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Smooth extended regularly varying functions

Žarko Mijajlović[a], Danijela Branković[b], Dragan Djurčić[c,*]

[a]*Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11158 Belgrade, Serbia*
[b]*School of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11120 Belgrade, Serbia*
[c]*Faculty of Techical Sciences Čačak, University of Kragujevac, Svetog Save 65, 32102 Čačak, Serbia*

Abstract. In this paper we analyzed the behaviour of smooth extended regularly varying functions at infinity. Also, we obtained some new properties and representations of this class of functions.

1. Introduction

This paper is motivated by using regularly varying functions theory to inspecting asymptotical behaviour of cosmological parameters, concretely the expansion scale factor a(t) in the standard model of the universe, known as ΛCDM model. Barrow and Shaw started in papers [3] and [4] studies of the standard ΛCDM model using asymptotic analysis based on the theory of Hardy fields. This approach is extended by Mijajlović, Pejović, Marić and others in [16], [15] and [13], using the theory of regular variation and properties of slowly varying functions.

As our main tool in this paper is the theory of regular variation, we give a brief account without proofs, collecting together the properties we need. Bingham, Goldie and Teugels in [5] is the definite treatment, it presents this theory in details there. Seneta in [17] gives a short, but a good account of this subject, as well. We shall use the standard notation as used in these books. The symbols R and N will denote respectively the set of real numbers and the set of nonnegative integers. We shall denote the derivative of a function $f(x)$ over a variable x by $\dot{f}(x)$. If it is clear from the context which letter denotes the argument of the function $f(x)$, we shall often write simply $\dot{f}$. The theory of regular variation refer to real and positive functions. Therefore, if not stated otherwise, it is assumed that all mentioned functions are real and positive. Now we proceed to the definition of regularly varying functions.

Definition 1.1. Let $F(t)$ be a real positive function defined for $t > t_0$. Then

1. $F(t)$ is slowly varying (SV) if

$$\frac{F(\lambda t)}{F(t)} \to 1 \quad \text{as} \quad t \to \infty, \quad \text{for each } \lambda > 0. \tag{1}$$

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* Corresponding author: Dragan Djurčić
Email addresses: zarkom@matf.bg.ac.rs (Žarko Mijajlović), danijela@etf.bg.ac.rs (Danijela Branković), dragan.djurcic@ftn.kg.ac.rs (Dragan Djurčić)

2. $F(t)$ is a regularly varying (RV) function of index r if

$$\frac{F(\lambda t)}{F(t)} \to \lambda^r \quad \text{as} \quad t \to \infty, \quad \text{for each } \lambda > 0. \tag{2}$$

Examples of slowly varying functions include logarithmic function $\ln(x)$ and the iterated logarithmic functions $\ln(\ldots \ln(x) \ldots)$. RV functions are exactly functions which satisfy the generalized power law

$$F(t) = t^r L(t), \tag{3}$$

where $L(t)$ is a slowly varying function and r is a real constant. Many physical phenomena satisfy power law, so RV functions make an important and useful class of functions in their studies.

Continuing works of Hardy, Littlewood and Landau, Karamata [9] originally defined and studied these notions for continuous functions. Later this theory was extended to measurable functions.

Theory of regularly varying functions is applied in physics (for example see [16], [15]), hence we restrict here our attention only to smooth functions, concretely those having at least the continuous second derivative. Following the notation in [16] and [15], by $\mathcal{R}_\alpha$ we denote the class of regularly varying functions of index α. Consequently, the class of slowly varying functions is denoted with $\mathcal{R}_0$. With $\mathcal{Z}_0$ is denoted the class of functions $\varepsilon(t)$ that satisfy the condition $\lim_{t\to\infty} \varepsilon(t) = 0$, i.e. zero functions at infinity. The following representation theorem [9] describes the fundamental property of these functions.

Theorem 1.2. (Representation theorem for slowly varying functions): *A function $L(t)$ is slowly varying if and only if there are measurable functions $g(x)$, $\varepsilon \in \mathcal{Z}_0$ and $b \in R$ so that*

$$L(x) = g(x) e^{\int_b^x \frac{\varepsilon(t)}{t} dt}, \quad x \geq b, \tag{4}$$

and $g(x) \to g_0$ as $x \to \infty$, g_0 is a positive constant.

There are various classes of positive measurable functions with similar asymptotic behavior to regularly varying functions. The class of ERV – extended regularly varying functions (or Matuszewska class of functions) will have particularly important role in this paper. ERV class of functions is derived from the results of [12] and [11]. Extensive literature on these functions is available, e.g. [1], [5], [6], [17], [2], [10], [7], [8]. We review here, following [5], their definitions and very basic notions and properties related to these functions. As in the case of regular variation, we assume that all mentioned functions are smooth.

The limit in (1) does not exist always for an arbitrary function, but the limit superior and the limit inferior do exist. So let

$$f^*(\lambda) = \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)}, \quad f_*(\lambda) = \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)}. \tag{5}$$

Note that the difference $f^*(\lambda) - f_*(\lambda)$ measures the oscillation of $f(\lambda x)/f(x)$ at infinity. It is said that a real positive measurable function f belongs to the ERV class if and only if there are constants d and c such that

$$\lambda^d \leq f_*(\lambda) \leq f^*(\lambda) \leq \lambda^c, \quad \lambda \geq 1. \tag{6}$$

We see at once that RV $\subseteq$ ERV. Note that f is RV if and only if $f_*(\lambda) = f^*(\lambda)$, $\lambda \geq 1$. Therefore, RV functions are exactly ERV functions which do not oscillate at infinity.

Proposition 1.3. *The following statements are equivalent for a real function f:*

1. *f belongs to the class ERV.*
2. *f has the representation:*

$$f(x) = \exp\left(C + \eta(x) + \int_1^x h(t)dt/t\right), \quad x \geq 1, \quad C \text{ is a constant},$$

where $\eta(x) \to 0$ as $x \to \infty$, $h(t)$ is bounded and $h(t)$, $\eta(t)$ are both measurable.

According to this proposition, an ERV function $f(x)$ obviously has the following representation for a function $g(x)$ and a constant $g_0 > 0$:

$$f(x) = g(x)e^{\int_1^x \frac{h(t)}{t}dt}, \quad \lim_{x\to\infty} g(x) \to g_0, \quad h(t) \text{ is a bounded function.} \tag{7}$$

An ERV function $f(x)$ is normalized if $g(x) = c_0$ is a constant function. We shall often take $c_0 = 1$, hence in this case, by the integral representation theorem, $f(1) = 1$.

2. ERV functions

In this section we develop further properties of smooth ERV functions.

If $f(x)$ is from above bounded function, then there is $b = \limsup_{x\to\infty} f(x)$. It means:

1. For any $\varepsilon > 0$ there is x_0 such that for all $x > x_0$, $f(x) < b + \varepsilon$.
2. For any $\varepsilon > 0$ and every y there is $x > y$ so that $f(x) > b - \varepsilon$.

The next statement uniformizes the above properties of bounded functions.

Lemma 2.1. *Suppose f(x) is a real, from above bounded function and let* $b = \limsup_{x\to\infty} f(x)$. *Then there is a smooth, decreasing zero function* $\varepsilon(x)$ *and a real number p, such that:*

1. *For all* $x > p$, $f(x) < b + \varepsilon(x)$.
2. *For every* $y > p$ *there is* $x > y$ *so that* $b - \varepsilon(x) < f(x)$.

Proof. Let η_n, $n \in N$, be an arbitrary decreasing zero sequence. Observe that from these assumptions it follows $\eta_n > 0$. By the first property of $b = \limsup_{x\to\infty} f(x)$, there is a monotonously increasing sequence x_n of real numbers such that $x_n \to \infty$ as $n \to \infty$ and for all $n \in N$ and all $x > x_n$ we have $f(x) < b + \eta_n$. Let x'_n be a sequence of real numbers such that $x_n < x'_n < x_{n+1}$. We define in a piecewise manner a zero function $\eta(x)$ in the following way.

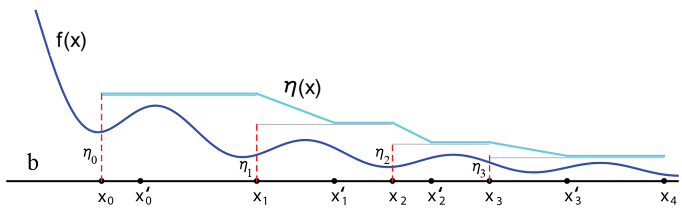

Figure 1: Zero function $\eta(x)$.

For $x \le x_1$ we take $\eta(x) = \eta_0$.

If $x_n \le x \le x'_n$, $n = 1, 2, 3, \ldots$, we take for $\eta(x)$ to be the line segment connecting points (x_n, η_{n-1}) and (x'_n, η_n), i.e.

$$\eta(x) = \eta_{n-1} + \frac{\eta_n - \eta_{n-1}}{x'_n - x_n}(x - x_n). \tag{8}$$

If $x'_n \le x \le x_{n+1}$, $n = 1, 2, 3, \ldots$, then $\eta(x) = \eta_n$.

Then it is easy to see that $\eta(x)$ fulfills the first required condition; it is a decreasing zero function and $f(x) < b + \eta(x)$ for all $x > x_0$.

Now we define a function $\xi(x)$ which will satisfy the second part of the Lemma. It is constructed in a similar way as it was $\eta(x)$, in a piecewise manner. Let ξ_n be an arbitrary decreasing zero sequence. Then by the second property of $b = \limsup_{x\to\infty} f(x)$, there is a monotonously increasing sequence y_n, $n \in N$, such that $\lim_{n\to\infty} y_n = \infty$ and $f(y_n) > b - \xi_n$. Let u_n and v_n be two sequences such that

$$y_n < u_n < v_n < y_{n+1}, \quad n \in N. \tag{9}$$

Then $\xi(x)$ is defined by cases as follows.

If $y_0 \le x \le u_0$ then $\xi(x) = \xi_0$.

If $u_n \le x \le v_n$, $n \in N$, then

$$\xi(x) = \xi_n + \frac{\xi_{n+1} - \xi_n}{v_n - u_n}(x - u_n). \tag{10}$$

If $v_n \le x \le u_{n+1}$, then $\xi(x) = \xi_{n+1}$.

Then it is easy to see that $\xi(x)$ fulfills the second required condition; it is a decreasing zero function and for every $y \ge y_0$, there is $x > y$ so that $b - \xi(x) < f(x)$.

Finally, we take $\varepsilon(x) = \eta(x) + \xi(x)$. Then it is easy to see that for $p = \max\{x_0, y_0\}$, $\varepsilon(x)$ is a decreasing zero function which satisfies conditions (1) and (2) of the Lemma.

We can make a smooth variant of $\varepsilon(x)$ if we replace the line (8) (and similarly the line (10)) connecting consecutive segments in the construction of $\varepsilon(x)$ by a segment of a four degree polynomial $g(x)$. The polynomial $g_n(x)$ which connects smoothly points (x_n, η_{n-1}) and (x'_n, η_n) is defined by the set of equations:

$$g_n(x_n) = \eta_{n-1},\ \dot{g}_n(x_n) = 0 \quad \text{and} \quad g_n(x'_n) = \eta_n,\ \dot{g}_n(x'_n) = 0. \tag{11}$$

In definition of $\varepsilon(x)$ we take the restriction of $g_n(x)$ on the real interval (x_n, x'_n). □

Note that $\varepsilon(x)$ and all its derivatives as well are bounded on each finite interval. In a similar manner one can prove the next statement.

Lemma 2.2. *Suppose f(x) is a real, from bellow bounded function and let* $a = \liminf_{x\to\infty} f(x)$. *Then there is a smooth, decreasing zero function* $\xi(x)$ *and a real number q, such that:*

1. *For all* $x > q$, $a - \xi(x) < f(x)$.
2. *For every* $y > q$ *there is* $x > y$ *so that* $f(x) < a + \xi(x)$.

If $\varepsilon(x) = \eta(x) + \xi(x)$, where $\eta(x)$ and $\xi(x)$ are functions constructed in the previous lemmas and taking $x_0 = \max\{p, q\}$, we see that the function $\varepsilon(x)$ for $x > x_0$ still satisfies the inequalities in these lemmas replacing $\eta(x)$ and $\xi(x)$ by $\varepsilon(x)$. Hence, we got the following statement.

Corollary 2.3. *Let* $f(x)$ *be a real bounded function and* $a = \liminf_{x\to\infty} f(x)$ *and* $b = \limsup_{x\to\infty} f(x)$. *Then there is a smooth, decreasing zero function* $\varepsilon(x)$ *and a real number* x_0, *such that*

1. *If* $x > x_0$ *then* $a - \varepsilon(x) < f(x) < b + \varepsilon(x)$.
2. *For every* $y > x_0$ *there is* $x > y$ *such that* $f(x) > b - \varepsilon(x)$.
3. *For every* $y > x_0$ *there is* $x > y$ *such that* $f(x) < a + \varepsilon(x)$.

Now we proceed to the proof that for each proper ERV function, i.e. ERV function which is not regularly varying, there are two narrow and disjoint strips defined by two pairs of regularly varying functions, which $f(x)$ visits infinitely many times at infinity. First we make precise definitions of these notions.

If u and v are real functions and for some x_0 for every $x > x_0$ we have $u(x) < v(x)$, then the domain $S \subseteq R^2$ defined by

$$S = \{(x, y) \in R^2 \colon x > x_0, u(x) \le y \le v(x)\} \tag{12}$$

is a strip defined by the pair (u, v). We identify the strip S with its defining pair (u, v) and we shall often write $S = (u, v)$.

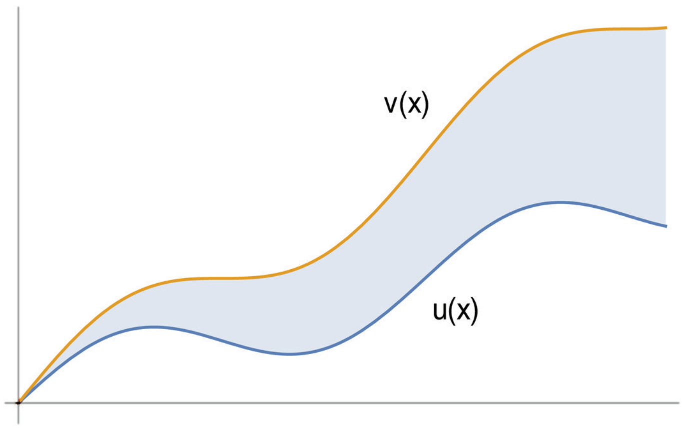

Figure 2: Strip defined by functions u and v.

In next definitions it is convenient to think of a function $f(x)$ as of the path of a point moving through the plain. We say that $f(x)$ visits a strip S at a point a, if $u(a) \leq f(a) \leq v(a)$. If $f(x)$ visits S at some point, then we simply say that $f(x)$ visits S. The set V of all contiguous points at which $f(x)$ visits S, we call the visit of $f(x)$ to the strip S. More formally, there are real numbers a and b, such that $a < b$ and

1. $V = \{(x, f(x)) \colon a \leq x \leq b\}$,
2. For some $\varepsilon > 0$, if $a - \varepsilon < x < a$, then the point $(x, f(x))$ does not belong to the strip V. In other words, $f(x) < u(x)$, or $f(x) > v(x)$. Similarly, for some $\varepsilon > 0$, if $b < x < b + \varepsilon$, then $f(x) < u(x)$, or $f(x) > v(x)$.

In this case we say that the visit V starts at a and ends at b, while $I_V = [a, b]$ is called the supporting interval of V.

A visit V is finite if there are numbers $a < b$, such that $f(x)$ visits S at a and $f(b) < u(b)$, or $f(b) > v(b)$. Here we consider functions having only finite visits to S.

We see that two visits V and V' are different, if and only if their supporting intervals I and I' are disjoint. This enables us to define an order in the set of all visits of $f(x)$ to S. We say that a visit V precedes V', or V' comes after V, if they are different and there are $a \in I$ and $a' \in I'$ such that $a < a'$, I and I' are the supporting intervals respectively of V and V'. Finally, if for every c there is $a > c$ at which $f(x)$ visits S, then we say that $f(x)$ visits S at infinity.

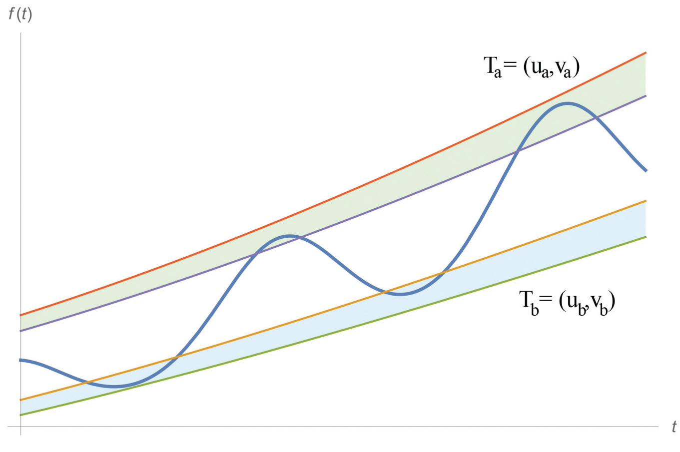

Figure 3: Function $f(x)$ visits strips T_a and T_b.

In order to infer new representation of ERV functions, as well as their new properties, we use the operator $\mathcal{L}$, defined in [14] in the following way.

Definition 2.4. $\mathcal{L}(h)(x) = \dfrac{1}{\ln(x)} \displaystyle\int_{x_0}^{x} \frac{h(t)}{t} dt, \quad x_0 > 1, h \in C^2(R).$

The exact value of x_0 is not so important, since our goal is to analyze asymptotical behaviour at infinity. Therefore, we can take $x_0 = 1$, since the following holds

$$\lim_{x \to 1} \mathcal{L}(h)(x) = h(1). \tag{13}$$

The following Theorem 2.5 is proved in [14] and gives us some interesting properties of the linear operator $\mathcal{L}$ that we use in this paper.

Theorem 2.5. *Let $\mathcal{L}'$ denote the restriction of $\mathcal{L}$ to the appropriate domain. Then*

1. $\mathcal{L}' : \mathcal{Z}_0 \to \mathcal{Z}_0$.
2. $\mathcal{L}' : \mathcal{R}_0 \to \mathcal{R}_0$.
3. $\mathcal{L}' : \mathcal{R}_\alpha \to \mathcal{R}_\alpha, \quad \alpha \in R$.
4. $\mathcal{L}' : B(R) \to B(R)$, *where $B(R)$ is the set of real bounded functions.*

In Lemma 2.6 we prove that the inverse function of an ERV function is also an ERV function.

Lemma 2.6. *Assume that $f_1 \in C^1(1,\infty)$ is normalized ERV function that satisfies $\dot{f}_1(t) \neq 0$ for every $t > 1$. Then the inverse function of f_1 exists and is also in the class of the normalized ERV functions.*

Proof. Since $f_1(t)$ is a normalized ERV function, substituting $g(t) = 1$ in (19), we infer $f_1(t) = t^{\mathcal{L}(h)(t)}$, where the function h is bounded function from the representation of the function $f_1(t)$. According to the global inverse function theorem, the inverse function $f_2(y)$ for the function $f_1(x)$ exists and is in the class $C^1(1,\infty)$. Therefore, for every $y \in f_1(1,\infty)$ we have

$$y = f_1(f_2(y)) = f_2(y)^{\mathcal{L}(h)(f_2(y))}. \tag{14}$$

Moreover

$$\ln(y) = \ln(f_2(y))\mathcal{L}(h)(f_2(y)), \tag{15}$$

wherefrom, using definition of the linear operator $\mathcal{L}$, we infer

$$\ln(y) = \int_1^{f_2(y)} \frac{h(x)}{x} dx. \tag{16}$$

Differentiating (16) over variable y, we obtain

$$\frac{1}{y} = \frac{h(f_2(y))}{f_2(y)} \dot{f}_2(y), \tag{17}$$

wherefrom, substituting $f_2(1) = 1$ (since $f_1(1) = 1$), we infer

$$f_2(y) = \exp\left(\int_1^y \frac{1/\big(h(f_2(x))\big)}{x} dx \right). \tag{18}$$

Since function h is bounded, we conclude that $f_2(y)$ is a normalized ERV function. □

Now, suppose $f(x)$ is an ERV function with representation (7). Then

$$f(x) = g(x)x^{\mathcal{L}(h)(x)}, \quad g(x) \to g_0 \text{ as } x \to \infty. \tag{19}$$

Here is an example of a proper ERV function, i.e. an ERV function which is not regularly varying function.

Example 2.7. Let $h(x) = \sin(\ln(\ln(x))) + \cos(\ln(\ln(x)))$ and $x_0 = e$. Then $\mathcal{L}(h)(x) = \sin(\ln(\ln(x)))$ and so $f(x) = x^{\sin(\ln(\ln(x)))}$. We see that $f(x)$ varies between functions $u(x) = x^{-1}$ and $v(x) = x$ (consider sequences $f(a_n)$ and $f(b_n)$, where $a_n = e^{e^{2\pi n+\pi/2}}$ and $b_n = e^{e^{2\pi n-\pi/2}}$). Hence $f(x)$ is a proper ERV function.

If $f(x)$ is an ERV function, then in the integral representation (7) the function $h(x)$ is bounded and by Theorem 2.5 $\mathcal{L}(h)(x)$ is bounded, too. Hence, if $f(x)$ is a proper ERV function, $\mathcal{L}(h)(x)$ does not converge at infinity. So, there are $\alpha = \liminf_{x\to\infty} \mathcal{L}(h)(x)$ and $\beta = \limsup_{x\to\infty} \mathcal{L}(h)(x)$. Next theorem shows that then $f(x)$ varies between functions $u(x) = x^{\alpha}$ and $v(x) = x^{\beta}$.

Theorem 2.8. *Assume $f(t)$ is a proper ERV function with representation (7) and let $\alpha = \liminf_{t\to\infty} \mathcal{L}(h)(t)$ and $\beta = \limsup_{t\to\infty} \mathcal{L}(h)(t)$, $\alpha < \beta$.*
Then we have

1. *If $f(t)$ is a normalized ERV function, then there is a smooth, decreasing zero function $\varepsilon(t)$, such that $f(t)$ visits at infinity eventually disjoint strips $T_\alpha = (t^{\alpha-\varepsilon(t)}, t^{\alpha+\varepsilon(t)})$ and $T_\beta = (t^{\beta-\varepsilon(t)}, t^{\beta+\varepsilon(t)})$.*
2. *If the attribute "normalized" is omitted, then for some arbitrary small $\delta > 0$ the function $f(t)$ visits at infinity eventually disjoint strips $S_\alpha = (t^{\alpha-\delta}, t^{\alpha+\delta})$ and $S_\beta = (t^{\beta-\delta}, t^{\beta+\delta})$.*

Proof. First we assume that $f(t)$ is normalized. Since $f(t)$ is a proper ERV function, it follows $\alpha < \beta$. Then by Corollary 2.3, there exists a smooth decreasing zero function $\varepsilon(t)$ and a real number t_0 such that for all $t > t_0$:

$$\alpha - \varepsilon(t) < \mathcal{L}(h)(t) < \beta + \varepsilon(t), \tag{20}$$

wherefrom, by the representation (19), we have

$$t^{\alpha-\varepsilon(t)} \le f(t) \le t^{\beta+\varepsilon(t)}, \quad \text{as } t \to \infty. \tag{21}$$

We also have that for every $s > t_0$ there is $t > s$ such that $\mathcal{L}(h)(t) > \beta - \varepsilon(t)$. Therefore, for every $s > t_0$ there is $t > s$ such that

$$\beta - \varepsilon(t) < \mathcal{L}(h)(t) < \beta + \varepsilon(t). \tag{22}$$

Using again (19), we conclude that for every $s > t_0$ there is $t > s$ such that

$$u_\beta(t) \equiv t^{\beta-\varepsilon(t)} \le f(t) \le t^{\beta+\varepsilon(t)} \equiv v_\beta(t). \tag{23}$$

In a similar way it is proved that for every $s > t_0$ there is $t > s$ such that

$$u_\alpha(t) \equiv t^{\alpha-\varepsilon(t)} \le f(t) \le t^{\alpha+\varepsilon(t)} \equiv v_\alpha(t). \tag{24}$$

Hence, $f(t)$ visits at infinity strips T_α and T_β, what proves the first part of the theorem. As $\varepsilon(t)$ is a zero function and $\alpha < \beta$, there is t_0 such that we can choose $\delta > \varepsilon(t)$, $t > t_0$, but $(\beta - \alpha) > 2\delta$. Hence,

$$t^{\alpha+\delta} < f(t) < t^{\beta-\delta}, \quad t > t_0, \tag{25}$$

and for arbitrary large $t > t_0$ we have $f(t) < t^{\alpha+\delta}$ and $t^{\beta-\delta} < f(t)$. In other words, $f(t)$ also visits at infinity disjoint strips

$$S_\alpha = \{(t^{\alpha-\delta}, t^{\alpha+\delta}) \colon t > t_0\}, \quad S_\beta = \{(t^{\beta-\delta}, t^{\beta+\delta}) \colon t > t_0\}. \tag{26}$$

If $f_0(t)$ is not normalized, then there is a constant $g_0 > 0$ and a zero function $\eta(t)$, such that $f_0(t) = (g_0 + \eta(t))f(t)$, where $f(t)$ is a normalized function. Assume derivation (21) – (26). Then, if we multiply for example inequalities (24) by $g_0 + \eta(t)$, we obtain

$$t^{\alpha-\delta} < (g_0 + \eta(t))t^{\alpha-\varepsilon(t)} \le f_0(t) \le (g_0 + \eta(t))t^{\alpha+\varepsilon(t)} < t^{\alpha+\delta} \tag{27}$$

for sufficiently large t. In a similar way one can prove the appropriate inequalities for $f_0(t)$ in (21) and (23). So, $f_0(t)$ visits at infinity both strips S_α and S_β, what proves the second part of the theorem. □

We see that the strips T_α and T_β appearing in the previous proof are narrow in the sense that $u_\alpha/v_\alpha \to 1$ and $u_\beta/v_\beta \to 1$ as $t \to \infty$.

We also note, that using the continuity of $\dot{f}(t)$ one can prove that to each visit V of $f(t)$, for example to the strip T_α, corresponds a real interval $I = [t', t'']$, $t' < t''$, such that the visit starts at t' and ends at t''. Namely, $f(t)$ intersects the graph of $v_\alpha(t) = t^{\alpha+\varepsilon(t)}$ at t' and t'' and we have

$$t^{\alpha-\varepsilon(t)} < f(t) < t^{\alpha+\varepsilon(t)}, \quad t' < t < t''. \tag{28}$$

Further, from these inequalities it follows $-\varepsilon(t) < \xi(t) < \varepsilon(t)$, where

$$\xi(t) = \frac{\ln(f(t))}{\ln(t)} - \alpha. \tag{29}$$

Hence, $\xi(t)$ is a smooth zero function and obviously

$$f(t) = t^{\alpha+\xi(t)}, \quad t' < t < t''. \tag{30}$$

By (29) we see that $\xi(t)$ is independent of the visit of $f(t)$ to the strip T_α. Hence, for a visit V of $f(t)$ to the strip T_α we have $f(t) = t^{\alpha+\xi(t)}$, $t \in I_V$, where I_V is the supporting interval of the visit V. If $(s, f(s))$ is not a visiting point of $f(t)$ to T_α, we take $\xi(t) = \varepsilon(t)$ and then $f(s) > s^{\alpha+\varepsilon(s)} \geq s^{\alpha+\xi(s)}$. Hence

$$t^{\alpha+\xi(t)} \leq f(t), \quad t \geq t_0, \tag{31}$$

and $f(t) = t^{\alpha+\xi(t)}$ for visiting points t of $f(t)$ to T_α. Similarly for a smooth zero function $\zeta(t)$ we have $f(t) \leq t^{\beta+\zeta(t)}$, $t \geq t_0$, and $f(t) = t^{\beta+\zeta(t)}$ for visiting points t of $f(t)$ to T_β. Hence for each t, $f(t)$ is a convex combination

$$f(t) = \alpha_t t^{\alpha+\xi(t)} + \beta_t t^{\beta+\zeta(t)}, \quad \alpha_t + \beta_t = 1, \quad \alpha_t, \beta_t \geq 0. \tag{32}$$

Obviously, there is a function $u(t)$ such that $\cos(u(t))^2 = \alpha_t$, $\sin(u(t))^2 = \beta_t$ and $u(t)$ takes all values in the interval $[0, \pi]$. Therefore we can conclude the above discussion with the following.

Theorem 2.9. *Assume $f(t)$ is a proper normalized ERV function defined for $t \geq t_0$. Then there are real numbers $\alpha < \beta$, zero functions $\xi(t)$ and $\zeta(t)$ and a real function $u(t)$ which takes all values in the interval $[0, \pi]$, so that*

$$f(t) = t^{\alpha+\xi(t)} \cos(u(t))^2 + t^{\beta+\zeta(t)} \sin(u(t))^2. \tag{33}$$

3. Examples

In this section we provide several examples that illustrate the presented material. The first example is a theorem which describes a large collection of bounded functions for which the exponential integral (7) is an RV function in spite of our primary expectation that there is a proper ERV function of this type. This class of functions is made up by bounded periodic functions.

Theorem 3.1. *Let $f\colon [1, +\infty) \to \mathbf{R}$ be a Lebesgue measurable and bounded function, $\mathbf{R}$ is the set of real numbers. Let $L \geq 1$ be a real constant and $E = \bigcup_{k=1}^{\infty}(kL, (k+1)L)$. Suppose $f|E\colon E \to E$ and that $f(x)$ is periodic on E with the period L. Then*

$$R(x) = e^{\int_1^x \frac{f(t)}{t} dt} \tag{34}$$

is RV (regularly varying) function.

Proof. In the next proof we suppose $x \geq L \geq 1$. By boundness of $f(x)$ there is a constant $M > 0$ such that $|f(x)| \leq M$, $x \geq 1$. Let $I(x) = \int_1^x \frac{f(t)}{t} dt$ and

$$I(x) = I_0 + J(x) + I_1(x) \tag{35}$$

where

$$I_0 = \int_1^L \frac{f(t)}{t} dt, \quad J(x) = \int_L^{nL} \frac{f(t)}{t} dt, \quad I_1(x) = \int_{nL}^x \frac{f(t)}{t} dt, \tag{36}$$

where $n = n(x) = \lfloor x/L \rfloor$. Throughout this proof, the symbol n is reserved to denote the function $n(x)$. We see that n is the greatest integer such that $nL \leq x$ and I_0 is a constant. Hence

$$1 \leq \frac{x/L}{\lfloor x/L \rfloor} \leq \frac{n+1}{n}. \tag{37}$$

We see that $n(x) \to +\infty$ as $x \to +\infty$, hence

$$\frac{x/L}{n(x)} \to 1 \quad \text{as} \quad x \to +\infty. \tag{38}$$

Observe that $f(x)/x$ is also Lebesgue integrable and bounded. In fact

$$\left| \frac{f(x)}{x} \right| \leq M, \quad x \geq 1. \tag{39}$$

Now we prove

$$\lim_{x \to +\infty} I_1(x) = 0. \tag{40}$$

Really, by definition of $I_1(x)$ we have

$$|I_1(x)| \leq M(\ln(x) - \ln(nL)) = M \ln\left(\frac{x/L}{n(x)}\right), \tag{41}$$

so by (38), $I_1(x) \to 0$ as $x \to +\infty$. By definition (34) of R we have

$$R(x) = c_0(x) e^{J(x)} \tag{42}$$

where $c_0(x) = e^{I_0 + I_1(x)}$ and $\lim_{x \to +\infty} c_0(x) = c_0$, $c_0 = e^{I_0}$. Further, by integrability and boundness of $f(t)/t$ and periodicity of $f(t)$, taking $s + kL = t + L$ we have

$$\begin{aligned} J(x) &= \sum_{k=1}^{n-1} \int_{kL}^{(k+1)L} \frac{f(t)}{t} dt = \sum_{k=1}^{n-1} \int_L^{2L} \frac{f(s + (k-1)L)}{s + (k-1)L} ds \\ &= \sum_{k=1}^{n-1} \int_L^{2L} \frac{f(t)}{t + (k-1)L} dt. \end{aligned} \tag{43}$$

Hence,

$$J(x) = \int_L^{2L} f(t) \sum_{k=1}^{n-1} \frac{1}{t + (k-1)L} dt \tag{44}$$

Let $F(x) = \int_L^x f(t)dt$ the indefinite Lebesgue integral of $f(x)$, $L \le x \le 2L$. Then $F(L) = 0$, so using integration by parts

$$\begin{aligned} J(x) &= \int_L^{2L} \sum_{k=1}^{n-1} \frac{1}{t+(k-1)L} dF(t) \\ &= F(t) \sum_{k=1}^{n-1} \frac{1}{t+(k-1)L}\Big|_L^{2L} + \int_L^{2L} F(t) \sum_{k=1}^{n-1} \frac{1}{(t+(k-1)L)^2} dt \\ &= \frac{F(2L)}{L} \sum_{k=1}^{n-1} \frac{1}{k+1} + \int_L^{2L} G(t) \sum_{k=1}^{n-1} \frac{1}{(t+(k-1)L)^2} dt = A + B. \end{aligned} \tag{45}$$

Here, A is the first and B is the second summand in the last sum, while $G(t)$ is a periodic function on E with period L defined by

$$G(t+kL) = F(t),\ t \in (L, 2L),\ G((k+1)L) = 0,\ k = 0, 1, 2, \ldots. \tag{46}$$

For $S_n = \sum_{k=1}^{n-1} \frac{1}{k+1}$ and harmonic sum $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ we have

$$S_n = H_n - 1. \tag{47}$$

It is known that

$$H_n = \ln(n) + \gamma + \frac{1}{2n} - \xi_n, \quad 0 \le \xi_n \le \frac{1}{8n^2}, \tag{48}$$

where γ is the Euler constant. By (37) we have

$$\ln(x/L) \ge \ln(n) \ge \ln\left(\frac{n}{n+1}\right) + \ln(x/L), \tag{49}$$

so

$$\ln(n) = \ln(x) - \ln(L) + \eta(x), \quad \eta(x) \to 0 \quad \text{as} \quad x \to +\infty. \tag{50}$$

Therefore

$$A = \frac{F(2L)}{L} \ln(x) + a + \delta(x), \tag{51}$$

where $a = \frac{F(2L)}{L}(\gamma - \ln(L) - 1)$ is a constant and

$$\delta(x) = \frac{F(2L)}{L}\left(\frac{1}{2n(x)} - \xi_{n(x)} + \eta(x)\right) \tag{52}$$

is a zero function at infinity. Therefore, see (42),

$$R(x) = c_1(x) x^k e^B \tag{53}$$

where $k = F(2L)/L$ and $c_1(x) = c_0(x)e^{a+\delta(x)}$ is a constant function at infinity. Now we determine the asymptotic behaviour of part B appearing in (45). The function $G(t)$ is bounded on $[L, 2L]$ as

$$|G(t)| = |F(t)| = \left|\int_L^x f(t)dt\right| \le \int_L^x |f(t)|dt \le LM. \tag{54}$$

By periodicity of $G(t)$ with period L, it follows that $G(t)$ is Lebesgue measurable and bounded on its domain. Hence $G(t)$ is Lebesgue integrable, bounded and periodic on $(L,+\infty)$ with period L, so

$$\begin{aligned} B &= \int_L^{2L} G(t) \sum_{k=1}^{n-1} \frac{1}{(t+(k-1)L)^2} dt = \sum_{k=1}^{n-1} \int_L^{2L} \frac{G(t+(k-1)L)}{(t+(k-1)L)^2} dt \\ &= \sum_{k=1}^{n-1} \int_{kL}^{(k+1)L} \frac{G(t)}{t^2} dt = \int_L^x \frac{\varepsilon(t)}{t} dt - K(x), \end{aligned} \tag{55}$$

where

$$\varepsilon(t) = G(t)/t, \quad K(x) = \int_{nL}^x \frac{G(t)}{t^2} dt. \tag{56}$$

Obviously $\varepsilon(t) \to 0$ as $t \to +\infty$, while $\lim_{x\to+\infty} K(x) = 0$ is proved in the same way as the statement (40). Hence, (53) can be rewritten as

$$R(x) = c(x) x^k e^{\int_L^x \frac{\varepsilon(t)}{t} dt}, \; x \geq L, \; \varepsilon(t) \to 0 \quad \text{as} \quad t \to +\infty, \tag{57}$$

where $c(x) = c_1(x)e^{-K(x)}$, $c(x) \to c_0 e^a$ as $x \to +\infty$. This proves the theorem. □

We note that the proof of this theorem can be easily adapted if assumed $0 < L < 1$.

Example 3.2. Functions

$$e^{\int_1^x \frac{\sin(t)}{t} dt}, \quad e^{\int_1^x \frac{|\sin(t)|}{t} dt} \tag{58}$$

are RV functions.

Obviously, we have more generally

Theorem 3.3. *If $f(x)$ is a trigonometric polynomial in* $\sin(x)$, $\cos(x)$, $|\sin(x)|$, $|\cos(x)|$, *then*

$$R(x) = e^{\int_1^x \frac{f(t)}{t} dt}, \quad x \geq 1, \tag{59}$$

is RV function.

In contrast to this theorem we have the following

Example 3.4. The function

$$V(x) = e^{\int_1^x \frac{2-\sin(\ln(t))}{t} dt} \tag{60}$$

is proper ERV function, i.e. $V(x)$ is ERV but not RV. This follows from the fact that $V(x)$ varies between functions $g_1(x) = e^{-2}x^2$ and $g_2(x) = x^2$, touching them periodically. More precisely, $V(x) = e^{-1}x^2 e^{\cos(\ln(x))}$ and it is easy to see that for $\lambda \neq e^{2k\pi}$, k is an integer, $\lim_{x\to+\infty} V(\lambda x)/V(x)$ does not exist.

We also tried to resolve the following

Problem 3.5. *Is there a proper monotonous ERV function having only finitely many inflection points,*

however we didn't succeed.

Solution of this problem may affect some questions on standard ΛCDM cosmological model, as the scale factor $a(t)$, the solution of Friedmann equation, under certain assumptions is an ERV function.

4. Conclusion

We inferred some new properties of smooth ERV (extended varying) functions. A particularly interesting feature is that for each proper ERV function $f(x)$ there are two narrow and disjoint strips defined by two pairs of regularly varying functions, which $f(x)$ visits infinitely many times at infinity. Using a linear operator $\mathcal{L}$, we obtained a new representations of this class of functions, formulas (19) and

$$f = t^{\alpha+\xi} \cos(u)^2 + t^{\beta+\zeta} \sin(u)^2, \quad \alpha < \beta \text{ and } \xi, \zeta \text{ are zero functions.}$$

We also proved that bounded periodic functions appearing in the integral representation of ERV functions yield in fact RV functions. Examples of proper ERV functions are also given.

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