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On trans-para-Sasakian manifolds

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Abstract. In this paper, we investigate the geometry of the trans-para-Sasakian manifolds. Finally, an example of a three-dimensional trans-para-Sasakian manifold is constructed to verify the results.

1. Introduction

In [11], Oubina has introduced two new classes of almost contact structures, called trans-Sasakian and almost trans-Sasakian structures, which are obtained from certain classes of Hermitian manifolds. Also, the author proved that an almost metric structure (ϕ, ξ, η, g) is a trans-Sasakian structure if and only if it is normal and

$$d\Phi = 2\beta\eta \wedge \Phi, \qquad d\eta = \alpha\Phi, \tag{1}$$

where $\alpha = \frac{1}{2n}\delta\Phi(\xi)$ and $\beta = \frac{1}{2n}div(\xi)$. This may be expressed as a condition [3]:

$$(\nabla_E\phi)F = \alpha[g(E, F)\xi - \eta(F)E] + \beta[g(\phi E, F)\xi - \eta(F)\phi E]. \tag{2}$$

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinea and Gonzales [6]. The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [10]. Different types of almost contact structures are defined [1, 2, 7, 11]. Many authors have studied some properties of trans-Sasakian structure [5, 12].

In geometry, one of the important idea is symmetry. It also plays a significant role in the nature. In local perspective, a *locally symmetric* manifold was defined independently by Shirokov [13] and Levy[9] satisfying

$$\nabla R = 0, \tag{3}$$

where R and ∇ are the Riemann curvature tensor and Levi-Civita connection on M, respectively.

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Many notions have been introduced to generalize locally symmetric manifolds. One of them is *semi-symmetric manifold* which was introduced by Cartan [4]. A Riemannian manifold is called semi-symmetric if

$$R(E, F).R = 0, \tag{4}$$

where $R(E, F)$ acts as a derivation on R.

The set of locally symmetric manifolds is a proper subset of the class of semi-symmetric manifolds. A Riemannian manifold is said to be *Ricci symmetric* if

$$\nabla S = 0, \tag{5}$$

where S and ∇ are the Ricci tensor of type (0,2) and Levi-Civita connection on M, resp. Semi-symmetric manifolds were classified by Szabo. The weakend notion of Ricci symmetry introduced by Szabo [14] as *Ricci semi-symmetric* satisfying

$$R(E, F).S = 0, \tag{6}$$

where $R(E, F)$ acts as a derivation on S.

The class of Ricci semi-symmetric manifolds includes the set of Ricci symmetric manifolds as a proper subset. Moreover, every semi-symmetric manifold is Ricci semi-symmetric but the converse is not true.

The study of trans-para-Sasakian manifold was initiated by Zamkovoy [16]. He introduced the trans-para-Sasakian manifolds and studied some curvature properties. A trans-para-Sasakian manifold is a trans-para-Sasakian structure of type (α, β), where α and β are smooth functions. The trans-para-Sasakian manifolds of types (α, β) are respectively the para-cosympletic, para-Sasakian and para-Kenmotsu for $\alpha = \beta = 0; \alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$. If α and β are constants, then trans-papa-Sasakian manifold of types $(\alpha, 0)$ and $(0, \beta)$ is called α-para-Sasakian and β-para-Kenmotsu, respectively. In literature, there are a lot of studies about trans-Sasakian manifolds. So, these considerations motivate us to study trans-para-Sasakian manifolds. The paper is organized in the following way. In section 2, we recall the common properties for $(2n + 1)$-dimensional trans-para-Sasakian manifolds. Section 3 deals with the curvature properties of trans-para-Sasakian manifolds. Moreover, we show that in a trans-para-Sasakian manifold, the Ricci operator Q does not commute with the structure tensor ϕ. In Section 4, especially we give the expressions of Ricci tensor and Riemannian curvature tensor in three dimensional trans-para-Sasakian manifolds. We find the sufficient and necessary condition for a three dimensional trans-para-Sasakian manifold to be η-Einstein. In the last section, we consider Ricci semi-symmetric trans-para-Sasakian manifolds and we present the Ricci tensor equation of Ricci semi-symmetric trans-para-Sasakian manifolds. Finally, a three dimensional trans-para-Sasakian manifold example that satisfies our results is constructed.

2. Preliminaries

A $(2n + 1)$-dimensional manifold M is called *almost paracontact manifold* if it admits a triple (ϕ, ξ, η) satisfying the followings:

$$\eta(\xi) = 1, \quad \phi^2 = I - \eta \otimes \xi \tag{7}$$

and ϕ induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta)$,where ϕ, ξ and η are $(1, 1)-$tensor field, vector field and $1-$form, resp. One can easily checked that $\phi\xi = 0, \eta \circ \phi = 0$ and $rank\phi = 2n$, by the definition. Here, ξ is a unique vector field dual to η and satisfying $d\eta(\xi, E) = 0$ for all E. When the tensor field $N_\phi := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, the almost paracontact manifold is said to be *normal* [15]. If the structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric such that

$$g(\phi E, \phi F) = -g(E, F) + \eta(E)\eta(F) \tag{8}$$

then we say that (M, ϕ, ξ, η, g) is an *almost paracontact metric manifold*. Note that any pseudo-Riemannian metric with a given almost paracontact metric manifold structure is necessarily of signature $(n+1, n)$. For an almost paracontant metric manifold, one can always find an orthogonal basis $\{E_1, \ldots, E_n, F_1, \ldots, F_n, \xi\}$, namely $\phi-$basis, such that $g(E_i, E_j) = -g(F_i, F_j) = \delta_{ij}$ and $F_i = \phi E_i$, for any $i, j \in \{1, \ldots, n\}$. Further, we can define a skew-symmetric tensor field (2-form), usually called fundamental form, Φ by

$$\Phi(E, F) = g(E, \phi F).$$

An almost paracontact metric manifold is said to be *η-Einstein* if its Ricci tensor S is of the form

$$S = \lambda g + \mu \eta \otimes \eta, \tag{9}$$

where λ and μ are smooth functions on the manifold. For the sake of the shortness, we denote the following tensors on trans-para-Sasakian manifolds

$$\begin{aligned} A(E, F, U) &= g(F, U)E - g(E, U)F, \\ A(E, F, U, V) &= g(A(E, F, U), V), \\ B_n(\alpha, \beta) &= \phi(grad\alpha) + (2n-1)grad\beta, \\ B_n(\alpha, \beta, E) &= -\phi E(\alpha) + (2n-1)E(\beta), \\ C_n(\alpha, \beta) &= -\phi(grad\beta) + (2n-1)grad\alpha, \end{aligned}$$

and

$$C_n(\alpha, \beta, E) = \phi E(\beta) + (2n-1)E(\alpha),$$

for all vector fields E, F, U and V on $\mathfrak{X}(M)$, where α and β are smooth functions. In case $n = 1$, we will say $B(\alpha, \beta) = B_1(\alpha, \beta)$, $C_(\alpha, \beta) = C_1(\alpha, \beta)$ and $B(\alpha, \beta, E) = B_1(\alpha, \beta, E)$, $C_(\alpha, \beta, E) = C_1(\alpha, \beta, E)$.

Definition 2.1. *[16] If*

$$(\nabla_E \phi)F = \alpha A(E, \xi, F) + \beta A(\phi E, \xi, F), \tag{10}$$

then the manifold $(M^{2n+1}, \phi, \eta, \xi, g)$ is said to be a trans-para-Sasakian manifold.

In a $(2n+1)$-dimensional trans-para-Sasakian manifold, the following identities hold [16]:

$$\nabla_E \xi = \alpha A(\xi, \phi E, \xi) + \beta A(\xi, E, \xi), \tag{11}$$

$$(\nabla_E \eta)F = \alpha A(\xi, E, \phi F, \xi) + \beta A(E, \xi, F, \xi), \tag{12}$$

$$\begin{aligned} R(E, F)\xi = &-(\alpha^2 + \beta^2)A(E, F, \xi) - 2\alpha\beta(A(\phi E, F, \xi) + A(E, \phi F, \xi)) \\ &+ \phi(A(E, F, grad\alpha)) + \phi^2(A(E, F, grad\beta)), \end{aligned} \tag{13}$$

$$\begin{aligned} \eta(R(E, F)U) = &(\alpha^2 + \beta^2)A(E, F, \xi, U) + 2\alpha\beta[A(\phi E, F, \xi, U) + A(E, \phi F, \xi, U)] \\ &+ A(F, E, \phi U, grad\alpha) + A(E, F, \phi^2 U, grad\beta), \end{aligned} \tag{14}$$

$$R(\xi, E)\xi = (\alpha^2 + \beta^2 - \xi(\beta))A(E, \xi, \xi), \tag{15}$$

$$S(E, \xi) = -(2n(\alpha^2 + \beta^2) - \xi(\beta))\eta(E) + B_n(\alpha, \beta, E), \tag{16}$$

$$S(\xi, \xi) = -2n(\alpha^2 + \beta^2 - \xi(\beta)), \tag{17}$$

$$2\alpha\beta - \xi(\alpha) = 0, \tag{18}$$

$$Q\xi = -(2n(\alpha^2 + \beta^2) - \xi(\beta))\xi + B_n(\alpha, \beta), \tag{19}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator defined by $S(E, F) = g(QE, F)$.

Corollary 2.2. *[16] If $B_n(\alpha, \beta) = 0$ in a $(2n+1)$-dimensional trans-para-Sasakian manifold, then*

$$\xi(\beta) = g(\xi, grad\beta) = -\frac{1}{2n-1} g(\xi, \phi(grad\alpha)) = 0. \tag{20}$$

3. Some Properties of Trans-para-Sasakian Manifolds

In this section, we discuss some curvature properties of trans-para-Sasakian manifolds. We start with the following relation for Riemannian curvature tensor.

Lemma 3.1. *In a trans-para-Sasakian manifold* $(M^{2n+1}, \phi, \xi, \eta, g)$ *the following relation holds:*

$$\begin{aligned} R(E,F)\phi U - \phi R(E,F)U = {} & (\alpha^2+\beta^2)[A(\phi E,F,U) + A(E,\phi F,U)] \\ & + 2\alpha\beta[A(E,F,U) + A(\phi E,\phi F,U)] \\ & - E(\alpha)A(\xi,F,U) + F(\alpha)A(\xi,E,U) \\ & - E(\beta)A(\xi,\phi F,U) + F(\beta)A(\xi,\phi E,U) \end{aligned} \tag{21}$$

for all E, F *and* U *on* $\mathfrak{X}(M)$.

Proof. From (10), (11) and the Ricci identity, we get (21) by a straightforward calculation. □

Lemma 3.2. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be a trans-para-Sasakian manifold. Then the following identity holds:*

$$\begin{aligned} g(\phi R(\phi E,\phi F)U,\phi V) = {} & g(R(E,F)U,V) + (\alpha^2+\beta^2)[A(E,F,U,V) \\ & + A(\phi E,\phi F,U,V)] - 2\alpha\beta[A(E,\phi F,U,V) + A(\phi E,F,U,V)] \\ & - U(\alpha)A(E,F,\xi,\phi V) - V(\alpha)A(E,F,\phi U,\xi) - U(\beta)A(E,F,\xi,V) \\ & - V(\beta)A(E,F,U,\xi) + \eta(V)[\phi E(\alpha)A(\xi,F,U,\xi) - \phi F(\alpha)A(\xi,E,U,\xi) \\ & - A(\phi E,\phi F,grad\beta,U)] \end{aligned} \tag{22}$$

for all vector fields E, F, U *and* V *on* $\mathfrak{X}(M)$.

Proof. Using (8), we get

$$g(\phi R(\phi E,\phi F)U,\phi V) = -g(R(\phi E,\phi F)U,V) + \eta(R(\phi E,\phi F)U)\eta(V).$$

Then by (7) and (14) and the Riemannian curvature tensor properties, we have

$$\begin{aligned} & g(\phi R(\phi E,\phi F)U,\phi V) \\ & = -g(R(U,V)\phi E,\phi F) + \eta(V)[\phi E(\alpha)A(\xi,F,U,\xi) \\ & \quad - \phi F(\alpha)A(\xi,E,U,\xi) - A(\phi E,\phi F,grad\beta,U)]. \end{aligned}$$

By virtue of (8) and (21), we find

$$\begin{aligned} g(\phi R(\phi E,\phi F)U,\phi V) = {} & -g(\phi R(U,V)E,\phi F) \\ & + (\alpha^2+\beta^2)[-g(U,E)A(\xi,V,F,\xi) + g(V,E)A(\xi,U,F,\xi) + A(\phi E,\phi F,U,V)] \\ & - 2\alpha\beta[A(E,\phi F,U,V) - g(U,\phi E)A(\xi,V,F,\xi) + g(V,\phi E)A(\xi,U,F,\xi)] \\ & + \eta(E)[A(V,U,\phi F,grad\alpha) + U(\beta)A(\xi,V,F,\xi) - V(\beta)A(\xi,F,U,\xi)] \\ & + \eta(V)[\phi E(\alpha)A(\xi,F,U,\xi) - \phi F(\alpha)A(\xi,E,U,\xi) - A(\phi E,\phi F,grad\beta,U)]. \end{aligned}$$

Finally, using (8), we get

$$\begin{aligned} & g(\phi R(\phi E,\phi F)U,\phi V) = \\ & \quad g(R(U,V)E,F) - \eta(R(U,V)E)\eta(F) \\ & \quad + (\alpha^2+\beta^2)[-g(U,E)A(\xi,V,F,\xi) + g(V,E)A(\xi,U,F,\xi) + A(\phi E,\phi F,U,V)] \\ & \quad - 2\alpha\beta[A(E,\phi F,U,V) - g(U,\phi E)A(\xi,V,F,\xi) + g(V,\phi E)A(\xi,U,F,\xi)] \\ & \quad + \eta(E)(A(V,U,\phi F,grad\alpha) + U(\beta)A(\xi,V,F,\xi) \\ & \quad - V(\beta)A(\xi,F,U,\xi)) + \eta(V)(\phi E(\alpha)A(\xi,F,U,\xi) \\ & \quad - \phi F(\alpha)A(\xi,E,U,\xi) - A(\phi E,\phi F,grad\beta,U)). \end{aligned}$$

One can get (22) by using (14) in the last equation. □

Lemma 3.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a trans-para-Sasakian manifold. Then the following relation holds:*

$$\begin{aligned} g(R(\phi E, \phi F)\phi U, \phi V) &= g(R(U, V)E, F) + (\alpha^2 + \beta^2)[A(E, F, U, \xi)\eta(V) \\ &- A(E, F, V, \xi)\eta(U)] - 4\alpha\beta[A(E, \phi F, U, V) + A(\phi E, F, U, V)] \\ &+ 2\alpha\beta[A(E, F, \phi V, \xi)\eta(U) - A(E, F, \phi U, \xi)\eta(V)] \\ &- U(\alpha)A(E, F, \xi, \phi V) - V(\alpha)A(E, F, \phi U, \xi) + U(\beta)A(E, F, V, \xi) \\ &- V(\beta)A(E, F, U, \xi) - \phi E(\alpha)A(U, V, F, \xi) + \phi F(\alpha)A(U, V, E, \xi) \\ &+ \phi E(\beta)A(V, U, \phi F, \xi) - \phi F(\beta)A(V, U, \phi E, \xi) \end{aligned} \tag{23}$$

for all vector fields E, F, U and V on $\mathfrak{X}(M)$.

Proof. Replacing in (21), E, F by $\phi E, \phi F$, resp, and taking the inner product with ϕV, we get

$$\begin{aligned} g(R(\phi E, \phi F)\phi U, \phi V) &= g(\phi R(\phi E, \phi F)U, \phi V) + (\alpha^2 + \beta^2)[A(\phi E, \phi^2 F, U, \phi V) \\ &+ A(\phi^2 E, \phi F, U, \phi V)] + 2\alpha\beta[A(\phi E, \phi F, U, \phi V) + A(\phi E, \phi F, \phi U, \phi^2 V)] \\ &+ \eta(U)[A(\phi E, \phi F, \phi V, grad\alpha) - A(\phi E, \phi F, \phi^2 V, grad\beta)]. \end{aligned} \tag{24}$$

On the other hand, using (7) and (8) in (24), we have

$$\begin{aligned} &g(R(\phi E, \phi F)\phi U, \phi V) \\ &= g(\phi R(\phi E, \phi F)U, \phi V) + (\alpha^2 + \beta^2)[A(\phi E, \phi F, V, U) \\ &\quad - A(E, F, U, V) + \eta(V)A(E, F, U, \xi) - \eta(U)A(E, F, V, \xi)] \\ &\quad + 2\alpha\beta[\eta(U)A(E, F, \phi V, \xi) - \eta(V)A(E, F, \phi U, \xi) - A(E, \phi F, U, V) \\ &\quad - A(\phi E, F, U, V)] + \eta(U)[A(\phi F, \phi E, V, grad\beta) - \phi E(\alpha)A(\xi, F, V, \xi) \\ &\quad + \phi F(\alpha)A(\xi, E, V, \xi)]. \end{aligned}$$

Finally, in order to prove (23), we use (22) in the last equation.
□

Theorem 3.4. *In a trans-para-Sasakian manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following relation holds:*

$$(Q\phi - \phi Q)E = B_n(\alpha, \beta, \phi E)\xi - \phi(B_n(\alpha, \beta))\eta(E) - 8\alpha\beta(n-1)\phi^2 E \tag{25}$$

for all vector field E on $\mathfrak{X}(M)$.

Proof. Let $\{e_i, \phi e_i, \xi\}$ $(i = 1, 2, \ldots, n)$ be a local orthonormal ϕ-basis. Setting $F = U = e_i$ in (23) and taking the summation over i, we have

$$\begin{aligned} -\sum_{i=1}^{n} \varepsilon_i g(\phi R(\phi E, \phi e_i)\phi e_i, V) = \varepsilon_i \sum_{i=1}^{n} &[g(R(E, e_i)e_i, V) - e_i(\alpha)g(V, \phi e_i)\eta(E) \\ &+ e_i(\beta)g(e_i, V)\eta(E) - \phi e_i(\alpha)g(E, e_i)\eta(V) \\ &+ \phi E(\alpha)g(e_i, e_i)\eta(V) - \phi e_i(\beta)g(\phi E, e_i)\eta(V) \\ &- 4\alpha\beta\{g(E, e_i)g(e_i, \phi V) - g(\phi E, e_i)g(e_i, V) \\ &- g(e_i, e_i)g(E, \phi V)\} - V(\beta)g(e_i, e_i)\eta(E) \\ &+ (\alpha^2 + \beta^2)g(e_i, e_i)\eta(E)\eta(V)]. \end{aligned} \tag{26}$$

On the other hand, putting $F = U = \phi e_i$ in (23) and using (7), we get

$$\begin{aligned} -\sum_{i=1}^{n} \varepsilon_i g(\phi R(\phi E, e_i)e_i, V) =& \varepsilon_i \sum_{i=1}^{n} [g(R(E, \phi e_i)\phi e_i, V) - \phi e_i(\alpha) g(V, e_i)\eta(E) \\ &+ \phi e_i(\beta) g(\phi e_i, V)\eta(E) - e_i(\alpha) g(E, \phi e_i)\eta(V) \\ &+ \phi E(\alpha) g(\phi e_i, \phi e_i)\eta(V) - e_i(\beta) g(\phi E, \phi e_i)\eta(V) \\ &- 4\alpha\beta\{g(E, \phi e_i) g(\phi e_i, \phi V) - g(\phi E, \phi e_i) g(\phi e_i, V) \\ &- g(\phi e_i, \phi e_i) g(E, \phi V)\} - V(\beta) g(\phi e_i, \phi e_i)\eta(E) \\ &+ (\alpha^2 + \beta^2) g(\phi e_i, \phi e_i)\eta(E)\eta(V)]. \end{aligned} \tag{27}$$

Using the definition of the Ricci operator, (26) and (27), we obtain by direct calculation

$$\begin{aligned} \phi(Q(\phi E) - R(\phi E, \xi)\xi) =& QE - R(E, \xi)\xi + 2n(\alpha^2 + \beta^2)\eta(E)\xi \\ &- 8\alpha\beta(n-1)\phi E + C_n(\alpha, \beta, \phi E)\xi \\ &- B_n(\alpha, \beta)\eta(E) - \eta(E)\xi(\beta)\xi. \end{aligned} \tag{28}$$

From (15), we have

$$R(\phi E, \xi)\xi = -(\alpha^2 + \beta^2 - \xi(\beta))\phi E. \tag{29}$$

With the help of (15) and (29), the relation (28) becomes

$$\begin{aligned} \phi(Q(\phi E)) =& QE + 2n(\alpha^2 + \beta^2)\eta(E)\xi - 8\alpha\beta(n-1)\phi E - \eta(E)\xi(\beta)\xi \\ &+ C_n(\alpha, \beta, \phi E)\xi - B_n(\alpha, \beta)\eta(E). \end{aligned} \tag{30}$$

Finally, applying ϕ to (30) and using (7), we have

$$\begin{aligned} Q(\phi E) - \phi(QE) =& S(\phi E, \xi)\xi - 8\alpha\beta(n-1)\phi^2 E \\ &- \phi(B_n(\alpha, \beta))\eta(E). \end{aligned} \tag{31}$$

From (16), we get

$$S(\phi E, \xi) = B_n(\alpha, \beta, \phi E). \tag{32}$$

In the sense of (31) and (32), we obtain the assertion. □

Proposition 3.5. *Let M be a three-dimensional trans-para-Sasakian manifold. If $B(\alpha, \beta) = 0$, then $Q\phi = \phi Q$.*

Proof. The assertion follows from (18) and (25). □

Theorem 3.4 gives following.

Corollary 3.6. *If M^{2n+1} is α-para-Sasakian, β-para-Kenmotsu or paracosymplectic manifold, then $Q\phi = \phi Q$.*

Proposition 3.7. *In a trans-para-Sasakian manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following relation holds:*

$$\begin{aligned} S(\phi E, \phi F) =& - S(E, F) + (2n-1)A(\xi, F, grad\beta, \xi)\eta(E) \\ &+ [-(2n(\alpha^2 + \beta^2) - \xi(\beta))\eta(E) + B_n(\alpha, \beta, E)]\eta(F) \\ &+ 8\alpha\beta(n-1)g(\phi E, F) - \phi F(\alpha)\eta(E), \end{aligned} \tag{33}$$

for all vector fields E, F on $\mathfrak{X}(M)$.

Proof. Taking the inner product of (25) with ϕF, we have

$$S(\phi E,\phi F)A(\xi,QE,F,\xi)=8\alpha\beta(1-n)g(E,\phi F)+(2n-1)A(\xi,F,grad\beta,\xi)\eta(E) - \phi F(\alpha)\eta(E).$$

By virtue of (16) we get (33). □

Proposition 3.8. *In an η-Einstein trans-para-Sasakian manifold $(M^{2n+1},\phi,\xi,\eta,g)$, the Ricci tensor is expressed as*

$$S(E,F)=[\frac{r}{2n}+(\alpha^2+\beta^2-\xi(\beta))]g(E,F) - [\frac{r}{2n}+(2n+1)(\alpha^2+\beta^2-\xi(\beta))]\eta(E)\eta(F), \tag{34}$$

for all vector fields E,F on $\mathfrak{X}(M)$.

Proof. From (9) we have

$$r=(2n+1)\lambda+\mu, \tag{35}$$

where r is the scalar curvature. On the other hand, (17) and (9) implies that

$$-2n(\alpha^2+\beta^2-\xi(\beta))=\lambda+\mu. \tag{36}$$

From (35) and (36), it follows that

$$\lambda=\frac{r}{2n}+(\alpha^2+\beta^2-\xi(\beta))$$

and

$$\mu=-\frac{r}{2n}-(2n+1)(\alpha^2+\beta^2-\xi(\beta)).$$

It completes the proof. □

4. 3-Dimensional Trans-para-Sasakian Manifolds

It is well-known that in a three-dimensional Riemannian manifold the curvature tensor is of the following form:

$$R(E,F)U=A(QE,F,U)-A(QF,E,U)-\frac{r}{2}A(E,F,U), \tag{37}$$

where r is the scalar curvature of the manifold.

Letting $U=\xi$ in (37) and using (13) and (16), we obtain

$$\begin{aligned}&[QE-(\frac{r}{2}-\xi(\beta)+(\alpha^2+\beta^2))E+2\alpha\beta\phi E-E(\beta)\xi]\eta(F)\\&\quad -F(\alpha)\phi E-\phi F(\alpha)E\\&=[QF-(\frac{r}{2}-\xi(\beta)+(\alpha^2+\beta^2))F+2\alpha\beta\phi F-F(\beta)\xi]\eta(E)\\&\quad -E(\alpha)\phi F-\phi E(\alpha)F.\end{aligned} \tag{38}$$

So, we can obtain the expression of the Ricci operator for a 3-dimensional trans-para-Sasakian manifold.

Theorem 4.1. *In a 3-dimensional trans-para-Sasakian manifold, the Ricci operator is given by*

$$\begin{aligned} QE =& [\frac{r}{2} - \xi(\beta) + (\alpha^2 + \beta^2)]E - [\frac{r}{2} - \xi(\beta) + 3(\alpha^2 + \beta^2)]\eta(E)\xi \\ &+ B(\alpha, \beta)\eta(E) + B(\alpha, \beta, E)\xi, \end{aligned} \tag{39}$$

for all vector field E on $\mathfrak{X}(M)$.

Proof. Setting $F = \xi$ in (38) and then using the formulas (18) and (19) we find (39). □

From (39), we have the following corollary.

Corollary 4.2. *In a 3-dimensional trans-para-Sasakian manifold, the Ricci tensor is given by*

$$\begin{aligned} S(E, F) =& [\frac{r}{2} - \xi(\beta) + (\alpha^2 + \beta^2)]g(E, F) - [\frac{r}{2} - \xi(\beta) + 3(\alpha^2 + \beta^2)]\eta(E)\eta(F) \\ &+ B(\alpha, \beta, F)\eta(E) + B(\alpha, \beta, E)\eta(F), \end{aligned} \tag{40}$$

for all vector fields E, F on $\mathfrak{X}(M)$.

Corollary 4.3. *If $B(\alpha, \beta) = 0$ then the 3-dimensional trans-para-Sasakian manifold is η-Einstein.*

Proof. From the assumption we have

$$E(\beta) = g(grad\beta, E) = -g(\phi(grad\alpha), E) = \phi E(\alpha). \tag{41}$$

Using (41) and Corollary 2.2 in (40), we get the result. □

Theorem 4.4. *A 3-dimensional trans-para-Sasakian manifold is an η-Einstein manifold if and only if*

$$B(\alpha, \beta, E) = \xi(\beta)\eta(E), \tag{42}$$

for all vector field $E \in \mathfrak{X}(M)$.

Proof. If a 3-dimensional trans-para-Sasakian manifold is an η-Einstein manifold, from (40) we get

$$B(\alpha, \beta, F)\eta(E) + B(\alpha, \beta, E)\eta(F) = ag(E, F) + b\eta(E)\eta(F), \tag{43}$$

where a and b are smooth functions and $E, F \in \mathfrak{X}(M)$. Letting $E = \phi E$ and $F = \phi F$ in (43), we obtain

$$0 = ag(\phi E, \phi F), \tag{44}$$

which implies $a = 0$. On the other hand, putting $E = F = \xi$ in (43), we get

$$2\xi(\beta) = b. \tag{45}$$

Therefore, using (44) and (45) in the equation (43), we have

$$B(\alpha, \beta, F)\eta(E) + B(\alpha, \beta, E)\eta(F) = 2\xi(\beta)\eta(E)\eta(F). \tag{46}$$

Letting $E = F$ in the above equation gives

$$B(\alpha, \beta, E) = \xi(\beta)\eta(E). \tag{47}$$

Conversely, if (42) holds, from (40) we have

$$S(E, F) = [\frac{r}{2} - \xi(\beta) + (\alpha^2 + \beta^2)]g(E, F) - [\frac{r}{2} - 3\xi(\beta) + 3(\alpha^2 + \beta^2)]\eta(E)\eta(F). \tag{48}$$

Hence, it completes the proof. □

We can note that Proposition 3.8 coincide with Theorem 4.4.

Corollary 4.5. *In a 3-dimensional trans-para-Sasakian manifold, the Riemannian curvature tensor is given by*

$$\begin{aligned} R(E,F)U =& [\frac{r}{2} - 2\xi(\beta) + 2(\alpha^2+\beta^2)]A(E,F,U) \\ &- g(F,U)([\frac{r}{2} - \xi(\beta) + 3(\alpha^2+\beta^2)]\eta(E)\xi \\ &- B(\alpha,\beta)\eta(E) - B(\alpha,\beta,E)\xi) \\ &+ g(E,U)([\frac{r}{2} - \xi(\beta) + 3(\alpha^2+\beta^2)]\eta(F)\xi \\ &- B(\alpha,\beta)\eta(F) - B(\alpha,\beta,F)\xi) \\ &- ([\frac{r}{2} - \xi(\beta) + 3(\alpha^2+\beta^2)]\eta(F)\eta(U) \\ &- B(\alpha,\beta,U)\eta(F) - B(\alpha,\beta,F)\eta(U))E \\ &+ ([\frac{r}{2} - \xi(\beta) + 3(\alpha^2+\beta^2)]\eta(E)\eta(U) \\ &- B(\alpha,\beta,U)\eta(E) - B(\alpha,\beta,E)\eta(U))F, \end{aligned} \tag{49}$$

for all vector fields E, F, U on $\mathfrak{X}(M)$.

Proof. By using (39) and (40) in (37), we have the Riemannian curvature tensor for a 3-dimensional trans-para-Sasakian manifold by (49). □

Moreover, from (42) and (49) we can state the following lemma.

Lemma 4.6. *In a 3-dimensional η-Einstein trans-para-Sasakian manifold, the curvature tensor is given by*

$$\begin{aligned} R(E,F)U =& [\frac{r}{2} - 2\xi(\beta) + 2(\alpha^2+\beta^2)]A(E,F,U) \\ &- [\frac{r}{2} - 3\xi(\beta) + 3(\alpha^2+\beta^2)](A(E,F,U,\xi)\xi + \eta(U)A(E,F,\xi)), \end{aligned} \tag{50}$$

for all vector fields E, F, U on $\mathfrak{X}(M)$.

Using (49) and Corollary 2.2, we can give the following lemma.

Lemma 4.7. *In a 3-dimensional trans-para-Sasakian manifold, if $B(\alpha,\beta) = 0$, then the curvature tensor is given by*

$$\begin{aligned} R(E,F)U =& [\frac{r}{2} + 2(\alpha^2+\beta^2)]A(E,F,U) \\ &- [\frac{r}{2} + 3(\alpha^2+\beta^2)](A(E,F,U,\xi)\xi + \eta(U)A(E,F,\xi)), \end{aligned} \tag{51}$$

for all vector fields E, F, U on $\mathfrak{X}(M)$.

5. Ricci semi-symmetric Trans-para-Sasakian Manifolds

In this section, we give the expression of Ricci tensor for Ricci semi-symmetric trans-para-Sasakian manifold. Then we construct a three dimensional trans-para-Sasakian manifold example which satisfies our results.

Definition 5.1. *[14] A semi-Riemannian manifold (M^{2n+1}, g) is said to be Ricci semi-symmetric if we have $R(E,F)S = 0$ on M, where $R(E,F)$ is the curvature operator.*

Theorem 5.2. *In a Ricci semi-symmetric trans-para-Sasakian manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the Ricci tensor satisfies*

$$\begin{aligned}(\alpha^2+\beta^2-\xi(\beta))S(E,F) =& [2n(\alpha^2+\beta^2)\{2\xi(\beta)-\alpha^2-\beta^2\}-(\xi(\beta))^2\\&-(2n-1)\|grad\beta\|^2+\phi(grad\beta)(\alpha)]g(E,F)\\&+[(2n-1)\|grad\beta\|^2-\phi(grad\beta)(\alpha)\\&+4\alpha^2\beta^2]\eta(E)\eta(F)+[\{(1-2n)F(\beta)+\frac{1}{2}\phi F(\alpha)\}\xi(\beta)\\&+(2n-1)\alpha\beta\phi F(\beta)-2\alpha\beta F(\alpha)]\eta(E)\\&+[\{(1-2n)E(\beta)+\frac{1}{2}\phi E(\alpha)\}\xi(\beta)\\&+(2n-1)\alpha\beta\phi E(\beta)-2\alpha\beta E(\alpha)]\eta(F)\\&+E(\alpha)F(\alpha)-\frac{1}{2}[F(\beta)\phi E(\alpha)+E(\beta)\phi F(\alpha)\\&+(2n-1)\{F(\alpha)\phi E(\beta)+E(\alpha)\phi F(\beta)\}]\\&+(2n-1)E(\beta)F(\beta),\end{aligned}\tag{52}$$

for all vector fields E, F on $\mathfrak{X}(M)$.

Proof. From the assumption, we have $R(E,F)S = 0$. This condition is equivalent to

$$S(R(E,F)\zeta, V) + S(\zeta, R(E,F)V) = 0.$$

From the above equation, we get

$$S(R(\xi,E)\xi,F) + S(\xi, R(\xi,E)F) = 0. \tag{53}$$

Using (14) and (15) in (53), we obtain

$$\begin{aligned}(\alpha^2+\beta^2-\xi(\beta))S(E,F) =& [(\alpha^2+\beta^2)g(E,F)-F(\beta)\eta(E)+2\alpha\beta g(E,\phi F)]\\&S(\xi,\xi)+[F(\beta)-(\alpha^2+\beta^2)\eta(F)]S(\xi,E)\\&+[2\alpha\beta\eta(F)-F(\alpha)]S(\xi,\phi E)\\&+g(\phi E,\phi F)S(\xi,grad\beta)-g(E,\phi F)S(\xi,grad\alpha)\\&+(\alpha^2+\beta^2-\xi(\beta))\eta(E)S(\xi,F).\end{aligned}\tag{54}$$

Since S is a symmetric tensor, we also have

$$\begin{aligned}(\alpha^2+\beta^2-\xi(\beta))S(E,F) =& [(\alpha^2+\beta^2)g(E,F)-E(\beta)\eta(F)+2\alpha\beta g(\phi E,F)]\\&S(\xi,\xi)+[E(\beta)-(\alpha^2+\beta^2)\eta(E)]S(\xi,F)\\&+[2\alpha\beta\eta(E)-E(\alpha)]S(\xi,\phi F)\\&+g(\phi F,\phi E)S(\xi,grad\beta)-g(\phi E,F)S(\xi,grad\alpha)\\&+(\alpha^2+\beta^2-\xi(\beta))\eta(F)S(\xi,E).\end{aligned}\tag{55}$$

Adding (54) and (55), we get

$$\begin{aligned}2(\alpha^2+\beta^2-\xi(\beta))S(E,F) =& [2(\alpha^2+\beta^2)g(E,F)-F(\beta)\eta(E)-E(\beta)\eta(F)]\\&S(\xi,\xi)+[F(\beta)-\xi(\beta)\eta(F)]S(\xi,E)\\&+[E(\beta)-\xi(\beta)\eta(E)]S(\xi,F)\\&+[2\alpha\beta\eta(F)-F(\alpha)]S(\xi,\phi E)\\&+[2\alpha\beta\eta(E)-E(\alpha)]S(\xi,\phi F)\\&+2g(\phi E,\phi F)S(\xi,grad\beta).\end{aligned}\tag{56}$$

By the help of (16), (17) and (18), one can get (52). □

Corollary 5.3. *[8] A three-dimensional para-Sasakian (para-Kenmotsu) manifold is Ricci semi-symmetric manifold if and only if the manifold is an Einstein manifold.*

Example 5.4. *We consider the three dimensional manifold M and the vector fields*

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = (x+y)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where

$$g = \begin{pmatrix} 1 & 0 & -\frac{x+y}{2} \\ 0 & -1 & \frac{x+y}{2} \\ -\frac{x+y}{2} & \frac{x+y}{2} & 1 \end{pmatrix},$$

$$\phi = \begin{pmatrix} 0 & 1 & -(x+y) \\ 1 & 0 & -(x+y) \\ 0 & 0 & 0 \end{pmatrix}.$$

One can observe that

$$g(e_1,e_1) = g(e_3,e_3) = 1, g(e_2,e_2) = -1, g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0$$

and

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

We get

$$[e_1,e_3] = e_1 + e_2, \quad [e_2,e_3] = e_1 + e_2, \quad [e_1,e_2] = 0.$$

Taking $e_3 = \xi$ and using Koszul formula, we can calculate

$$\begin{aligned} \nabla_{e_1}e_1 &= -\xi, \quad \nabla_{e_2}e_1 = 0, \quad \nabla_{e_3}e_1 = -e_2 \\ \nabla_{e_1}e_2 &= 0, \quad \nabla_{e_2}e_2 = \xi, \quad \nabla_{e_3}e_2 = -e_1 \\ \nabla_{e_1}e_3 &= e_1, \quad \nabla_{e_2}e_3 = e_2, \quad \nabla_{e_3}e_3 = 0. \end{aligned}$$

We also see that

$$\begin{aligned} (\nabla_{e_1}\phi)e_1 &= \nabla_{e_1}\phi(e_1) - \phi(\nabla_{e_1}e_1) = -0 \\ &= 0(-g(e_1,e_1)\xi + \eta(e_1)e_1) - 1(g(e_1,\phi(e_1))\xi + \eta(e_1)\phi(e_1)), \end{aligned}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_2 &= \nabla_{e_1}\phi(e_2) - \phi(\nabla_{e_1}e_2) = -\xi \\ &= 0(-g(e_1,e_2)\xi + \eta(e_2)e_1) - 1(g(e_1,\phi(e_2))\xi + \eta(e_2)\phi(e_1)), \end{aligned}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_3 &= \nabla_{e_1}\phi(e_3) - \phi(\nabla_{e_1}e_3) = -e_2 \\ &= 0(-g(e_1,e_3)\xi + \eta(e_3)e_1) - 1(g(e_1,\phi(e_3))\xi + \eta(e_3)\phi(e_1)). \end{aligned}$$

In the above equations, we see that the manifold satisfies (10) for $X = e_1$, $\alpha = 0, \beta = -1$ and $e_3 = \xi$. Similarly, it is also true for $X = e_2$ and $X = e_3$.

The 1-form $\eta = dz$ and the fundamental 2-form $\Phi = dx \wedge dy - (x+y)dx \wedge dz + (x+y)dy \wedge dz$ defines a trans-para-Sasakian manifold, where $d\eta = \alpha\Phi$, $d\Phi = -2\beta\eta \wedge \Phi$. Hence, the manifold is a trans-para-Sasakian manifold of type $(0,-1)$.

Then the expressions of the curvature tensor is given by

$$\begin{aligned} &R(e_1,e_2)e_3 = 0, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_1,e_3)e_3 = -e_1, \\ &R(e_1,e_2)e_2 = e_1, \quad R(e_2,e_3)e_2 = -\xi, \quad R(e_1,e_3)e_2 = 0, \\ &R(e_1,e_2)e_1 = e_2, \quad R(e_2,e_3)e_1 = 0, \quad R(e_1,e_3)e_1 = \xi. \end{aligned}$$

Therefore, we have $S(e_1, e_1) = -2, S(e_2, e_2) = 2$ *and* $S(e_3, e_3) = -2$. *It implies that the scalar curvature* $r = -6$. *Then the equation (39) becomes*

$$QX = -2X. \tag{57}$$

From the equation (57), we have $Q\phi X = \phi(QX) = -2\phi X$ *and hence the Theorem 3.4 is verified.*

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