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Toughness and Q-spectral radius of graphs involving minimum degree

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Abstract. In 1973, Chvátal initially proposed the concept of toughness, which serves as a simple way to measure how tightly various pieces of a graph hold together. Let G be a non-complete graph and let t be a real number. If for every vertex cut set S of G, $|S| \geq tc(G-S)$, then we say that G is *t-tough*. The largest t such that G is t-tough is called the *toughness* of G and is denoted by $t(G)$. Recently, Fan, Lin and Lu [European J. Combin. 110 (2023) 103701] presented sufficient conditions based on the spectral radius for a graph to be 1-tough with minimum degree δ and t-tough with $t \geq 1$ being an integer, respectively. Inspired by their work, we in this paper consider the Q-spectral versions of the above two problems. Moreover, we also provide a sufficient condition in terms of the Q-spectral radius for a graph to be t-tough with $\frac{1}{t}$ being a positive integer.

1. Introduction

All graphs considered in this paper are undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* and *size* of G are denoted by $|V(G)| = n$ and $|E(G)| = e(G)$, respectively. A graph with one vertex is referred to as a trivial graph. For a vertex $v \in V(G)$, let $N_G(v)$ and $d_G(v)$ denote the neighbors and degree of v in G, respectively. We denote by $\delta(G)$ the minimum degree (δ for short) of G. Let $c(G)$ be the number of components of a graph G. For a vertex subset S of G, we denote by $G-S$ and $G[S]$ the subgraph of G obtained from G by deleting the vertices in S together with their incident edges and the subgraph of G induced by S, respectively. For two vertex-disjoint graphs G_1 and G_2, we denote by $G_1 + G_2$ the *disjoint union* of G_1 and G_2. The *join* $G_1 \vee G_2$ is the graph obtained from $G_1 + G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. For undefined terms and notions, one can refer to [3].

For a graph G of order n, the *adjacency matrix* of G is the symmetric matrix $A(G) = (a_{ij})_{n\times n}$ indexed by the vertex set $V(G)$ of G, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$, denote by $\rho(G)$, is called the *spectral radius* of G. The *signless Laplacian matrix* of G is defined as $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. The largest eigenvalue $q(G)$ of $Q(G)$ is called the *Q-spectral radius* of G. By the Perron-Frobenius theorem, $q(G)$ is always positive (unless G is trivial). Furthermore, when G is connected, there exists a unique positive unit eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ corresponding to $q(G)$, which is called the *Perron vector* of $Q(G)$.

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In the past decades, the connections between the structural properties and the Q-spectral radius of graphs have been well studied. Pan, Li and Zhao[13] investigated the relations between the fractional matching number and the Q-spectral radius of a graph. Fan et al.[8] presented sufficient conditions in terms of the Q-spectral radius to guarantee the existence of a spanning k-tree and a perfect matching in graphs, respectively. Recently, Ao et al.[2] provided sufficient conditions for a graph to be k-leaf connected in terms of the Q-spectral radius of G or its complement. Ao, Liu and Yuan[1] presented tight Q-spectral conditions for the existence of a spanning k-ended-tree and a spanning tree with leaf degree at most k, respectively. Very recently, Hao and Li[12] provided lower bounds for the Q-spectral radius to ensure that a graph has a path-factor and is a path-factor covered graph, respectively. Zheng et al.[14] established a sufficient condition with given minimum degree based on the Q-spectral radius to guarantee that a graph is k-factor-critical.

Chvátal[5] introduced the concept of toughness in 1973. Let G be a non-complete graph and let t be a real number. If for every vertex cut set S of G, $|S| \geq tc(G-S)$, then we say that G is *t-tough*. The largest t such that G is t-tough is called the *toughness* of G and is denoted by $t(G)$. If $G \cong K_n$, $t(G)$ is defined as $n-1$. Note that $\delta \geq 2$ is a trivial necessary condition for a graph to be 1-tough. Very recently, Fan, Lin and Lu[9] presented a sufficient condition in terms of the spectral radius to ensure that a connected graph to be 1-tough for $\delta \geq 2$.

Theorem 1.1 (Fan, Lin and Lu [9]). *Suppose that G is a connected graph of order $n \geq \max\{5\delta, \frac{2}{5}\delta^2+\delta\}$ with minimum degree $\delta \geq 2$. If*

$$\rho(G) \geq \rho(K_\delta \vee (K_{n-2\delta} + \delta K_1)),$$

then G is 1-tough unless $G \cong K_\delta \vee (K_{n-2\delta} + \delta K_1)$.

Motivated by the above result, we consider the Q-version of Theorem 1.1.

Theorem 1.2. *Let G be a connected graph of order $n \geq \max\{\frac{7}{2}\delta+2, \frac{1}{4}\delta^2+2\delta\}$ with minimum degree $\delta \geq 2$. If*

$$q(G) \geq q(K_\delta \vee (K_{n-2\delta} + \delta K_1)),$$

then G is 1-tough unless $G \cong K_\delta \vee (K_{n-2\delta} + \delta K_1)$.

In the same paper, Fan, Lin and Lu[9] also provided a spectral condition for a connected graph to be t-tough, where t is a positive integer.

Theorem 1.3 (Fan, Lin and Lu [9]). *Let t be a positive integer. If G is a connected graph of order $n \geq 4t^2+6t+2$ with*

$$\rho(G) \geq \rho(K_{2t-1} \vee (K_{n-2t} + K_1)),$$

then G is t-tough unless $G \cong K_{2t-1} \vee (K_{n-2t} + K_1)$.

Inspired by their result, we consider the sufficient condition of t-tough graphs from the Q-spectral radius perspective with t or $\frac{1}{t}$ is a positive number, which generalizes the scope of t in Theorem 1.3.

Theorem 1.4. *Let G be a connected graph of order n. Each of the following holds.*
(i) Let t be a positive integer and $n \geq 4t^2+6t+1$. If

$$q(G) \geq q(K_{2t-1} \vee (K_{n-2t} + K_1)),$$

then G is t-tough unless $G \cong K_{2t-1} \vee (K_{n-2t} + K_1)$.

(ii) Let $\frac{1}{t}$ be a positive integer and $n \geq \frac{2}{t}+9$. If

$$q(G) \geq q(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)),$$

then G is t-tough unless $G \cong K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$.

2. Proof of Theorem 1.2

Before presenting our proof, we introduce some necessary lemmas. We first introduce the matrix $A_a(G) = aD(G) + A(G)$ and denote by $\rho_a(G)$ the largest eigenvalue of $A_a(G)$, where $a \geq 0$. It is obvious that $A_0(G) = A(G)$ and $A_1(G) = Q(G)$. Motivated by the technique of Lemma 3.1 in [8], one can obtain the following result.

Lemma 2.1. *Let $a \geq 0$ and $n = \sum_{i=1}^{t} n_i + s$. If $n_1 \geq n_2 \geq \cdots \geq n_t \geq p \geq 1$, then*

$$\rho_a(K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_t})) \leq \rho_a(K_s \vee (K_{n-s-p(t-1)} + (t-1)K_p))$$

with equality if and only if $(n_1, n_2, \ldots, n_t) = (n - s - p(t-1), p, \ldots, p)$.

Next we prove the following useful lemma.

Lemma 2.2. *For $n \geq 2\delta + 1$ and $\delta \geq 2$, we have*

$$q(K_1 \vee (K_{n-\delta-1} + K_\delta)) < q(K_\delta \vee (K_{n-2\delta} + \delta K_1)).$$

Proof. Let $\tilde{G} = K_1 \vee (K_{n-\delta-1} + K_\delta)$. We can partition the vertex set of $\tilde{G}$ as $V(\tilde{G}) = V(K_1) \cup V(K_{n-\delta-1}) \cup V(K_\delta)$. Let $V(K_1) = \{u_1\}, V(K_{n-\delta-1}) = \{v_1, v_2, \ldots, v_{n-\delta-1}\}$ and $V(K_\delta) = \{w_1, w_2, \ldots, w_\delta\}$. Let $\mathbf{x}$ be the Perron vector of $Q(\tilde{G})$ corresponding to $\rho(\tilde{G})$. By symmetry, $\mathbf{x}$ takes the same value on the vertices of $V(K_1), V(K_{n-\delta-1})$ and $V(K_\delta)$, respectively. We denote the entries of $\mathbf{x}$ by x_1, x_2 and x_3 corresponding to the vertices in the above three vertex subsets, respectively. By $q(\tilde{G})\mathbf{x} = Q(\tilde{G})\mathbf{x}$, we have

$$\begin{cases} q(\tilde{G})x_2 = x_1 + (2n - 2\delta - 3)x_2, \\ q(\tilde{G})x_3 = x_1 + (2\delta - 1)x_3, \end{cases}$$

which leads to

$$[q(\tilde{G}) - (2n - 2\delta - 3)]x_2 = [q(\tilde{G}) - (2\delta - 1)]x_3.$$

Note that $K_{n-\delta}$ is a proper subgraph of $\tilde{G}$. Then $q(\tilde{G}) > q(K_{n-\delta}) = 2(n - \delta - 1)$. Combining this with $n \geq 2\delta + 1$, we have

$$x_2 = \frac{q(\tilde{G}) - (2\delta - 1)}{q(\tilde{G}) - (2n - 2\delta - 3)} x_3 = \left[1 + \frac{2n - 4\delta - 2}{q(\tilde{G}) - (2n - 2\delta - 3)}\right] x_3 \geq x_3.$$

Let $G^* = K_\delta \vee (K_{n-2\delta} + \delta K_1)$. Define $E_1 = \{v_i w_j | 1 \leq i \leq \delta - 1, 1 \leq j \leq \delta\}$ and $E_2 = \{w_i w_j | 1 \leq i \leq \delta - 1, i + 1 \leq j \leq \delta\}$. One can check that $G^* \cong \tilde{G} + E_1 - E_2$. Hence

$$\begin{aligned} q(G^*) - q(\tilde{G}) &\geq \mathbf{x}^T (Q(G^*) - Q(\tilde{G}))\mathbf{x} \\ &= \sum_{i=1}^{\delta-1} \sum_{j=1}^{\delta} (x_i + x_j)^2 - \sum_{i=1}^{\delta-1} \sum_{j=i+1}^{\delta} (x_i + x_j)^2 \\ &= \delta(\delta - 1)(x_2 + x_3)^2 - \frac{1}{2}\delta(\delta - 1)(x_3 + x_3)^2 \\ &= \delta(\delta - 1)[(x_2 + x_3)^2 - 2x_3^2] \\ &> 0, \end{aligned}$$

where the last inequality follows from $\delta \geq 2$ and $x_2 \geq x_3$. Hence $q(\tilde{G}) < q(G^*)$. □

Finally, we introduce the concepts of quotient matrices and equitable partitions. Let M be a real $n \times n$ matrix. Assume that M can be written as the following matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The quotient matrix $R(M)$ of the matrix M (with respect to the given partition) is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The above partition is called *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.3 (Brouwer and Haemers [4], Godsil and Royle [10], Haemers [11])**.** *Let M be a real symmetric matrix and let $R(M)$ be its equitable quotient matrix. Then the eigenvalues of the quotient matrix $R(M)$ are eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then the spectral radius of the quotient matrix $R(M)$ equals to the spectral radius of M.*

Now, we are in a position to present the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose to the contrary that G is not a 1-tough graph. By the definition of 1-tough graphs, then there exists a vertex subset $S \subseteq V(G)$ such that $c(G - S) > |S|$. Let $|S| = s$ and $c(G - S) = c$. Then $c \geq s + 1$, and hence $n \geq 2s + 1$. Note that G is a spanning subgraph of $G_1 = K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_{s+1}})$ for some integers $n_1 \geq n_2 \geq \cdots \geq n_{s+1} \geq 1$ and $\sum_{i=1}^{s+1} n_i = n - s$. Then we have

$$q(G) \leq q(G_1), \tag{1}$$

where equality holds if and only if $G \cong G_1$. Note that S is a vertex cut set. Then $s \geq 1$, and hence $1 \leq s \leq \frac{n-1}{2}$. Next we divide the proof into the following three cases according to the different values of s.

Case 1. $1 \leq s \leq \delta - 1$.

Note that $\delta(G_1) \geq \delta(G) = \delta$. Then $n_1 \geq n_2 \geq \cdots \geq n_{s+1} \geq \delta - s + 1$. Let $G_2 = K_s \vee (K_{n-s-(\delta-s+1)s} + sK_{\delta-s+1})$. By Lemma 2.1, we have

$$q(G_1) \leq q(G_2), \tag{2}$$

with equality holding if and only if $(n_1, n_2, \ldots, n_{s+1}) = (n - s - (\delta - s + 1)s, \delta - s + 1, \ldots, \delta - s + 1)$. If $s = 1$, then $G_2 = K_1 \vee (K_{n-\delta-1} + K_\delta)$. By Lemma 2.2, we have

$$q(G_2) < q(K_\delta \vee (K_{n-2\delta} + \delta K_1)). \tag{3}$$

Next we consider $s \geq 2$. Let $R(Q(G_2))$ be the quotient matrix of $Q(G_2)$ with respect to the partition $(V(K_s), V(K_{n-s-(\delta-s+1)s}), V(sK_{\delta-s+1}))$. One can see that

$$R(Q(G_2)) = \begin{pmatrix} n+s-2 & n+s^2-\delta s-2s & \delta s-s^2+s \\ s & 2n+2s^2-2\delta s-3s-2 & 0 \\ s & 0 & 2\delta-s \end{pmatrix}.$$

By simple calculation, the characteristic polynomial of $R(Q(G_2))$ is

$$\begin{aligned} P(R(Q(G_2)), x) \;=\; & x^3 - (3n + 2s^2 - 2\delta s - 3s + 2\delta - 4)x^2 + [2n^2 + (2s^2 - 2\delta s - 5s + 6\delta \\ & -6)n + 4\delta s^2 - 4s^2 - 4\delta^2 s + 8s - 8\delta + 4]x + (2s - 4\delta)n^2 - (4\delta s^2 \\ & -4\delta^2 s - 4\delta s + 6s - 12\delta)n - 2s^5 + (4\delta + 6)s^4 - (2\delta^2 + 8\delta + 6)s^3 \\ & +(2\delta^2 + 12\delta + 2)s^2 - (8\delta^2 + 8\delta - 4)s - 8\delta. \end{aligned} \tag{4}$$

Note that the above partition is equitable. By Lemma 2.3, we know that $q(G_2) = \lambda_1(R(Q(G_2)))$ is the largest root of the equation $P(R(Q(G_2)), x) = 0$. Define that $G^* = K_\delta \vee (K_{n-2\delta} + \delta K_1)$. Note that $Q(G^*)$ has the equitable quotient matrix with respect to the partition $(V(K_\delta), V(K_{n-2\delta}), V(\delta K_1))$

$$R(Q(G^*)) = \begin{pmatrix} n+\delta-2 & n-2\delta & \delta \\ \delta & 2n-3\delta-2 & 0 \\ \delta & 0 & \delta \end{pmatrix}.$$

Then the characteristic polynomial of $R(Q(G^*))$ is

$$\begin{aligned} P(R(Q(G^*)),x) &= x^3-(3n-\delta-4)x^2+(2n^2+\delta n-6n-4\delta^2+4)x-2\delta n^2+4\delta^2 n+6\delta n \\ &\quad -2\delta^3-6\delta^2-4\delta. \end{aligned} \tag{5}$$

By Lemma 2.3, $q(G^*)=\lambda_1(R(Q(G^*)))$ is the largest root of the equation $P(R(Q(G^*)),x)=0$. Note that G^* contains $K_{n-\delta}$ as a proper subgraph. Then $q(G^*)>q(K_{n-\delta})=2(n-\delta-1)$. Combining (4) and (5), we have

$$\begin{aligned} & P(R(Q(G_2)),2(n-\delta-1))-P(R(Q(G^*)),2(n-\delta-1)) \\ = \; & 2(\delta-s)[2(s-1)n^2-(8\delta s+2s-9\delta-2)n+s^4-(\delta+3)s^3+(\delta+3)s^2 \\ & +(8\delta^2+5\delta-1)s-9\delta^2-5\delta] \\ \triangleq \; & 2(\delta-s)f(n). \end{aligned}$$

Note that $s\geq 2$. Then the symmetry axis of $f(n)$ is

$$n=\frac{8\delta s+2s-9\delta-2}{4(s-1)}=2\delta+\frac{1}{2}-\frac{\delta}{4(s-1)}<2\delta+\frac{1}{2}<\frac{1}{4}\delta^2+2\delta,$$

where the last two inequalities follow from the fact that $\delta\geq s+1\geq 3$. This implies that $f(n)$ is monotonically increasing with respect to $n\in[\frac{1}{4}\delta^2+2\delta,+\infty)$. Since $s\geq 2$ and $\delta\geq s+1\geq 3$, we have

$$\begin{aligned} f(n) &\geq f(\frac{1}{4}\delta^2+2\delta) \\ &= \frac{\delta}{8}[\delta((s-1)\delta^2+2\delta-4s+12)-8s^3+8s^2+8s-8]+s(s-1)^3 \\ &\geq \frac{\delta}{8}[\delta(s^3+s^2-3s+13)-8s^3+8s^2+8s-8]+s(s-1)^3 \\ &= \frac{\delta}{8}(s^4-6s^3+6s^2+18s+5)+s(s-1)^3 \\ &> 0. \end{aligned}$$

Combining this with $\delta\geq s+1$, we obtain that

$$P(R(Q(G_2)),2(n-\delta-1))>P(R(Q(G^*)),2(n-\delta-1)). \tag{6}$$

Next we take derivatives of $P(R(Q(G_2)),x)$ and $P(R(Q(G^*)),x)$, respectively. Note that $\delta\geq s+1\geq 3, s\geq 2$ and $n\geq\frac{1}{4}\delta^2+2\delta$. For $x\geq 2(n-\delta-1)$, we have

$$\begin{aligned} P'(R(Q(G_2)),x)-P'(R(Q(G^*)),x) &= (\delta-s)[(4s-6)x-4\delta s-2ns+4\delta+5n+4s-8] \\ &\geq (\delta-s)[(6s-7)n-12\delta s+16\delta-4s+4] \\ &\geq (\delta-s)\left[\left(\frac{3}{2}\delta^2-4\right)s-\frac{7}{4}\delta^2+2\delta+4\right] \\ &\geq (\delta-s)\left(\frac{5}{4}\delta^2+2\delta-4\right) \\ &> 0. \end{aligned}$$

Hence we have $P'(R(Q(G_2)),x)>P'(R(Q(G^*)),x)$ for $x\geq 2(n-\delta-1)$. Combining this with (6), we deduce that

$$q(G_2)<q(G^*). \tag{7}$$

By (1), (2), (3) and (7), we have

$$q(G)\leq q(G_1)\leq q(G_2)<q(G^*),$$

which contradicts the assumption.

Case 2. $s = \delta$.

Recall that $G_1 = K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_{s+1}})$. At this case, $G_1 = K_\delta \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_{\delta+1}})$. By Lemma 2.1, we obtain that

$$q(G_1) \leq q(K_\delta \vee (K_{n-2\delta} + \delta K_1)),$$

with equality holding if and only if $G_1 \cong K_\delta \vee (K_{n-2\delta} + \delta K_1)$. Combining this with (1), we have

$$q(G) \leq q(K_\delta \vee (K_{n-2\delta} + \delta K_1)),$$

where equality holds if and only $G \cong K_\delta \vee (K_{n-2\delta} + \delta K_1)$. By the assumption $q(G) \geq q(K_\delta \vee (K_{n-2\delta} + \delta K_1))$, we have $q(G) = q(K_\delta \vee (K_{n-2\delta} + \delta K_1))$, and hence $G \cong K_\delta \vee (K_{n-2\delta} + \delta K_1)$ (see Fig. 1). Take $S = V(K_\delta)$. Then

$$\frac{|S|}{c(K_\delta \vee (K_{n-2\delta} + \delta K_1) - S)} = \frac{\delta}{\delta + 1} < 1,$$

which implies that $K_\delta \vee (K_{n-2\delta} + \delta K_1)$ is not 1-tough. So $G \cong K_\delta \vee (K_{n-2\delta} + \delta K_1)$.

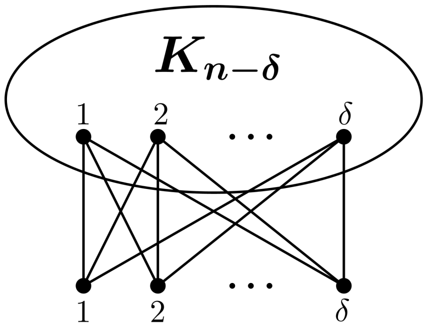

Figure 1: Graph $K_\delta \vee (K_{n-2\delta} + \delta K_1)$.

Case 3. $\delta + 1 \leq s \leq \frac{n-1}{2}$.

Let $G_3 = K_s \vee (K_{n-2s} + sK_1)$. By Lemma 2.1, we have

$$q(G_1) \leq q(G_3), \tag{8}$$

with equality holding if and only if $G_1 \cong G_3$. Recall that $G^* = K_\delta \vee (K_{n-2\delta} + \delta K_1)$. Next we prove that $q(G_3) < q(G^*)$. Observe that the vertex set of G_3 can be divided into $V(G_3) = V(\delta K_1) \cup V(K_{n-2s}) \cup V(K_s)$. Then the quotient matrix of $Q(G_3)$ with respect to this partition is

$$R(Q(G_3)) = \begin{pmatrix} n+s-2 & n-2s & s \\ s & 2n-3s-2 & 0 \\ s & 0 & s \end{pmatrix}.$$

By simple calculation, the characteristic polynomial of $R(Q(G_3))$ is

$$\begin{aligned} P(R(Q(G_3)), x) \quad = \quad & x^3 - (3n - s - 4)x^2 + (2n^2 + sn - 6n - 4s^2 + 4)x - 2sn^2 + 4s^2 n + 6sn \\ & -2s^3 - 6s^2 - 4s. \end{aligned} \tag{9}$$

Note that the above partition is equitable. By Lemma 2.3, $q(G_3) = \lambda_1(R(Q(G_3)))$ is the largest root of the equation $P(R(Q(G_3)), x) = 0$. By (5) and (9), we have

$$\begin{aligned} & P(R(Q(G_3)), x) - P(R(Q(G^*)), x) \\ = \quad & (s - \delta)[x^2 + (n - 4s - 4\delta)x - 2n^2 + (4s + 4\delta + 6)n - 2s^2 - 2\delta s - 6s - 2\delta^2 - 6\delta - 4] \\ \triangleq \quad & (s - \delta)g(x). \end{aligned}$$

Note that $n \geq 2s+1, n \geq \frac{7}{2}\delta+2$ and $\delta \geq 2$. The symmetry axis of $g(x)$ is

$$\begin{aligned} x &= -\frac{1}{2}n+2s+2\delta \\ &= 2(n-\delta-1)-\left(\frac{5}{2}n-2s-4\delta-2\right) \\ &= 2(n-\delta-1)-\left[(n-2s)+\left(\frac{3}{2}n-4\delta-2\right)\right] \\ &\leq 2(n-\delta-1)-\left(\frac{5}{4}\delta+2\right) \\ &< 2(n-\delta-1). \end{aligned}$$

This implies that $g(x)$ is monotonically increasing with respect to $x \in [2(n-\delta-1), +\infty)$. Since $\delta+1 \leq s \leq \frac{n-1}{2}$, $n \geq \frac{7}{2}\delta+2$ and $\delta \geq 2$, we obtain that

$$\begin{aligned} g(x) &\geq g(2(n-\delta-1)) \\ &= -2s^2-(4n-6\delta-2)s+4n^2-14\delta n-4n+10\delta^2+10\delta \\ &\geq -2(\delta+1)^2-(4n-6\delta-2)(\delta+1)+4n^2-14\delta n-4n+10\delta^2+10\delta \\ &= 4n^2-(18\delta+8)n+14\delta^2+14\delta \\ &\geq 4\left(\frac{7}{2}\delta+2\right)^2-(18\delta+8)\left(\frac{7}{2}\delta+2\right)+14\delta^2+14\delta \\ &= 6\delta > 0. \end{aligned}$$

Since $s \geq \delta+1$, we have $P(R(Q(G_3)), x) > P(R(Q(G^*)), x)$ for $x \geq 2(n-\delta-1)$. Note that G^* contains $K_{n-\delta}$ as a proper subgraph. Hence $q(G^*) > q(K_{n-\delta}) = 2(n-\delta-1)$, and so $q(G_3) < q(G^*)$. Combining this with (1) and (8), we have

$$q(G) \leq q(G_1) \leq q(G_3) < q(G^*),$$

a contradiction. □

3. Proof of Theorem 1.4

In order to prove Theorem 1.4, we present the following lemma.

Lemma 3.1 (Das[6]). *Let G be a graph with n vertices and $e(G)$ edges. Then*

$$q(G) \leq \frac{2e(G)}{n-1}+n-2.$$

Now we are ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume to the contrary that G is not a t-tough graph. By the definition of t-tough graphs, there exists a vertex subset $S \subseteq V(G)$ such that $tc(G-S) > |S|$. Let $|S| = s$ and $c(G-S) = c$. Then $tc > s$.

(i) When t is a positive integer, we have $tc \geq s+1$. Note that G is a spanning subgraph of $G' = K_{tc-1} \vee (K_{n_1}+K_{n_2}+\cdots+K_{n_c})$, where $n_1 \geq n_2 \geq \cdots \geq n_c \geq 1$ and $\sum_{i=1}^{c} n_i = n-tc+1$. Hence we have

$$q(G) \leq q(G'), \tag{10}$$

where equality holds if and only if $G \cong G'$. Let $G'' = K_{tc-1} \vee (K_{n-(t+1)c+2}+(c-1)K_1)$. By Lemma 2.1, we have

$$q(G') \leq q(G''), \tag{11}$$

with equality holding if and only if $G' \cong G''$. Note that G is a connected graph and S is a vertex cut set. This implies that $c \geq 2$. Next we divide the proof into two cases according to different values of $c \geq 2$.

Case 1. $c = 2$.

Then $G'' = K_{2t-1} \vee (K_{n-2t} + K_1)$. By (10) and (11), we deduce that

$$q(G) \leq q(K_{2t-1} \vee (K_{n-2t} + K_1)),$$

where equality holds if and only if $G \cong K_{2t-1} \vee (K_{n-2t} + K_1)$. By the assumption $q(G) \geq q(K_{2t-1} \vee (K_{n-2t} + K_1))$, we have $q(G) = q(K_{2t-1} \vee (K_{n-2t} + K_1))$, and hence $G \cong K_{2t-1} \vee (K_{n-2t} + K_1)$ (see Fig. 2). Take $S = V(K_{2t-1})$. Then

$$\frac{|S|}{c(K_{2t-1} \vee (K_{n-2t} + K_1) - S)} = \frac{2t-1}{2} < t,$$

which implies that $K_{2t-1} \vee (K_{n-2t} + K_1)$ is not t-tough. So $G \cong K_{2t-1} \vee (K_{n-2t} + K_1)$.

Case 2. $c \geq 3$.

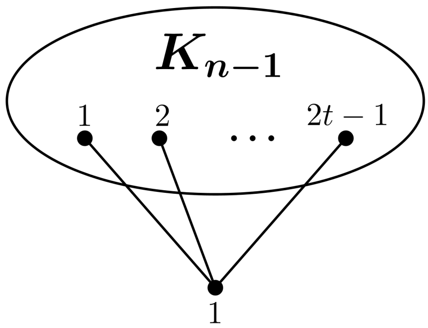

Figure 2: Graph $K_{2t-1} \vee (K_{n-2t} + K_1)$.

Recall that $G'' = K_{tc-1} \vee (K_{n-(t+1)c+2} + (c-1)K_1)$. It follows that

$$e(G'') = \left(t + \frac{1}{2}\right)c^2 - \left(n + t + \frac{3}{2}\right)c + \frac{1}{2}n^2 + \frac{1}{2}n + 1.$$

By Lemma 3.1, we have

$$q(G'') \leq \frac{2e(G'')}{n-1} + n - 2 = \frac{(2t+1)c^2 - (2n+2t+3)c + 2n^2 - 2n + 4}{n-1}. \tag{12}$$

Define $\varphi(c) = (2t+1)c^2 - (2n+2t+3)c + 2n^2 - 2n + 4$. Note that $n_1 \geq n_2 \geq \cdots \geq n_c \geq 1$. Hence $n \geq (t+1)c - 1$. Note that $3 \leq c \leq \frac{n+1}{t+1}$. According to $n \geq 4t^2 + 6t + 1$, by simple calculation, we obtain that

$$\begin{aligned} \varphi(3) - \varphi\left(\frac{n+1}{t+1}\right) &= \frac{n^2 - (4t^2 + 9t + 3)n + 12t^3 + 26t^2 + 15t + 2}{(t+1)^2} \\ &= \frac{(n - 4t^2 - 6t - 1)(n - 3t - 2)}{(t+1)^2} \\ &\geq 0. \end{aligned}$$

This implies that the maximum value of $\varphi(c)$ for $3 \leq c \leq \frac{n+1}{t+1}$ is attained at $c = 3$. Combining this with (12), we deduce that

$$q(G'') \leq \frac{\varphi(3)}{n-1} = \frac{2n^2 - 8n + 12t + 4}{n-1} = 2(n-2) - \frac{2n - 12t}{n-1} < 2(n-2).$$

Observe that K_{n-1} is a proper subgraph of $K_{2t-1} \vee (K_{n-2t} + K_1)$. Hence $q(K_{2t-1} \vee (K_{n-2t} + K_1)) > q(K_{n-1}) = 2(n-2)$. Therefore, we have $q(G'') < 2(n-2) < q(K_{2t-1} \vee (K_{n-2t} + K_1))$. Combining this with (10) and (11), we have

$$q(G) \leq q(G') \leq q(G'') < q(K_{2t-1} \vee (K_{n-2t} + K_1)).$$

a contradiction.

(ii) When $1/t$ is a positive integer, we have $c \geq \frac{s}{t} + 1$. It is obvious that G is a spanning subgraph of $G^{(1)} = K_s \vee (K_{n_1} + K_{n_2} + \cdots + K_{n_{\frac{s}{t}+1}})$ for $n_1 \geq n_2 \geq \cdots \geq n_{\frac{s}{t}+1} \geq 1$ and $\sum_{i=1}^{\frac{s}{t}+1} n_i = n - s$. Then we have

$$q(G) \leq q(G^{(1)}), \tag{13}$$

with equality holding if and only if $G \cong G^{(1)}$. Let $G^{(2)} = K_s \vee (K_{n-s-\frac{s}{t}} + \frac{s}{t}K_1)$. By Lemma 2.1, we have

$$q(G^{(1)}) \leq q(G^{(2)}), \tag{14}$$

where equality holds if and only if $G^{(1)} \cong G^{(2)}$. Since S is a vertex cut set, $s \geq 1$. Next we consider the following two cases depending on the different values of $s \geq 1$.

Case 1. $s = 1$.

Then $G^{(2)} = K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$. By (13) and (14), we conclude that

$$q(G) \leq q(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)),$$

with equality holding if and only if $G \cong K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$. By the assumption $q(G) \geq q(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1))$, we have $q(G) = q(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1))$, and hence $G \cong K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$ (see Fig. 3). Take $S = V(K_1)$. Then

$$\frac{|S|}{c(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1) - S)} = \frac{1}{1 + \frac{1}{t}} < t,$$

which implies that $K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$ is not t-tough. So $G \cong K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$.

Case 2. $s \geq 2$.

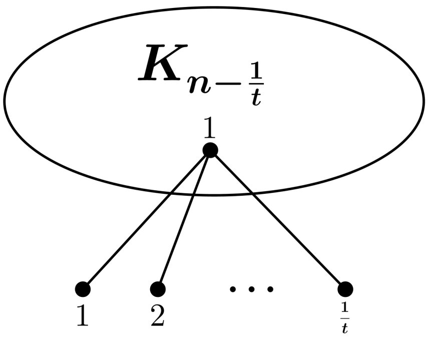

Figure 3: Graph $K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$.

Recall that $G^{(2)} = K_s \vee (K_{n-s-\frac{s}{t}} + \frac{s}{t}K_1)$. For convenience, we take $r = \frac{1}{t}$. It is easy to see that $r \geq 1$. By Lemma 3.1, we have

$$\begin{aligned} q(G^{(2)}) &\leq \frac{2e(G^{(2)})}{n-1} + n - 2 \\ &= \frac{(r^2 + 2r)s^2 - (2rn - r)s + 2n^2 - 4n + 2}{n-1} \\ &\triangleq \frac{\psi(s)}{n-1}. \end{aligned}$$

Note that $2 \le s \le \frac{n-1}{r+1}$. Since $n \ge \frac{2}{t} + 9 = 2r + 9$ and $r \ge 1$, we have

$$\psi(2) - \psi(\frac{n-1}{r+1}) = \frac{r(n-2r-3)(rn-2r^2-6r-3)}{(r+1)^2} \ge 0.$$

This implies that $\max_{2\le s\le \frac{n-1}{r+1}} \psi(s) = \psi(2)$. Hence

$$\begin{aligned} q(G^{(2)}) &\le \frac{\psi(2)}{n-1} \\ &= \frac{2n^2-(4r+4)n+4r^2+10r+2}{n-1} \\ &= 2(n-r-1) - \frac{2r(n-2r-4)}{n-1} \\ &< 2(n-r-1). \end{aligned}$$

Hence we have $q(G^{(2)}) < 2(n - \frac{1}{t} - 1)$. Since $K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)$ contains $K_{n-\frac{1}{t}}$ as a proper subgraph, $q(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1)) > 2(n - \frac{1}{t} - 1)$. It follows that $q(G^{(2)}) < 2(n - \frac{1}{t} - 1) < q(K_1 \vee (K_{n-\frac{1}{t}-1} + \frac{1}{t}K_1))$. Combining this with (13) and (14), we deduce that

$$q(G) \le q(G^{(1)}) \le q(G^{(2)}) < q(K_1 \vee (K_{n-1-\frac{1}{t}} + \frac{1}{t}K_1)),$$

which contradicts the assumption. □

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