then every $S \in \mathcal{L}(\ell_{p^*}, E)$ is finite-dimensional (indeed, since $E = \bigcup_{n\in\omega} \mathbb{F}^n$ it follows that $\ell_{p^*} = \bigcup_{n\in\omega} S^{-1}(\mathbb{F}^n)$ and hence, by the Baire property of ℓ_{p^*}, $S^{-1}(\mathbb{F}^m)$ is an open linear subspace of ℓ_{p^*} for some $m \in \omega$ that is possible only if $S^{-1}(\mathbb{F}^m) = \ell_{p^*}$; so S is finite-dimensional). Therefore, by Lemma 3.5, $T \circ S$ is limited for each $S \in \mathcal{L}(\ell_{p^*}, E)$. However, T is not weakly p-convergent and hence, by (ii) of Proposition 3.1, T is not weak* p-convergent. Indeed, for every $n \in \omega$, let $x_n = e^*_n \in E$ and $\eta_n = e^*_n \in L'$. Then the sequence $\{x_n\}_{n\in\omega}$ is weakly p-summable in E (for every $\chi = (a_n) \in \ell_p = E'$, we have $\sum_{n\in\omega} |\langle \chi, x_n\rangle|^p = \sum_{n\in\omega} |a_n|^p < \infty$) and the sequence $\{\eta_n\}_{n\in\omega}$ is even weakly q-summable in $\ell_{q^*} = L'_\beta$ (for every $x = (b_n) \in \ell_q = (L'_\beta)'$, we have $\sum_{n\in\omega} |\langle x, \eta_n\rangle|^q = \sum_{n\in\omega} |b_n|^q < \infty$). Since for every $n \in \omega$, $\langle \eta_n, T(x_n)\rangle = 1$ it follows that T is not weakly p-convergent. □

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