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Disjoint topologically super-recurrent operators

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Abstract. In this article, we introduce and study the notion of disjoint topologically super-recurrent operators for finitely many operators acting on a complex Banach space. As an application, we characterize the disjoint topological super-recurrence of finitely many different powers of unilateral and bilateral weighted shifts.

1. Introduction and preliminaries

Throughout this paper, we use X to represent a Banach space over the field $\mathbb{C}$ of complex numbers. An operator is defined as a map that is both linear and continuous on X. The collection of all such operators is denoted by $\mathcal{B}(X)$. An operator $T \in \mathcal{B}(X)$ is considered hypercyclic (respectively, supercyclic) if there exists a vector $x \in X$ such that the orbit $\text{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ (respectively, the projective orbit $\mathbb{C}\text{Orb}(T, x) = \{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$) is dense in X. Such a vector x is called a hypercyclic vector (respectively, supercyclic vector) for T.

An operator T on a separable Banach space X is hypercyclic if and only if it exhibits topological transitivity in dynamical systems. This implies that for any two non-empty open subsets U and V of X, there is an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. The first hypercyclic operator on a Banach space was demonstrated by Rolewicz in 1969 [34], showing that the scaled shift λB of the unweighted unilateral backward shift B on $\ell^2(\mathbb{N})$ is hypercyclic if and only if $|\lambda| > 1$.

Salas [35] offered a comprehensive characterization of hypercyclic unilateral weighted backward shifts on $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$ and hypercyclic bilateral weighted shifts on $\ell^p(\mathbb{Z})$ for $1 \leq p < \infty$ based on their weight sequences. Using this characterization, Fernando León-Saavedra and Montes-Rodríguez [28, p. 544] demonstrated that these weighted shifts are hypercyclic if and only if they fulfill the hypercyclicity criterion, which is fundamental in hypercyclic operator theory.

In 1974, Hilden and Wallen [21] showed that any unilateral backward weighted shift is supercyclic. Later, Salas [36] characterized supercyclic bilateral weighted shifts on $\ell^p(\mathbb{Z})$ for $1 \leq p < \infty$ using their weight sequences. In the same work, he introduced a supercyclicity criterion similar to the hypercyclicity criterion, proving that weighted shift operators are supercyclic if and only if they satisfy this supercyclicity criterion.

For recent results in this field, see [5, 7, 22–25].

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In [33] Poincaré proposed the concept of recurrence, which has a lengthy history in the linear dynamical system. Furstenberg in [13], Gottschalk and Hedlund [16] later studied it. Recently, recurrent operators have been studied in [11]. If $n \in \mathbb{N}$ exists for each open subset U of X, such that

$$T^n(U) \cap U \neq \emptyset,$$

then $T \in \mathcal{B}(X)$ is recurrent. If there exists an increasing sequence $(n_k) \subset \mathbb{N}$ such that

$$T^{n_k}x \longrightarrow x,$$

then $x \in X$ is called a recurrent vector for T. We denoted by $Rec(T)$ the set of all recurrent vectors for T, and we have that T is recurrent if and only if $Rec(T)$ is dense in X. For more information about this classes of operators, see [1–3, 11–13, 26, 27].
Recurrent operators have been extended to a broad class of operators known as super-recurrent operators recently in [4]. We say that $T \in \mathcal{B}(X)$ is super-recurrent if, for each open subset U of X, there exists $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that

$$\lambda T^n(U) \cap U \neq \emptyset.$$

A vector $x \in X$ is called a super-recurrent vector for T if there exists an increasing sequence $(n_k) \subset \mathbb{N}$ and a sequence $(\lambda_k) \subset \mathbb{C}$ such that

$$\lambda_k T^{n_k}x \longrightarrow x.$$

The set of all super-recurrent vectors for T is denoted by $SRec(T)$. T is considered super-recurrent if and only if $SRec(T)$ is dense in X.

Now if given $N \geq 2$ operators $T_1, T_2, ..., T_N \in \mathcal{B}(X)$, it has been natural to study whether the hypercyclic (or supercyclic) properties of their direct sum $T_1 \oplus ... \oplus T_N$ may inherit from those of $T_1, T_2, ..., T_N$. In 2007, Bernal [8] and Bès [9] independently looked into the orbits' properties

$$\{(z, z, ..., z), (T_1z, T_2z, ..., T_Nz), (T_1^2z, T_2^2z, ..., T_N^2z), ...\}(z \in X).$$

They studied the case when one of these orbits is dense in X^N endowed with the product topology. The operators $T_1, T_2, ..., T_N$ are said to be disjoint hypercyclic (d-hypercyclic) if there exists a vector $z \in X$ that satisfies the above condition. Research on the d-hypercyclicity of a finite number of weighted shifts has been conducted in several papers, as noted in [8, 9, 30].

Likewise, if there exists a vector $\hat{z} \in X$ such that the projective orbit

$$\mathbb{C}\{(\hat{z}, \hat{z}, \ldots, \hat{z}), (T_1\hat{z}, T_2\hat{z}, \ldots, T_N\hat{z}), (T_1^2\hat{z}, T_2^2\hat{z}, \ldots, T_N^2\hat{z}), \ldots\}(\hat{z} \in X).$$

is dense in the product space X^N, then the operators $T_1, T_2, \ldots, T_N$ are termed disjoint supercyclic (d-supercyclic). Recent literature has examined the d-supercyclicity of finite sets of weighted shifts. Interested readers can explore Section 4 in [9] and Chapter 4 in [31] for further details.

Definition 1.1. *([30, Definition 1.2]) We say the operators $T_1, T_2, ..., T_N \in \mathcal{B}(X)$ are d-topologically transitive for supercyclicity provided for every non-empty open subsets $V_0, V_1, ..., V_N$ of X, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that*

$$V_0 \cap (\lambda T_1^{-n})(V_1) \cap ... \cap (\lambda T_N^{-n})(V_N) \neq \emptyset.$$

The following results connect d-topologically transitive supercyclicity with d-supercyclicity.

Proposition 1.2. *([30, Proposition 1.3]) Given $N \geq 2$ and the operators $T_1, T_2, ..., T_N \in \mathcal{B}(X)$, they are d-topologically transitive for supercyclicity if and only if the set of d-supercyclic vectors for $T_1, T_2, ..., T_N$ is a dense G_δ set.*

The following criterion plays an important role in our main results: This criterion is due to Ö. Martin and R. Sanders [32].

Definition 1.3. *Let X be a Banach space, $(n_k)_k$ be a strictly increasing sequence of positive integers and $N \geq 2$. We say that $T_1, T_2, ..., T_N \in \mathcal{B}(X)$ satisfy the d-Supercyclicity Criterion with respect to $(n_k)_k$ provided there exist dense subsets $X_0, X_1, ..., X_N$ of X and mappings $S_{l,k} : X_l \longrightarrow X (1 \leq l \leq N, k \in \mathbb{N})$ such that for $1 \leq l \leq N$,*

(i) $(T_l^{n_k} S_{i,k} - \delta_{i,l} I_{X_i}) \longrightarrow 0$ *pointwise on* $X_i (1 \leq i \leq N)$;

(ii) $\lim_{k \longrightarrow \infty} \left\| T_l^{n_k} x \right\| . \left\| \sum_{j=1}^{N} S_{j,k} y_j \right\| = 0$ *for* $x \in X_0$ *and* $y_j \in Y_j$.

The d-Supercyclicity Criterion is satisfied by operators $T_1, T_2, ..., T_N$ if there exists a sequence $(n_k)_{k \in \mathbb{N}}$ satisfying (i) and (ii).

Let $N \geq 2$ and the operators $T_1, T_2, ..., T_N \in \mathcal{B}(X)$ satisfy the d-Supercyclicity Criterion. Then $T_1, T_2, ..., T_N$ have a dense set of d-supercylic vectors (see [32, Proposition 1.12]).

We introduce a new concept in this paper, called disjoint topologically super-recurrent operators, in the linear dynamical system by association between d-topologically transitive for supercyclicity and super-recurrent operators.

The article is organized as follows: In Section 2 we provide some basic definitions associated with disjoint topologically super-recurrence. In addition, the related properties are obtained, which play a key role in the theory of disjoint topologically super-recurrent. In Section 3 we characterize the disjoint topologically super-recurrent for distinct powers of weighted bilateral(unilateral) shifts.

2. Disjoint topologically super-recurrent operators

Definition 2.1. *For $N \geq 2$, the operators $T_1, T_2, ..., T_N$ in $\mathcal{B}(X)$ are disjoint topologically super-recurrent if for every nonempty open subset U of X, there exists $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that*

$$U \cap \left(\lambda T_1^{-m}\right)(U) \cap ... \cap \left(\lambda T_N^{-m}\right)(U) \neq \emptyset.$$

A vector $x \in X \setminus \{0\}$ is called a disjoint super-recurrent for $T_1, T_2, ..., T_N$ if $(x, ..., x) \in X^N$ is a super-recurrent vector for the direct sums $\bigoplus_{i=1}^{N} T_i$.

Remark 2.2. *(a) Obviously, the constant λ in Definition 2.1 is nonzero. So, let U be a non-empty open subset of X and assume first that $0 \notin U$. Suppose that $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$ are such that*

$$U \cap \lambda T_1^{-m}(U) \cap ... \cap \lambda T_N^{-m}(U) \neq \emptyset,$$

if λ is 0, we must have that

$$U \cap \lambda T_1^{-m}(U) \cap ... \cap \lambda T_N^{-m}(U) = \{0\}.$$

Thus, U contains 0, which is a contradiction, so we must have that $\lambda \neq 0$.
Assume now that $0 \in U$. Then it follows trivially that

$$0 \in U \cap \alpha T_1^{-m}(U) \cap ... \cap \alpha T_N^{-m}(U)$$

for all non-zero $\alpha \in \mathbb{C}$ since $0 \in (T_i^{-m})(U)$ because $0 \in U$ and $T_i^{m}(0) = 0$ (for all $i \in \{1, ..., N\}$).
Therefore, we may without loss of generality assume that $\lambda \neq 0$ in Definition 2.1.

(b) It is easy to see that if $T_1, T_2, ..., T_N$ ($N \geq 2$) are disjoint topologically super-recurrent on X, then $(T_i)_{1 \leq i \leq N}$ are super-recurrent operators, and every disjoint super-recurrent vector for $T_1, T_2, ..., T_N$ is a super-recurrent vector for any operator T_i, such that $1 \leq i \leq N$.

For $N \geq 2$ operators $T_1, T_2, ..., T_N$, we explain the relationship between the set of disjoint super-recurrent vectors and the disjoint topologically super-recurrent in the following result.

Theorem 2.3. *Let $T_1, T_2, ..., T_N$ be $N \geq 2$ operators acting on X. Then, the following are equivalent:*

(i) $T_1, T_2, ..., T_N$ are disjoint topologically super-recurrent operators;

(ii) $T_1, T_2, ..., T_N$ admits a dense subset of disjoint super-recurrent vectors.

Proof. $(ii) \Rightarrow (i)$. Let U be a nonempty open subset of X, then there is a disjoint super-recurrent vector x for $T_1, ..., T_N$ such that $x \in U$. So, there exist an increasing sequence (n_k) of positive integers and a sequence (λ_k) of complex numbers such that

$$\lambda_k (T_1 \oplus ... \oplus T_N)^{n_k} (x, ..., x) \longrightarrow (x, ..., x)$$

as $k \longrightarrow +\infty$. Since U is nonempty open and $x \in U$, it follows that there exists $m \in \mathbb{N}$ such that $\lambda_m T_i^m x \in U$ for all $1 \leq i \leq N$. Thus,

$$x \in U \cap \left(\lambda_m^{-1} T_1^{-m}\right)(U) \cap ... \cap \left(\lambda_m^{-1} T_N^{-m}\right)(U).$$

Then, $T_1, T_2, ...T_N$ are disjoint topologically super-recurrent.

$(i) \Rightarrow (ii)$. For a fixed element $x \in X$ and a fixed strictly positive number $\varepsilon > 0$, let $B := B(x, \varepsilon)$. We need to show that there is a disjoint super-recurrent vector for $T_1, T_2, ..., T_N$ which belongs to B. Since $T_1, T_2, ..., T_N$ are disjoint topologically super-recurrent, there exist some positive integer k_1 and some complex number λ_1 such that

$$B \cap \lambda_1 T_1^{-k_1}(B) \cap ... \cap \lambda_1 T_N^{-k_1}(B) \neq \emptyset.$$

Let $x_1 \in X$ such that $x_1 \in B \cap \lambda_1 T_1^{-k_1}(B) \cap ... \cap \lambda_1 T_N^{-k_1}(B)$. Since $T_1, ..., T_N$ are continuous, there exists $\varepsilon_1 < \frac{1}{2}$ such that

$$B_2 := B(x_1, \varepsilon_1) \subset B \cap \bigcap_{i=1}^{N} \lambda_1 T_i^{-k_1}(B).$$

Again, since $T_1, ..., T_N$ are disjoint topologically super-recurrent, there exist some $k_2 \in \mathbb{N}$ and $\lambda_2 \in \mathbb{C}$ such that

$$B_2 \cap \lambda_2 T_1^{-k_2}(B_2) \cap ... \cap \lambda_2 T_N^{-k_2}(B_2) \neq \emptyset.$$

Let $x_2 \in X$ such that $x_2 \in B_2 \cap \lambda_2 T_1^{-k_2}(B_2) \cap ... \cap \lambda_2 T_N^{-k_2}(B_2)$. Since $T_1, T_2, ..., T_N$ are continuous, there exists $\varepsilon_2 < \frac{1}{2^2}$ such that

$$B_3 := B(x_2, \varepsilon_2) \subset B_2 \cap \bigcap_{i=1}^{N} \lambda_2 T_i^{-k_2}(B_2).$$

Continuing inductively, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X, a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of complex numbers, a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ and a sequence of positive real numbers $\varepsilon_n < \frac{1}{2^n}$, such that

$$B(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1}) \quad \textit{and} \quad \bigcup_{i=1}^{N} \lambda_n^{-1} T_i^{k_n}(B(x_n, \varepsilon_n)) \subset B(x_{n-1}, \varepsilon_{n-1}).$$

Since X is a Banach space, we conclude by Cantor's theorem that

$$\bigcap_{n\in\mathbb{N}} B(x_n, \varepsilon_n) = \{y\},$$

for some $y \in X$. It readily follows that $\lambda_n T_i^{k_n} y \longrightarrow y$ for all $1 \leq i \leq N$. Then,

$$\lambda_n (T_1 \oplus ... \oplus T_N)^{k_n} (y, ..., y) \longrightarrow (y, ..., y),$$

and so y is a disjoint topologically super-recurrent vector for $T_1, T_2, ..., T_N$ such that $y \in B$. $\square$

Remark 2.4. *By using a proof similar to that of Proposition 1.2, we can also see that if $T_1, T_2, ..., T_N$ are disjoint topologically super-recurrent, then the set of disjoint super-recurrent vectors for $T_1, T_2, ..., T_N$ is a dense G_δ set.*

The condition topologically super-recurrent for $N \geq 2$ operators $T_1, T_2, ..., T_N$ implies that $T_1, T_2, ..., T_N$ has disjoint super-recurrent vectors, but the converse is not true in general, as we show in the next example.

Example 2.5. *Let X be a Banach space, and let $(e_i)_{i\in I}$ be a basis of X. Let $i_0 \in I$ and $\lambda \in \mathbb{C}$ be nonzero fixed numbers. We define an operator T on X by:*

$$Te_{i_0} = \lambda e_{i_0} \text{ and } Te_i = 0, \text{for all } i \in I \setminus \{i_0\}.$$

We have that (e_{i_0}, e_{i_0}) is a super-recurrent vector for $T \oplus T$, but T is not super-recurrent (see [4]). Then, $T \oplus T$ is not super-recurrent. And so T, T is not topologically disjoint super-recurrent.

Remark 2.6. *If $T_1, T_2, ..., T_N$ are disjoint topologically super-recurrent operators on a Banach space X, then $\lambda T_1, \lambda T_2, ..., \lambda T_N$ are disjoint topologically super-recurrent operators for all $\lambda \in \mathbb{C}^*$. Moreover, they share the disjoint super-recurrent vectors.*

Theorem 2.7. *Let p be a nonzero positive integer and $T_1, T_2, ..., T_N$ be $N \geq 2$ operators acting on a Banach space X. Then, $T_1, T_2, ..., T_N$ are disjoint topologically super-recurrent if and only if $T_1^p, T_2^p, ..., T_N^p$ are disjoint topologically super-recurrent. Moreover, $T_1, T_2, ..., T_N$ and $T_1^p, T_2^p, ..., T_N^p$ have the same disjoint super-recurrent vectors.*

Proof. It is easy to show that if $T_1^p, T_2^p, ..., T_N^p$ are disjoint topologically super-recurrent operators, then $T_1, T_2, ..., T_N$ are also disjoint topologically super-recurrent operators. We now prove that every disjoint super-recurrent vector for $T_1, T_2, ..., T_N$ is a disjoint super-recurrent vector for $T_1^p, T_2^p, ..., T_N^p$. So, let x be a disjoint super-recurrent vector for $T_1, T_2, ..., T_N$. Then there exist a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of complex numbers such that

$$\lambda_n (T_1 \oplus ... \oplus T_N)^{k_n} (x, ..., x) \longrightarrow (x, ..., x)$$

as $n \longrightarrow +\infty$. Without loss of generality we may suppose that $k_n > p$ for all $n \in \mathbb{N}$. Hence, for all n, there exist $l_n \in \mathbb{N}$ and $v_n \in \{0, ..., p-1\}$ such that

$$k_n = pl_n + v_n.$$

Since $v_n \in \{0, ..., p-1\}$ for all n, it is obvious that (v_n) contains a subsequence that consists of a single (repeated) element, so we can call this element v. Thus,

$$\lambda_n (T_1 \oplus ... \oplus T_N)^{pl_n+v} (x, ..., x) \longrightarrow (x, ..., x)$$

for some subsequence of (l_n) and a subsequence (λ_n) which we call them again (l_n) and (λ_n). Let U be a nonempty open subset of X such that $x \in U$. Since

$$\lambda_n (T_1 \oplus ... \oplus T_N)^{pl_n+v} (x, ..., x) \longrightarrow (x, ..., x),$$

we have $\lambda_n T_i^{pl_n+v} x \longrightarrow x$ for all $1 \leq i \leq N$, and there exists a positive integer $m_1 := l_{n_1}$ such that $\lambda_{n_1} T_i^{pm_1+v} x \in U$ for all $1 \leq i \leq N$. Then,

$$\lambda_n \lambda_{n_1} T_i^{p(l_n+m_1)+2v} x = \lambda_{n_1} T_i^{pm_1+v} (\lambda_n T_i^{pl_n+v} x) \longrightarrow \lambda_{n_1} T_i^{pm_1+v} x \in U$$

for all $1 \leq i \leq N$. Thus, we can find a positive integer $m_2 := m_1 + l_{n_2} > m_1$ such that $\lambda_{n_1} \lambda_{n_2} T_i^{pm_2+2v} x \in U$ for all $1 \leq i \leq N$. Continuing inductively we can find a positive integer $m_p = m_{p-1} + l_{n_p}$ such that

$$\lambda_{n_1} ... \lambda_{n_p} T_i^{pm_p+pv} x \in U$$

for all $1 \leq i \leq N$. Put $\lambda = \lambda_{n_1} ... \lambda_{n_p}$, then $\lambda \left(T_i^p\right)^{m_p+v} x \in U$ for all $1 \leq i \leq N$. Hence x is a disjoint super-recurrent vector for $T_1^p, T_2^p, ..., T_N^p$. Now we just apply Theorem 2.3 to complete the proof. □

Remark 2.8. *The d-topologically transitive for supercyclicity implies the disjoint topologically super-recurrent. However, the converse does not hold in general. Indeed, let* $\alpha_1, ..., \alpha_4$ *be four nonzero complex numbers such that* $|\alpha_1| = ... = |\alpha_4| = R$ *for some strictly positive real number R. We define two operators* T_1 *and* T_2 *on* $\mathbb{C}^2$ *by*

$$\begin{array}{rcl} T_1 : \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x_1, x_2) & \longmapsto & (\alpha_1 x_1, \alpha_2 x_2) \end{array}$$

and

$$\begin{array}{rcl} T_2 : \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x_1, x_2) & \longmapsto & (\alpha_3 x_1, \alpha_4 x_2). \end{array}$$

Let U be a nonempty open subset of $\mathbb{C}^2$ *and* $x \in U$*. We will prove that x is a disjoint super-recurrent vector for* T_1, T_2*. Since* $\left|R^{-1}\alpha_1\right| = ... = \left|R^{-1}\alpha_4\right| = 1$*, it follows that there exists a strictly increasing sequence of positive integers* $(k_n)_{n\in\mathbb{N}}$ *such that* $\left(R^{-1}\alpha_i\right)^{k_n} \longrightarrow 1$*, for all* $1 \leq i \leq 4$*. Let* $\lambda_k = R^{-k_n}$*, for all k, then*

$$\lambda_k (T_1 \oplus T_2)^{k_n}(x, x) \longrightarrow (x, x).$$

So x is a disjoint super-recurrent vector for T_1, T_2*. We now apply Theorem 2.3 to provide that* T_1, T_2 *are disjoint topologically super-recurrent, but* T_1 *and* T_2 *are not supercyclic (see [4]). Then,* T_1, T_2 *are not d-topologically transitive for supercyclicity.*

3. Disjoint topologically super-recurrent weighted shifts

In this section, we characterize the disjoint topologically super-recurrent for distinct powers of weighted bilateral (unilateral) shifts in terms of their weight sequences by showing that the disjoint topologically super-recurrence and d-topologically transitive for supercyclicity are equivalent in this case.

3.1. *Unilateral shifts*

As shown below, $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $(1 \leq p < \infty)$ over the complex scalar field $\mathbb{C}$. Given a bounded sequence $a = (a_k)_{k\geq 1}$ with nonzero weights, define the unilateral weighted shift $T_a : X \longrightarrow X$ as follows:

$$T_a(x_0, x_1, ...) = (a_1 x_1, a_2 x_2, ...).$$

Theorem 3.1. *Let* $X = c_0(\mathbb{N})$ *or* $\ell^p(\mathbb{N})$, $(1 \leq p < \infty)$*. For* $N \geq 2$ *and* $1 \leq l \leq N$*, let* $a_l = (a_{l,n})_{n=1}^{\infty}$ *be a bounded sequence of nonzero scalars and let* T_l *be the associated unilateral backward shift on X:*

$$T_l(x_0, x_1, ...) = (a_{l,1} x_1, a_{l,2} x_2, ...).$$

For any integers $1 \leq r_1 < r_2 < ... < r_N$*, the following are equivalent:*

a) $T_1^{r_1}, T_2^{r_2}, ..., T_N^{r_N}$ *are disjoint topologically super-recurrent operators;*
b) $T_1^{r_1}, T_2^{r_2}, ..., T_N^{r_N}$ *have a dense set of d-supercyclic vectors;*
c) For each $\varepsilon > 0$ *and* $q \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *satisfying, for each* $0 \leq j \leq q$

$$\frac{|\prod_{i=j+1}^{j+r_l m} a_{l,i}|}{|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{l,i}|} > \frac{1}{\varepsilon} \quad (1 \leq s < l \leq N) \tag{1}$$

d) $T_1^{r_1}, T_2^{r_2}, ..., T_N^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Proof. Let us first prove that *a*) implies *c*). Fix a positive integer N and let $\varepsilon > 0$. Take also a positive integer q. Then consider a positive number δ such that $\frac{\delta}{(1-\delta)} < \varepsilon$ and $\delta < 1$. Consider the open ball $B\left(\sum_{j=0}^{q} e_j, \delta\right)$. There exists $m \in \mathbb{N}$ ($m > q$) and $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) such that

$$B\left(\sum_{j=0}^{q} e_j, \delta\right) \cap \left(\lambda T_1^{-mr_1}\right)\left(B\left(\sum_{j=0}^{q} e_j, \delta\right)\right) \cap \ldots \cap \left(\lambda T_N^{-mr_N}\right)\left(B\left(\sum_{j=0}^{q} e_j, \delta\right)\right) \neq \emptyset.$$

Hence there exists $x = \sum_{k=0}^{\infty} x_k e_k \in X$ such that for every $l = 1, \ldots, N$, in particular,

$$\left|\lambda\left(\prod_{i=j+1}^{j+r_l m} a_{l,i}\right) x_{j+r_l m} - 1\right| \quad < \quad \delta \quad if \quad (0 \leq j \leq q), \tag{2}$$

and

$$\left|\lambda\left(\prod_{i=k+1}^{k+r_l m} a_{l,i}\right) x_{k+r_l m}\right| \quad < \quad \delta \quad if \quad (k > q). \tag{3}$$

Having at our hands the above inequalities we argue as in the proof of [29, Theorem 3.2] and we conclude that for all $0 \leq j \leq q$ we have:

$$if \quad 1 \leq s < l \leq N \quad \frac{|\prod_{i=j+1}^{j+r_l m} a_{l,i}|}{|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{l,i}|} \quad > \quad \frac{1}{\varepsilon}.$$

Hence we proved that *a*) implies *c*). Condition *b*), *c*) and *d*) are known to be equivalent from [29, Theorem 3.2]. Finally the implication *b*) $\Rightarrow$ *a*) hold trivially and this completes the proof of the equivalence of statements *a*) − *d*) of the theorem. □

In the following, we obtain special cases of Theorem 3.1.

Corollary 3.2. *Let* $X = c_0(\mathbb{N})$ *or* $\ell^p(\mathbb{N})$, $(1 \leq p < \infty)$. *For* $N \geq 2$, *let* $r_l \in \mathbb{N}$ *and* $\lambda_l \in \mathbb{C}$ $(1 \leq l \leq N)$ *with* $1 \leq r_1 < r_2 < \ldots < r_N$. *Let* $B : X \longrightarrow X$ *be the backward shift defined as follows:*

$$B(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

The following statements are equivalent:

a) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \ldots, \lambda_N B^{r_N}$ *are disjoint topologically super-recurrent operators;*
b) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \ldots, \lambda_N B^{r_N}$ *have a dense set of d-supercyclic vectors;*
c) For each $\varepsilon > 0$ *there exists* $m \in \mathbb{N}$ *satisfying,*

$$\left|\frac{\lambda_s}{\lambda_l}\right|^m \quad < \quad \varepsilon \quad (1 \leq s < l \leq N);$$

d) $\lambda_1 B^{r_1}, \lambda_2 B^{r_2}, \ldots, \lambda_N B^{r_N}$ *satisfy the d-Supercyclicity Criterion.*

Proof. For each $0 \leq l \leq N$, let λ_l^{1/r_l} denote a fixed root of $z^{r_l} - \lambda_l$, and let T_l denote the unilateral backward shift with constant weight sequence $a_l = (a_{l,n})_{n=1}^{\infty} = \left(\lambda_l^{1/r_l}\right)_{n=1}^{\infty}$. Then $T_l^{r_l} = \lambda_l B^{r_l}$. It is clear that the result follows from Theorem 3.1. □

3.2. Bilateral shifts

Recall that $1 \leq p < \infty$, the space $\ell^p(\mathbb{Z})$ denotes the Banach space of bilateral sequences that are p-summable. Let $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, and let $(e_k)_{k\in\mathbb{Z}}$ be the canonical basis of X. If $a = (a_k)_{k\in\mathbb{Z}}$ is a bounded weight sequence of nonzero scalars in $\mathbb{C}$, then the bilateral backward weighted shift $B_a : X \longrightarrow X$ is the bounded operator defined by

$$B_a e_n = a_n e_{n-1} \quad (n \in \mathbb{Z}).$$

Theorem 3.3. *Let $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, $(1 \leq p < \infty)$. For $N \geq 2$ and $l = 1, ..., N$, let $a_l = \left(a_{l,j}\right)_{j\in\mathbb{Z}}$ be a bounded bilateral sequence of nonzero scalars, and let B_l be the associated backward shift on X given by $B_l e_k = a_{l,k} e_{k-1}$ $(k \in \mathbb{Z})$. For any integers $1 \leq r_1 < r_2 < ... < r_N$, the following are equivalent:*

a) $B_1^{r_1}, B_2^{r_2}, ..., B_N^{r_N}$ are disjoint topologically super-recurrent operators;

b) $B_1^{r_1}, B_2^{r_2}, ..., B_N^{r_N}$ have a dense set of d-supercyclic vectors;

c) For each $\varepsilon > 0$ and $q \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > 2q)$, such that, for each $|j|, |k| \leq q$ and $1 \leq l, s \leq N$, we have that

$$\left|\prod_{i=j-r_l m+1}^{j} a_{l,i}\right| < \varepsilon \left|\prod_{i=k+1}^{k+r_s m} a_{s,i}\right| \quad (1 \leq l, s \leq N), \tag{4}$$

and for $1 \leq s < l \leq N$

$$\begin{cases} \left|\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} a_{s,i}\right| < \varepsilon \left|\prod_{i=j+1}^{j+r_l m} a_{l,i}\right| \\ \left|\prod_{i=j-(r_l-r_s)m+1}^{j+r_s m} a_{l,i}\right| < \varepsilon \left|\prod_{i=j+1}^{j+r_s m} a_{s,i}\right| \end{cases}. \tag{5}$$

d) $B_1^{r_1}, B_2^{r_2}, ..., B_N^{r_N}$ satisfy the d-Supercyclicity Criterion.

Proof. Let us first prove that *a*) implies *c*). Fix a positive integer N and let $\varepsilon > 0$. Take also a positive integer q. Then consider a positive number δ such that $\frac{\delta}{(1-\delta)} < \varepsilon$ and $0 < \delta < \frac{1}{2}$. Consider the open ball $B\left(\sum_{|j|\leq q} e_j, \delta\right)$. There exists $m \in \mathbb{N}$ $(m > 2q)$ and $0 \neq \lambda \in \mathbb{C}$ such that

$$B\left(\sum_{|j|\leq q} e_j, \delta\right) \cap \left(\lambda B_1^{-mr_1}\right)\left(B\left(\sum_{|j|\leq q} e_j, \delta\right)\right) \cap ... \cap \left(\lambda B_N^{-mr_N}\right)\left(B\left(\sum_{|j|\leq q} e_j, \delta\right)\right) \neq \emptyset.$$

Hence there exists $x = \sum_{k\in\mathbb{Z}} x_k e_k \in X$ such that

$$\begin{aligned} &\left\|x - \sum_{|j|\leq q} e_j\right\| < \delta \\ &\left\|\lambda B_l^{r_l m} x - \sum_{|j|\leq q} e_j\right\| < \delta \quad (1 \leq l \leq N). \end{aligned} \tag{6}$$

It follows that

$$\left|x_j - 1\right| < \delta \quad if \quad \left|j\right| \leq q, \tag{7}$$

$$\left|x_j\right| < \delta \quad if \quad \left|j\right| > q. \tag{8}$$

By (6), so

$$\left|\lambda\left(\prod_{i=j+1}^{j+r_lm}a_{l,i}\right)x_{j+r_lm}-1\right|<\delta \quad if \quad |j|\le q, \tag{9}$$

$$\left|\lambda\left(\prod_{i=k+1}^{k+r_lm}a_{l,i}\right)x_{k+r_lm}\right|<\delta \quad if \quad |k|>q. \tag{10}$$

Having at our hands the above inequalities we argue as in the proof of [29, Theorem 4.1] and we conclude that for all $|j|\le q$ we have:

$$if \quad 1\le l,s\le N, \quad |\prod_{i=j-r_lm+1}^{j}a_{l,i}| \quad < \quad \varepsilon|\prod_{i=k+1}^{k+r_sm}a_{s,i}|,$$

and

$$if \quad 1\le s<l\le N, \quad \begin{cases}|\prod_{i=j+(r_l-r_s)m+1}^{j+r_lm}a_{s,i}|<\varepsilon|\prod_{i=j+1}^{j+r_lm}a_{l,i}|\\ |\prod_{i=j-(r_l-r_s)m+1}^{j+r_sm}a_{l,i}|<\varepsilon|\prod_{i=j+1}^{j+r_sm}a_{s,i}|\end{cases}.$$

Hence we proved that *a*) implies *c*). Condition *b*), *c*) and *d*) are known to be equivalent from [29, Theorem 4.1]. Finally the implication *b*) ⇒ *a*) hold trivially and this completes the proof of the equivalence of statements *a*) − *d*) of the theorem. □

When the shifts in Theorem 3.3 are invertible, we have

Corollary 3.4. *Let $X=c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, $(1\le p<\infty)$. For $N\ge 2$ and $l=1,\dots,N$, let $B_le_k=a_{l,k}e_k$ such that $(k\in\mathbb{Z})$ be an invertible bilateral backward shift on X, with weight sequence $(a_{l,j})_{j\in\mathbb{Z}}$. Let $1\le r_1<r_2<\dots<r_N$ be a positive integers. Then the following are equivalent:*

a) $B_1^{r_1},B_2^{r_2},\dots,B_N^{r_N}$ are disjoint topologically super-recurrent operators;
b) $B_1^{r_1},B_2^{r_2},\dots,B_N^{r_N}$ have a dense set of d-supercyclic vectors;
c) There exists integers $1\le n_1<n_2<\dots$ so that for $1\le s<l\le N$ and $j\in\mathbb{N}$

$$\begin{aligned}&\lim_{q\to\infty}\frac{\left|\prod_{i=j+1}^{j+r_ln_q}a_{l,i}\right|}{\left|\prod_{i=j+(r_l-r_s)n_q+1}^{j+r_ln_q}a_{s,i}\right|}=\infty\\ &\lim_{q\to\infty}\frac{\left|\prod_{i=j-(r_l-r_s)n_q+1}^{j+r_sn_q}a_{l,i}\right|}{\left|\prod_{i=j+1}^{j+r_sn_q}a_{s,i}\right|}=0,\end{aligned} \tag{11}$$

and such that

$$\lim_{q\to\infty}\max\left\{\frac{\left|\prod_{i=-r_ln_q}^{1}a_{l,i}\right|}{\left|\prod_{i=1}^{r_sn_q}a_{s,i}\right|}:1\le l,s\le N\right\}=0; \tag{12}$$

d) $B_1^{r_1},B_2^{r_2},\dots,B_N^{r_N}$ satisfy the d-Supercyclicity Criterion.

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