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Krasnoselskii-type best proximity point theorem in Banach algebras

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Abstract. The main purpose of this paper is to present some new existence of hybrid best proximity point theorems involving the sum of two non-self operators in Banach spaces as well as the product of two non-self operators in Banach algebras. In this way, we extend and revisited the main conclusions of a recent paper by Kar and Veeramani [S. Kar, P. Veeramani, Best proximity version of Krasnoselskii's fixed point theorem, Acta Sci. Math. (Szeged), 86 (2020), 265-271]. Illustrative examples are given to support our results.

1. Introduction and Preliminaries

In 1955, Krasnoselskii ([14]) combined the Banach contraction principle and the Schauder's fixed point problem and obtained the following important fixed point theorem.

Theorem 1.1. (Krasnoselskii's fixed point theorem) *Let A be a nonempty, closed and convex subset of a Banach space X. Assume that $T : A \to X$ and $S : A \to X$ are two operators such that*

(i) *T is a contraction, that is, there exists a real number $k \in (0, 1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for any $x, y \in A$;*

(ii) *S is a continuous and compact operator;*

(iii) *$T(A) + S(A) \subseteq A$.*

Then there is an element $u \in A$ for which $Tu + Su = u$.

There are a huge number of papers contributing generalizations or modifications of the Krasnoselskii's fixed point theorem and their applications (see for example [1, 2, 6, 7]).

Let (A, B) be a nonempty pair in a metric space (X, d) and $T : A \to B$ be a non-self mapping. A point $p^\star \in A$ is called a *best proximity point* of T if

$$d(p^\star, Tp^\star) = \text{dist}(A, B) := \inf_{(a,b) \in A \times B} d(a, b).$$

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In fact best proximity point theorems are studied to find necessary conditions to guarantee the existence of a solution to the minimization problem

$$\min_{x\in A} d(x,Tx). \tag{1}$$

We refer to [4, 5, 13, 19] for some discussions on the existence, uniqueness and convergence of a best proximity point.

Let (A,B) be a nonempty pair in a Banach space X. Set

$$A_0 = \{x \in A : \|x-y\| = \text{dist}(A,B), \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : \|x-y\| = \text{dist}(A,B), \text{ for some } x \in A\}.$$

The pair (A_0,B_0) is said to be a *proximal pair* of (A,B). It is remarkable to note that (A_0,B_0) may by empty, but in particular if (A,B) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X, then it's proximal pair (A_0,B_0) is also nonempty, closed and convex (see [13] for more details). In what follows by $\mathcal{S}_X$ we denote the unit sphere in a Banach space X, that is,

$$\mathcal{S}_X := \Big\{x \in X : \|x\| = 1\Big\}.$$

Definition 1.2. *A Banach space X is said to be strictly convex if for any two distinct elements $x,y \in \mathcal{S}_X$, we have*

$$\Big\|\frac{x+y}{2}\Big\| < 1.$$

Hilbert spaces and ℓ^p spaces $(1 < p < \infty)$ are instances of strictly convex Banach spaces whereas the Banach spaces ℓ^1 and ℓ^∞ are not strictly convex.

In 2011, Snkar Raj introduced the following geometric notion and used to investigate an interesting extension of the Banach contraction principle.

Definition 1.3. ([17]) *Let (A,B) be a nonempty pair in a metric space (X,d) with $A_0 \neq \emptyset$. The pair (A,B) is said to have the* P*-property* (d-property in some literatures) *if and only if*

$$\begin{cases} d(x_1,y_1) = \text{dist}(A,B) \\ d(x_2,y_2) = \text{dist}(A,B) \end{cases} \Rightarrow d(x_1,x_2) = d(y_1,y_2),$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

It was announced in [11] that every nonempty, bounded, closed and convex pair in reflexive and Busemann convex spaces has the P-property. It is interesting to note that the concept of P-property characterizes the strict convexity of Banach spaces. Indeed, a Banach space X is strictly convex if and only if every nonempty, closed and convex pair in X has the P-property (see Theorem 3.1 of [18]).

The next lemmas regarding to the notion of P-property will be used in our coming discussions.

Lemma 1.4. (see Lemma 3.1 of [8]) *Let (A,B) be a nonempty, closed pair in a complete metric space (X,d) such that A_0 is nonempty and (A,B) has the* P*-property. Then the proximal pair (A_0,B_0) is closed.*

It is well-known in approximation theory that if A is a nonempty subset of a Banach space X a *metric projection operator* $\mathcal{P}_A : X \rightrightarrows A$ is defined as

$$\mathcal{P}_A(x) := \{a \in A : \|x-a\| = \text{dist}(\{x\},A)\}.$$

We mention that if A is a nonempty, closed and convex subset of a reflexive and strictly convex Banach space X, then the metric projection $\mathcal{P}_A$ is single-valued from X to A.

Lemma 1.5. ([9, 10]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X such that (A, B) has the* P*-property. Define a projection mapping $\mathcal{P} : A_0 \cup B_0 \to A_0 \cup B_0$ as*

$$\mathcal{P}(x) = \begin{cases} \mathcal{P}_{A_0}(x); & \text{if} \quad x \in B_0, \\ \mathcal{P}_{B_0}(x); & \text{if} \quad x \in A_0. \end{cases} \tag{2}$$

Then the following statements hold:

(i) $\|x - \mathcal{P}x\| = \operatorname{dist}(A, B)$ *for any $x \in A_0 \cup B_0$ and so $\mathcal{P}$ is cyclic on $A_0 \cup B_0$, that is, $\mathcal{P}(A_0) \subseteq B_0$ and $\mathcal{P}(B_0) \subseteq A_0$;*

(ii) $\mathcal{P}|_{A_0}$ *and* $\mathcal{P}|_{B_0}$ *are isometry;*

(iii) $\mathcal{P}|_{A_0}$ *and* $\mathcal{P}|_{B_0}$ *are affine;*

(iv) $\mathcal{P}^2|_{A_0} = i_{A_0}$ *and* $\mathcal{P}^2|_{B_0} = i_{B_0}$*, where i_E denotes the identity mapping on a nonempty subset E of X.*

The first existence result of a best proximity point by using the concept of P-property was established by Sankr Raj in [17] as follows.

Theorem 1.6. *Let (A, B) be a nonempty and closed pair in a complete metric space (X, d) and $T : A \to B$ be a contraction non-self mapping, that is,*

$$\exists\, k \in (0, 1) \ : \ d(Tx, Ty) \le kd(x, y), \quad \forall x, y \in A.$$

If (A, B) has the P*-property and $T(A_0) \subseteq B_0$, then T has a best proximity point.*

Recently, an extension version of Krasnoselskii's fixed point theorem was presented in [12] in order to study the existence of a best proximity point for sum of two non-self mappings by applying the geometric concept of P-property, as below.

Theorem 1.7. (Theorem 2.1 of [12]) *Let (A, B) be a nonempty, closed and convex pair in a Banach space X such that A_0 is nonempty and (A, B) has the* P*-property. Assume that $T : A \to X$ and $S : B \to X$ are two operators such that*

(i) *T is a contraction, that is, there exists a real number $k \in (0, 1)$ such that $\|Tx - Ty\| \le k\|x - y\|$ for any $x, y \in A$;*

(ii) *S is a continuous and compact operator;*

(iii) $T(A_0) + S(B_0) \subseteq A_0$.

Then the mapping $(I - T)^{-1}S$ has a best proximity point in B_0, that is,

$$\exists\, y^\star \in B_0 \text{ s.t. } \|y^\star - (I - T)^{-1}Sy^\star\| = \operatorname{dist}(A, B).$$

It is worth mention that if in Theorem 1.7 $A = B$, then we get the Krasnoselskii's fixed point theorem.

In this paper we give a generalization of Theorem 1.7 by using a Meir-Keeler contractive condition. We also establish a counterpart result of Theorem 1.7 for multiplication of two non-self mappings in Banach algebras .

2. An extension of Karsnoselskii's fixed point problem

We begin our main discussions by recalling the following extension of the Banach contraction principle due to Meir and Keeler (see [16]).

Theorem 2.1. (Meir-Keeler fixed point theorem) *Let (X, d) be a complete metric space and $T : X \to X$ be an $\mathcal{MK}$ contraction mapping, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ for which*

$$\forall x, y \in X, \quad \varepsilon \le d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon.$$

Then T has a unique fixed point and the Picard iteration sequence $\{T^n x_0\}$ converges to the fixed point of T for any $x_0 \in X$.

Unfortunately, Theorem 2.1 has no an appropriate application in nonlinear differential and integral equations. Thereby, Lim ([15]) and Suzuki ([20]) presented a more practical and equivalent contractive conditions w.r.t $\mathcal{MK}$ condition.

Definition 2.2. *A function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be a strictly $\mathcal{L}$-function provided that $\psi(0) = 0$ and for every $s > 0$ there exists a $\delta > 0$ such that $0 < \psi(r) < s$ for all $r \in [s, s + \delta)$.*

It was announced in [20] that the $\mathcal{MK}$ contractive condition of the self mapping T defined on a metric space X is equivalent to the following contractive condition:

$$d(Tx, Ty) \le \psi\Big(d(x, y)\Big), \quad \forall x, y \in X,$$

where ψ is a nondecreasing function and right continuous strictly $\mathcal{L}$-function on $[0, +\infty)$.

In the next theorem, we generalize Theorem 1.7 by considering the contractive assumption presented by Meir and Keeler.

Theorem 2.3. *Let (A, B) be a nonempty, closed and convex pair in a Banach space X such that A_0 is nonempty and (A, B) has the* P*-property. Assume that $T : A \to X$ and $S : B \to X$ are two operators such that*

(i) *T is an $\mathcal{MK}$ contraction;*

(ii) *S is a continuous and compact operator;*

(iii) *$T(A_0) + S(B_0) \subseteq A_0$.*

Then the mapping $(I - T)^{-1}S$ has a best proximity point in B_0.

Proof. By the fact that T is an $\mathcal{MK}$ contraction, there exists a nondecreasing function and right continuous strictly $\mathcal{L}$-function ψ such that $\|Tx - Ty\| \le \psi(\|x - y\|)$ for all $x, y \in A$. Thus for any distinct elements $x, y \in A$ we have

$$\begin{aligned} \|(I - T)x - (I - T)y\| &= \|(x - y) - (Tx - Ty)\| \ge \|x - y\| - \|Tx - Ty\| \\ &\ge \|x - y\| - \psi(\|x - y\|) > 0. \end{aligned}$$

Also,

$$\begin{aligned} \|(I - T)x - (I - T)y\| &= \|(x - y) - (Tx - Ty)\| \le \|x - y\| + \|Tx - Ty\| \\ &\le \|x - y\| + \psi(\|x - y\|) < 2\|x - y\|, \end{aligned}$$

which implies that the mapping $I - T$ is a homeomorphism on A (see Theorem 2.3 of [2]). Now let $y \in B_0$ be an arbitrary element and define $T_y : A_0 \to A_0$ with

$$T_y(x) = Tx + Sy, \quad \forall x \in A_0.$$

Note that by (iii), T_y is well-defined and

$$\|T_y(x) - T_y(z)\| = \|Tx - Tz\| \le \psi(\|x - z\|), \quad \forall x, z \in A_0,$$

that is, T_y is an $\mathcal{MK}$ contraction on the set A_0, where A_0 is complete because of Lemma 1.4. It now follows from Theorem 2.1 that T_y has a unique fixed point, say $h(y) \in A_0$. Therefore,

$$h(y) = T_y\Big(h(y)\Big) = T\Big(h(y)\Big) + Sy, \quad \forall y \in B_0,$$

and so,

$$Sy = (I - T)\Big(h(y)\Big) \in (I - T)(A_0), \quad \forall y \in B_0.$$

Hence, $(I-T)^{-1}\Big(S(B_0)\Big) \subseteq A_0$. Now by considering the projection mapping $\mathcal{P}$ in Lemma 1.5, we obtain

$$\mathcal{P}(I-T)^{-1}S(B_0) \subseteq \mathcal{P}(A_0) \subseteq B_0,$$

where the mapping $\mathcal{P}(I-T)^{-1}S : B_0 \to B_0$ is a continuous and compact operator and so, by the well-known Schauder's fixed point result, there exists an element $y^\star \in B_0$ for which $\mathcal{P}(I-T)^{-1}Sy^\star = y^\star$. Hence, by the property (i) of Lemma 1.5 we conclude that

$$\text{dist}(A, B) = \|(I-T)^{-1}Sy^\star - \mathcal{P}(I-T)^{-1}Sy^\star\| = \|(I-T)^{-1}Sy^\star - y^\star\|,$$

and this completes the proof. □

Corollary 2.4. *Theorem 1.7 is a particular case of Theorem 2.3.*

Proof. It is sufficient to consider $\psi(t) = kt$ for some $k \in (0, 1)$ and for all $t \geq 0$ in Theorem 2.3. □

Let us illustrate Theorem 2.3 with the following examples.

Example 2.5. *Consider the Banach space $X = \ell^1$ with the canonical basis $\{e_n\}$ and let $A = \mathcal{B}(0; \frac{1}{2})$ and $B = \mathcal{B}(0; 1)$. Then* dist$(A, B) = 0$ *and so, (A, B) has the* P*-property and $A_0 = B_0 = A$. Define the non-self mappings $T : A \to X$ and $S : B \to X$ with*

$$T(x_1, x_2, x_3, \dots) = (0, \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \dots), \quad \forall\, \mathbf{x} =: (x_1, x_2, x_3, \dots) \in A,$$
$$S(y_1, y_2, y_3, \dots) = (\frac{1 - \|\mathbf{y}\|}{4})e_1, \quad \forall\, \mathbf{y} := (y_1, y_2, y_3, \dots) \in B.$$

Clearly, T is an $\mathcal{MK}$ contraction and S is a continuous and compact operator. Now for any $\mathbf{x} \in A$ and $\mathbf{y} \in B$ we have

$$T\mathbf{x} + S\mathbf{y} = (\frac{1 - \|y\|}{4}, \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \dots),$$

and so,

$$\|T\mathbf{x} + S\mathbf{y}\| = \frac{1 - \|y\|}{4} + \sum_{j=1}^{\infty} \frac{|x_j|}{2} \leq \frac{1}{2},$$

which ensures that $T(A_0) + S(B_0) \subseteq A_0$. Thus the fixed point equation $Tp + Sp = p$ has a solution. Indeed, if $p = (p_1, p_2, p_3, \dots) \in A$ is a fixed point of $T + S$, then we must have

$$(p_1, p_2, p_3, \dots) = Tp + Sp = (\frac{1 - \|p\|}{4}, \frac{p_1}{2}, \frac{p_2}{2}, \frac{p_3}{2}, \dots),$$

which concludes that

$$p_1 = \frac{1 - \|p\|}{4}, \quad \& \quad p_{n+1} = \frac{p_1}{2^n}, \quad \forall n \in \mathbb{N}.$$

A simple calculation shows that $p_1 = \frac{1}{6}$ and so,

$$p = \Big(\frac{1}{6}, \frac{1}{6 \times 2}, \frac{1}{6 \times 2^2}, \frac{1}{6 \times 2^3}, \dots\Big).$$

Next example shows that Theorem 2.3 is a real extension of Theorem 1.7.

Example 2.6. *Consider the Banach space* $C[0,1]$ *renormed according to*

$$\|f\| = \|f\|_2 + \|f\|_\infty, \quad \forall f \in C[0,1].$$

Then $(C[0,1], \|.\|)$ *is strictly convex and it is easy to see that*

$$\|f\|_\infty \le \|f\| \le 2\|f\|_\infty, \quad \forall f \in C[0,1]. \tag{3}$$

Let

$$A = \Big\{f \in C[0,1] \ s.t. \ t \le f(t) \le 2\Big\},$$
$$B = \Big\{g \in C[0,1] \ s.t. \ 0 \le g(t) \le \frac{t}{t+1}\Big\}.$$

Then (A,B) *is a bounded, closed and convex pair and so,* (A,B) *has the* P*-property. Also, for any* $(f,g) \in A \times B$ *we have* $|f(t) - g(t)| \ge t - \frac{t}{t+1}$ *for all* $t \in [0,1]$ *and so,*

$$\begin{aligned}\|f - g\| \ge& \Big(\int_0^1 \Big(t - \frac{t}{t+1}\Big)^2 dt\Big)^{\frac{1}{2}} + \sup_{t\in[0,1]} |t - \frac{t}{t+1}| \\ &= \sqrt{\frac{17 - 24\ln 2}{6}} + \frac{1}{2} \simeq 0.744,\end{aligned}$$

which implies that $\operatorname{dist}(A,B) \simeq 0.744$*. It is worth noticing that*

$$A_0 = \Big\{t\Big\}, \quad \& \quad B_0 = \Big\{\frac{t}{t+1}\Big\}.$$

Now define the mappings $T : A \to C[0,1]$ *and* $S : B \to C[0,1]$ *with*

$$\Big(Tf\Big)(t) = \frac{1}{2}\ln\Big(f(t) + 1\Big), \quad \& \quad \Big(Sg\Big)(t) = \frac{1}{2}t + \frac{1}{2}\int_0^t g(s)ds, \quad \forall (f,g) \in A \times B.$$

Then T *is an* $\mathcal{MK}$ *contraction. In fact for all* $f_1, f_2 \in A$ *we have*

$$\begin{aligned}\Big|\Big(Tf_1\Big)(t) - \Big(Tf_2\Big)(t)\Big| &= \frac{1}{2}\Big|\ln\Big(f_1(t) + 1\Big) - \ln\Big(f_2(t) + 1\Big)\Big| \\ &= \frac{1}{2}\Big|\ln\Big(\frac{f_1(t) + 1}{f_2(t) + 1}\Big)\Big| \\ &= \frac{1}{2}\Big|\ln\Big(1 + \frac{f_1(t) - f_2(t)}{f_2(t) + 1}\Big)\Big| \\ &\le \frac{1}{2}\ln\Big(1 + \Big|f_1(t) - f_2(t)\Big|\Big), \quad \forall\, t \in [0,1].\end{aligned}$$

which implies that

$$\|Tf_1 - Tf_2\|_\infty \le \frac{1}{2}\ln\Big(1 + \|f_1 - f_2\|_\infty\Big).$$

and using the relation (3), we obtain

$$\begin{aligned}\|Tf_1 - Tf_2\| &\le 2\|Tf_1 - Tf_2\|_\infty \\ &\le \ln\Big(1 + \|f_1 - f_2\|_\infty\Big) \\ &\le \ln\Big(1 + \|f_1 - f_2\|\Big).\end{aligned}$$

Now if we set $\psi(t) = \ln(1+t)$ for all $t \in [0,1]$, then ψ is a nondecreasing function and right continuous strictly $\mathcal{L}$-function and we conclude that

$$\|Tf_1 - Tf_2\| \le \psi(\|f_1 - f_2\|), \quad \forall\, f_1, f_2 \in A,$$

which deduces that T is an $\mathcal{MK}$ contraction. Besides, S is a continuous and compact operator. Moreover for $(f,g) = (t, \frac{t}{t+1}) \in A_0 \times B_0$ we have

$$\Big(Tf\Big)(t) + \Big(Sg\Big)(t) = \frac{1}{2}\ln\Big(t+1\Big) + \frac{1}{2}t + \frac{1}{2}\int_0^t \frac{s}{s+1}ds = t \in A_0,$$

that is, $T(A_0) + S(B_0) \subseteq A_0$. It now follows from Theorem 2.3 that $(I-T)^{-1}S$ has a best proximity point which is a point $y^\star(t) = \frac{t}{t+1}$, where $t \in [0,1]$. It is worth noticing that the existence of a best proximity point for the mapping $(I-T)^{-1}S$ cannot be concluded from Theorem 1.7 due to the fact that T is not a contraction on A.

3. More existence results in Banach algebras

In this section, motivated by Theorem 2.3, we present a best proximity point result for multiplication of two non-self mappings in the framework of Banach algebras.

We recall that a normed linear space X is said to be an *algebra* provided that there exists an operator $(.) : X \times X \to X$ which is associative and bilinear and that the norm of X satisfies the following condition:

$$\|x.y\| \le \|x\|\|y\|, \quad \forall x, y \in X.$$

A complete normed algebra is called a *Banach algebra*.

Suppose (A,B) is a nonempty pair in a Banach algebra X. Set

$$AB = \Big\{x.y \ : \ (x,y) \in A \times B\Big\},$$
$$\|A\| = \sup\Big\{\|x\| \ : \ x \in A\Big\}.$$

In what follows $\mathcal{D}$ denotes the class of all upper semi-continuous and nondecreasing functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that $\varphi(0) = 0$. A mapping $T : A \subseteq X \to X$ is called $\mathcal{D}$-Lipschitz (in the sense of Dhage) if there exists $\varphi \in \mathcal{D}$ such that

$$\|Tx - Ty\| \le \varphi\Big(\|x-y\|\Big), \quad \forall x, y \in A.$$

It is worth mentioning that , if $\varphi(t) = kt$ for some $k > 0$, then T is a Lipschitz operator.

We mention that the class of $\mathcal{D}$-Lipschitz functions was introduced by B.C. Dhage ([3]) in order to investigate the multiplication version of Karsnoselskii's fixed point problem. Using this idea, we give the following existence theorem.

Theorem 3.1. *Let (A,B) be a nonempty, bounded, closed and convex pair in a Banach algebra X such that (A,B) has the* P*-property. Let $T : A \to X$ and $S : B \to X$ be two operators satisfy the following conditions:*

(D1) *T is $\mathcal{D}$-Lipschitz with $\varphi \in \mathcal{D}$;*

(D2) *S is a continuous and compact operator on B;*

(D3) *$T(A_0)S(B_0) \subseteq A_0$;*

(D4) *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $L\varphi(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta)$, where $L := \|S(B)\|$.*

Then there is a point $y^\star \in B_0$ such that

$$\|y^\star - \Big(T(\mathcal{P}y^\star)\Big).Sy^\star\| = \text{dist}(A,B). \qquad (4)$$

Proof. By Lemma 1.4 the pair (A_0, B_0) is closed. Let $y \in B_0$ be an arbitrary and fixed element and define $T_y : A_0 \to A_0$ with

$$T_y(x) = (Tx).(Sy), \quad \forall x \in A_0.$$

By the condition (D3), T_y is well-defined. Using (D1), if $x_1, x_2 \in A_0$ and $\varepsilon > 0$ be given such that $\|x_1 - x_2\| \geq \varepsilon$, then

$$\|T_y(x_1) - T_y(x_2)\| = \|(Tx_1).(Sy) - (Tx_2).(Sy)\| \leq \|Sy\|\|Tx_1 - Tx_2\| \leq L\varphi(\|x_1 - x_2\|).$$

It now follows from the assumption (D4) that there exists $\delta > 0$ such that if $\|x_1 - x_2\| \in [\varepsilon, \varepsilon + \delta)$, then $L\varphi(\|x_1 - x_2\|) < \varepsilon$. So, $\|T_y(x_1) - T_y(x_2)\| < \varepsilon$ which ensures that the mapping T_y is an $\mathcal{MK}$ contraction. Thus from Theorem 2.1, T_y has a unique fixed point, say $\hbar(y) \in A_0$, that is,

$$\hbar(y) = T_y\big(\hbar(y)\big) = \big(T(\hbar y)\big).(Sy).$$

In this situation $\hbar$ maps the set B_0 to A_0. We shall prove that $\hbar$ is continuous. Suppose that $\{y_n\}$ is a sequence in B_0 such that $y_n \to q \in B_0$. Then

$$\begin{aligned}\|\hbar y_n - \hbar q\| &= \|\big(T(\hbar y_n)\big)(Sy_n) - \big(T(\hbar q)\big)(Sq)\| \\ &\leq \|\big(T(\hbar y_n)\big)(Sy_n) - \big(T(\hbar q)\big)(Sy_n)\| + \|\big(T(\hbar q)\big)(Sy_n) - \big(T(\hbar q)\big)(Sq)\| \\ &\leq \|T(\hbar y_n) - T(\hbar q)\|\|Sy_n\| + \|T(\hbar q)\|\|Sy_n - Sq\| \\ &\leq L\varphi(\|\hbar y_n - \hbar q\|) + \|T(\hbar q)\|\|Sy_n - Sq\|\end{aligned}$$

which deduces that

$$\begin{aligned}\limsup_{n\to\infty} \|\hbar y_n - \hbar q\| &\leq \limsup_{n\to\infty} \Big(L\varphi(\|\hbar y_n - \hbar q\|) + \|T(\hbar q)\|\|Sy_n - Sq\|\Big) \\ &= \limsup_{n\to\infty} L\varphi(\|\hbar y_n - \hbar q\|).\end{aligned}$$

Now, if $\limsup_{n\to\infty} \|\hbar y_n - \hbar q\| = r > 0$, then by the fact that φ is upper semi-continuous, we obtain

$$r \leq L\varphi(r),$$

which is a contradiction. So, we must have $r = 0$, that is, $\hbar : B_0 \to A_0$ is continuous.

We claim that h is a compact operator. Notice that for a fixed element $z \in A_0$ we have

$$\begin{aligned}\|Tx\| &\leq \|Tz\| + \|Tx - Tz\| \\ &\leq \|Tz\| + \varphi(\|x - z\|) \\ &\leq \|Tz\| + \varphi\big(\operatorname{diam}(A_0)\big), \quad \forall x \in A_0.\end{aligned}$$

Let $M := \|Tz\| + \varphi\big(\operatorname{diam}(A_0)\big)$. Then we have $\|Tx\| \leq M$ for any $x \in A_0$.
Now assume that $\{v_n\}$ is a sequence in the set B_0. We prove that $\{\hbar v_n\}$ has a Cauchy subsequence. By this reality that S is compact, $\{Sv_n\}$ has a convergent subsequence. We may assume that $\limsup_{m,n\to\infty} \|Sv_m - Sv_n\| = 0$. Therefore,

$$\begin{aligned}\|\hbar v_m - \hbar v_n\| &= \|\big(T(\hbar v_m)\big).(Sv_m) - \big(T(\hbar v_n)\big).(Sv_n)\| \\ &\leq \|\big(T(\hbar v_m)\big).(Sv_m) - \big(T(\hbar v_n)\big).(Sv_m)\| + \|\big(T(\hbar v_n)\big).(Sv_m) - \big(T(\hbar v_n)\big).(Sv_n)\| \\ &\leq \|(Sv_m)\|\|T(\hbar v_m) - T(\hbar v_n)\| + \|T(\hbar v_n)\|\|Sv_m - Sv_n\| \\ &\leq L\varphi(\|\hbar v_m - \hbar v_n\|) + M\|Sv_m - Sv_n\|.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{m,n\to\infty}\|\hbar v_m - \hbar v_n\| &\le \limsup_{m,n\to\infty}\Big(L\varphi(\|\hbar v_m - \hbar v_n\|) + M\|Sv_m - Sv_n\|\Big)\\ &= \limsup_{m,n\to\infty} L\varphi(\|\hbar v_m - \hbar v_n\|).\end{aligned}$$

If $\limsup_{m,n\to\infty}\|\hbar v_m - \hbar v_n\| = \varepsilon$ for some $\varepsilon > 0$, then by the fact that φ is an upper-semi continuous function and by the above inequality, $\varepsilon \le L\varphi(\varepsilon)$. Besides, from the condition (D4) there exists $\delta > 0$ such that $L\varphi(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta)$ which is a contradiction and so, $\limsup_{m,n\to\infty}\|\hbar v_m - \hbar v_n\| = 0$ i.e., $\{\hbar v_n\}$ is a Cauchy sequence which concludes that $\hbar(B_0)$ is boundedly compact. Thus $\hbar : B_0 \to A_0$ is compact. Now the self mapping $\mathcal{P}\hbar : B_0 \to B_0$ is continuous and compact and by applying Schauder's fixed point theorem there is an element $y^\star \in B_0$ for which $\mathcal{P}\hbar y^\star = y^\star$. Using the property (iv) of the Lemma 1.5 we have

$$\hbar y^\star = \mathcal{P}(\mathcal{P}\hbar y^\star) = \mathcal{P}y^\star,$$

which yields that

$$\begin{aligned}\|y^\star - \Big(T(\mathcal{P}y^\star)\Big).S(y^\star)\| &= \|y^\star - \Big(T(\hbar y^\star)\Big).S(y^\star)\|\\ &= \|y^\star - \hbar y^\star\| = \|y^\star - \mathcal{P}y^\star\|\\ &= \operatorname{dist}(A, B),\end{aligned}$$

and this completes the proof.

□

It is worth mentioning that if in Theorem 3.1, $A = B$, then the projection mapping $\mathcal{P}\big|_{A_0}$ is identity and we get the following corollary which is the main result of [3].

Corollary 3.2. ([3]) *Let A be a nonempty, bounded, closed and convex subset of a Banach algebra X and let $T, S : A \to X$ be two mappings such that*

(i) *There exists a real number $k \in [0, 1)$ such that $\|Tx - Ty\| \le k\|x - y\|$ for any $x, y \in A$;*

(ii) *S is a continuous and compact operator;*

(iii) *$T(A)S(A) \subseteq A$.*

If $kL < 1$, then

$$\exists\, x^\star \in A \text{ s.t. } Tx^\star Sx^\star = x^\star, \tag{5}$$

where $L := \|S(A)\|$.

Let us illustrate Theorem 3.1 with the following example.

Example 3.3. *Consider the Banach algebra $X = (L^\infty[0, 1], \|.\|_\infty)$ with the pointwise multiplication $(fg)x = f(x)g(x)$. Let*

$$\begin{aligned}A &= \Big\{f \in X\,;\ 0 \le f(x) \le \frac{1}{4}\ \text{(a.e.)}\Big\},\\ B &= \Big\{g \in X\,;\ 0 \le g(x) \le x^2\ \text{(a.e.)}\Big\}.\end{aligned}$$

Then (A, B) is bounded, closed and convex with $\operatorname{dist}(A, B) = 0$. *Moreover,*

$$A_0 = B_0 = \Big\{h \in X\,;\ 0 \le h(x) \le x^2\ \text{(a.e.) on } [0, \frac{1}{2}],\quad 0 \le h(x) \le \frac{1}{4}\ \text{(a.e.) on } [\frac{1}{2}, 1]\Big\}.$$

Let $T : A \to X$ and $S : B \to X$ be defined as

$$(Tf)x = \frac{1}{8}, \quad (Sg)x = x^2 + \int_0^x g(t)dt, \quad \forall x \in [0,1].$$

Then T is contraction and S is a continuous and compact operator. On the other hand, if $0 \le x \le \frac{1}{2}$, then for any $h_1, h_2 \in A_0$ we have

$$\begin{aligned}\Big(Th_1Sh_2\Big)(x) &= \frac{1}{8}\Big(x^2 + \int_0^x h_2(t)dt\Big) \le \frac{1}{8}x^2 + \int_0^x t^2 dt\\ &= \frac{1}{8}x^2 + \frac{1}{3}x^3 \le x^2.\end{aligned}$$

Also, for $x \in [\frac{1}{2}, 1]$ we have

$$\begin{aligned}\Big(Th_1Sh_2\Big)(x) &= \frac{1}{8}\Big(x^2 + \int_0^{\frac{1}{2}} h_2(t)dt + \int_{\frac{1}{2}}^x h_2(t)dt\Big)\\ &\le \frac{1}{8}\Big(x^2 + \int_0^{\frac{1}{2}} t^2 dt + \int_{\frac{1}{2}}^x \frac{1}{4}dt\Big)\\ &\le \frac{7}{48}.\end{aligned}$$

Hence, $Th_1Sh_2 \in A_0$ for all $h_1, h_2 \in A_0$ which yields that $T(A_0)S(B_0) \subseteq A_0$. Notice that

$$L := \|S(B)\|_\infty \le \sup_{x\in[0,1]} \Big(x^2 + \int_0^x t^2 dt\Big) = \sup_{x\in[0,1]} (x^2 + \frac{x^3}{3}) = \frac{4}{3},$$

so for $k \in (0, \frac{3}{4})$ we have $kL < 1$. It now follows from the Theorem 3.1

$$\exists\, h \in B_0 \quad s.t. \quad Th.Sh = h.$$

It is worth noticing that the function $h \in B_0$ is a solution of the following integral equation:

$$h(x) = \frac{1}{8}x^2 + \frac{1}{8}\int_0^x h(t)dt,$$

which is $h(x) = 16e^{\frac{1}{8}x} - 2x - 16$.

Example 3.4. *Consider the Banach algebra $X = \mathcal{M}_{2\times 2}$ consist of all complex 2×2 matrices equipped with the matrix multiplication and the norm*

$$\Big\|[a_{ij}]_{2\times 2}\Big\| = \sum_{i,j=1}^{2} |a_{ij}|, \quad \forall\, [a_{ij}]_{2\times 2} \in \mathcal{M}_{2\times 2}.$$

Let

$$A = \Big\{\begin{bmatrix} a_{11} & a_{12}\\ 0 & 0\end{bmatrix}; \quad |a_{ij}| \le 1\Big\}, \quad \& \quad B = \Big\{\begin{bmatrix} b_{11} & b_{12}\\ b_{21} & -i\end{bmatrix}; \quad |b_{ij}| \le 1\Big\}.$$

Then (A, B) is a bounded, closed and convex pair in X with $\mathrm{dist}(A, B) = 1$. Also,

$$A_0 = A, \quad B_0 = \Big\{\begin{bmatrix} b_{11} & b_{12}\\ 0 & -i\end{bmatrix}; \quad |b_{ij}| \le 1\Big\},$$

and that (A, B) *has the* P*-property. In this case the projection mapping* $\mathcal{P} : B_0 \to A_0$ *is defined as*

$$\mathcal{P}\Big(\begin{bmatrix} b_{11} & b_{12} \\ 0 & -i \end{bmatrix}\Big) = \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix}.$$

For $\mathbf{a} := \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} \in A$ *and* $\mathbf{b} := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & -i \end{bmatrix} \in B$ *define* $T\mathbf{a} = k\mathbf{a},\ S\mathbf{b} = \mathbf{b}^*$*, where* $k \in (0, \frac{1}{4})$*. Then* T *is a contraction and* S *is a continuous and compact operator. Moreover, it is easy to see that* $T(A_0)S(B_0) \subseteq A_0$*. Using Theorem 3.1 there exists a point* $y^\star \in B_0$ *such that* $\|y^\star - \Big(T(\mathcal{P}y^\star)\Big).Sy^\star\| = \operatorname{dist}(A, B)$ *and this point is* $y^\star = \begin{bmatrix} 0 & 0 \\ 0 & -i \end{bmatrix}$ *is a best proximity point for the mapping* $(T\mathcal{P}).S$*.*

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