Filomat 38:30 (2024), 10675–10686
https://doi.org/10.2298/FIL2430675F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The minimum incidence energy of k-uniform hypergraphs

Yanjie Fu[a], Yubin Gao[a,*], Zhihua Li[a]

[a] *School of Mathematics, North University of China, Taiyuan, Shanxi 030051, China*

Abstract. For a k-uniform hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ of order $n = |V(\mathcal{H})|$ and size $r = |E(\mathcal{H})|$, let $B(\mathcal{H})$ be the incidence matrix of $\mathcal{H}$. The incidence energy $BE(\mathcal{H})$ of $\mathcal{H}$ is the energy of $B(\mathcal{H})$. In this article, we determine the unique hypergraph with the minimum incidence energy among all k-uniform hypertrees of size r with fixed number of pendent edges. We also determine the unique hypergraph with the minimum incidence energy among all k-uniform unicyclic hypergraphs of size r.

1. Introduction

Spectral graph theory has a long history behind its development. In spectral graph theory, we analyse the eigenvalues of a connectivity matrix which is uniquely defined on a graph. Many researchers have had a great interest to study the eigenvalues of different connectivity matrices, such as, adjacency matrix, Laplacian matrix, etc. Now, a recent trend has been developed to explore spectral hypergraph theory. In 2005, L. Qi [11] introduced the concept of eigenvalues of a real supersymmetric tensor. Then spectral theory for tensors started to develop. Afterward, many researchers analyzed different eigenvalues of several connectivity tensors, such as, adjacency tensor, Laplacian tensor, normalized Laplacian tensors, etc. It is known, however, that to obtain eigenvalues of tensors has a high computational and theoretical cost. Perhaps for this reason, recently,some authors have renewed the interest to study the matrix representations of a hypergraph, as for example in [1–5, 9, 12, 13].

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph with vertex set $V(\mathcal{H})$ and hyperedge set $E(\mathcal{H})$, where $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$ and $2^{V(\mathcal{H})}$ stands for the power set of $V(\mathcal{H})$. A hypergraph $\mathcal{H}$ is said to k-uniform hypergraph (or a k-graph) if $|e| = k$ for every $e \in E(\mathcal{H})$. Especially, 2-uniform hypergraph is the ordinary graph. For convenience, let $[n] = \{1, 2, \ldots, n\}$. Let $E(v) = \left\{e \,\middle|\, v \in e \in E(\mathcal{H})\right\}$, and $d(v) = |E(v)|$ be the degree of v. A edge e in a hypergraph is said to be a pendent edge at a vertex $v \in e$ of degree greater than or equal two, if the other vertices in e are 1-degree vertices. For $u, v \in V(\mathcal{H})$, a walk from u to v in $\mathcal{H}$ is defined to be a sequence of vertices and edges $v_0e_1v_1e_2 \cdots e_pv_p$ with $v_0 = u$ and $v_p = v$ such that edge e_i contains vertices v_{i-1} and v_i, and $v_{i-1} \neq v_i$ for $i \in [p]$. The value p is the length of this walk. A path $\mathcal{P} = v_0e_1v_1e_2 \cdots e_pv_p$ is a walk with all v_i distinct and all e_i distinct. If $d(v_i) = 2$ for $i \in [p-1]$, and the other vertices in $V(\mathcal{P})$ are 1-degree vertices, then $\mathcal{P}$ is a loose path. If $d(v_0) \geq 3$, $d(v_i) = 2$ for $i \in [p-1]$ and the others vertices in $V(\mathcal{P})$ are 1-degree

2020 *Mathematics Subject Classification*. Primary 05C50; Secondary 15A18.
Keywords. Uniform hypergraph; Uniform hypertree; Unicyclic hypergraph; Incidence energy.
Received: 08 May 2024; Accepted: 08 July 2024
Communicated by Paola Bonacini
Research supported by Natural Science Foundation of Shanxi Province of China (No. 202303021211154).
* Corresponding author: Yubin Gao
Email addresses: yjfu27@163.com (Yanjie Fu), ybgao@nuc.edu.cn (Yubin Gao), lizhihua@nuc.edu.cn (Zhihua Li)

vertices, then $\mathcal{P}$ is a pendent path. A cycle $C = v_0e_1v_1e_2\cdots e_pv_p$ is a walk containing at least two edges, all e_i are distinct and all v_i are distinct except $v_0 = v_p$.

If there is a path from u to v for any $u, v \in V(\mathcal{H})$, then we say that $\mathcal{H}$ is connected. A hypertree is a connected hypergraph with no cycles. A unicyclic hypergraph is a connected hypergraph with exactly one cycle. For $m \geq 2$ and $k \geq 3$, let $C_{m,k} = v_1e_1v_2e_2v_3\cdots v_me_mv_{m+1}$ be a k-uniform cycle of length m, where $v_{m+1} = v_1$, for $i \in [m]$, $e_i = \{v_i, u_{i1}, u_{i2}, \ldots, u_{ik-2}, v_{i+1}\}$, $V_0 = \{v_i : i \in [m]\}$. Let $\mathcal{U}_r^m$ be the k-uniform unicyclic hypergraph of size r obtained from $C_{m,k}$ by attaching $r - m$ pendent edges to v_1 in $C_{m,k}$. For a hypergraph $\mathcal{H}$, the subdivision graph $S(\mathcal{H})$ is a graph obtained by adding a new vertex v_e and making it adjacent to all vertices of e for each edge e of $\mathcal{H}$.

For a matrix M, its energy $E(M)$ is defined as the sum of its singular values ([10]). Let $\mathcal{H}$ be a k-uniform hypergraph and $B(\mathcal{H}) = (b(v,e))_{|V(\mathcal{H})|\times|E(\mathcal{H})|}$ be the incidence matrix of $\mathcal{H}$, where $b(v,e) = 1$ if $v \in e$, and $b(v,e) = 0$ otherwise. Cardoso and Trevisan [3] defined the energy of $B(\mathcal{H})$ as the incidence energy $BE(\mathcal{H})$ of $\mathcal{H}$, and proposed the relation

$$BE(\mathcal{H}) = \frac{1}{2}E(A_s), \tag{1}$$

where A_s is the adjacency matrix of $S(\mathcal{H})$.

On this basis, the authors of [13] obtained the lower and upper bounds on $BE(\mathcal{H})$ for k-uniform hypertrees and characterized their corresponding extremal hypergraphs. The authors of [5] characterized the k-uniform hypertrees with the minimum incidence energy among all k-uniform hypertrees of order n with diameter $3 \leq d \leq r - 1$. Motivated by the above research, in this article, we determine the unique hypergraph with the minimum incidence energy among all k-uniform hypertrees of size r with fixed number of pendent edges. We also determine the unique hypergraph with the minimum incidence energy among all k-uniform unicyclic hypergraphs of size r.

2. Preliminaries

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G, denoted by $A(G)$, is an $n\times n$ matrix (a_{ij}) in which $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $A(G)$, denoted by $\phi_A(G, \lambda) = |\lambda I - A(G)|$, is called the characteristic polynomial of G. The n roots of the equation $\phi_A(G, \lambda) = 0$, denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are called the eigenvalues of G. The energy $E(G)$ of G is defined [6] as

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

If G is a bipartite graph, then its characteristic polynomial can be written as

$$\phi(G) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i b_{2i} x^{n-2i},$$

where $b_0 = 1$ and $b_{2i} \geq 0$. Let $a_{2i}(G) = (-1)^i b_{2i}(G)$. The Sachs theorem [7] states that for $i \geq 1$,

$$a_{2i} = \sum_{S \in L_{2i}} (-1)^{\omega(S)} 2^{c(S)}, \tag{2}$$

where L_{2i} denotes the set of Sachs subgraphs of G with $2i$ vertices, that is, the subgraphs in which every component is either K_2 or a cycle; $\omega(S)$ is the number of connected components of S, and $c(S)$ is the number of cycles contained in S.

In particularly, if G is a tree, then $b_{2i} = m(G, i)$ for all $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, where $m(G, i)$ equals to the number of i-matchings of G (see [7]). For two bipartite graphs G_1 and G_2 of order n, we define $G_1 \preceq G_2$ if and only if

$b_{2i}(G_1) \leq b_{2i}(G_2)$ for all $i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, if there exists a index i such that $b_{2i}(G_1) < b_{2i}(G_2)$, we write $G_1 \prec G_2$. The following result was proven (see [7]).

$$G_1 \preceq G_2 \Rightarrow E(G_1) \leq E(G_2),$$
$$G_1 \prec G_2 \Rightarrow E(G_1) < E(G_2).$$

A tree is said to be starlike if it has exactly one vertex of degree greater than 2. Let $S_{r,x}$ denote the starlike tree with r edges and x pendent paths of almost equal length and $S^a_{r,x-1}$ be the graph obtained from $S_{r-a,x-1}$ by joining a pendent vertex of path P_a to the center of $S_{r-a,x-1}$. Let $P_{n,k}$ be the tree obtained from $P_n = v_0v_1 \cdots v_{n-1}$ by adding $k-2$ pendent edges to each vertex v_{2j} with $j = 0, 1, 2, \ldots, \lfloor \frac{n-2}{2} \rfloor$, and $\tilde{P}_{n,k}$ be the tree obtained from $P_n = v_0v_1 \cdots v_{n-1}$ by adding $k-2$ pendent edges to each vertex v_{2j+1} with $j = 0, 1, 2, \ldots, \lfloor \frac{n-2}{2} \rfloor$.

The definition of a power graph was introduced in [2] as follows:

Definition 2.1 ([2]). *Let $G = (V, E)$ be a graph and let $k \geq 2$ be an integer. We define the power graph $\mathcal{G}^k$ as the k-graph with the following sets of vertices and edges*

$$V\left(\mathcal{G}^k\right) = V(G) \cup \left(\bigcup_{e \in E(G)} \varsigma_e \right) \text{and } E\left(\mathcal{G}^k\right) = \{e \cup \varsigma_e : e \in E(G)\},$$

where $\varsigma_e = \left\{v_1^e, \ldots, v_{k-2}^e\right\}$ is a set of additional vertices of degree one for each edge $e \in E(G)$.

Let $(\mathcal{S}_{r,x})^k$ denote the power graph of $S_{r,x}$ and $(\mathcal{S}^a_{r,x-1})^k$ denote the power graph of $S^a_{r,x-1}$.

In order to obtain our main results we need the following lemmas.

Lemma 2.2 ([7]). *Let $e = uv$ be an edge of a tree T with n vertices. Then*

$$m(T, i) = m(T - uv, i) + m(T - u - v, i - 1)$$

for $i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, where $m(T, 0) = 1$. Moreover, if u is a pendent vertex, then

$$m(T, i) = m(T - u, i) + m(T - u - v, i - 1).$$

Lemma 2.3 ([8]). *Let uv be an edge of a bipartite graph G, then*

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v) + 2 \sum_{C_l \in C(uv)} (-1)^{1+\frac{l}{2}} b_{2i-l}(G - C_l),$$

where $C(uv)$ is the set of cycles containing uv. In particular, if uv is a pendent edge of G with the pendent vertex v, then

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v).$$

3. The minimum incidence energy of k-uniform hypertrees with fixed number of pendent edges

Lemma 3.1. *Let $n = 4t$ or $n = 4t + 2$. Then for $j = 2, 3, \ldots, t$ and $1 \leq i \leq \frac{n}{2}$,*

$$m(P_{2j-2,k} \cup P_{n-2j+2,k}, i) \geq m(P_{2j,k} \cup P_{n-2j,k}, i).$$

Proof. We first consider $j = 2$. It is obvious to see that $P_{2,k} \cup P_{n-2,k} \cong P_{n,k} - v_1v_2$, $P_{4,k} \cup P_{n-4,k} \cong P_{n,k} - v_3v_4$. By Lemma 2.2 we have

$$m\left(P_{2,k} \cup P_{n-2,k}, i\right) = m\left(P_{n,k} - v_1v_2 - v_3v_4, i\right) + m\left(P_{n,k} - v_1v_2 - v_3 - v_4, i - 1\right),$$

and

$$m\left(P_{4,k} \cup P^k_{n-4}, i\right) = m\left(P_{n,k} - v_3v_4 - v_1v_2, i\right) + m\left(P_{n,k} - v_3v_4 - v_1 - v_2, i-1\right).$$

So

$$m\left(P_{2,k} \cup P_{n-2,k}, i\right) - m\left(P_{4,k} \cup P_{n-4,k}, i\right) = m\left(P_{n,k} - v_1v_2 - v_3 - v_4, i-1\right) - m\left(P_{n,k} - v_3v_4 - v_1 - v_2, i-1\right).$$

Note that for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $m\left(P_{2,k} \cup P_{n-2,k}, i\right) \ge m\left(P_{4,k} \cup P_{n-4,k}, i\right)$, and there exists $i = 3$ such that $m\left(P_{2,k} \cup P_{n-2,k}, i\right) > m\left(P_{4,k} \cup P_{n-4,k}, i\right)$.

For $j = 3, \ldots, t$, it is obvious to see that $P^k_{2j-2} \cup P^k_{n-2j+2} \cong P_{n,k} - v_{2j-3}v_{2j-2}$, and $P^k_{2j} \cup P^k_{n-2j} \cong P_{n,k} - v_{2j-1}v_{2j}$. By repeatedly utilizing Lemma 2.2 we have

$$\begin{aligned}
&m\left(P_{n,k} - v_{2j-3}v_{2j-2}, i\right)\\
=&m\left(P_{n,k} - v_{2j-3}v_{2j-2} - v_{2j-1}v_{2j}, i\right)\\
&+m\left(P_{n,k} - v_{2j-3}v_{2j-2} - v_{2j-1} - v_{2j} - v_{2j-4}v_{2j-3}, i-1\right)\\
&+m\left(P_{n,k} - v_{2j-3}v_{2j-2} - v_{2j-1} - v_{2j} - v_{2j-4} - v_{2j-3} - v_{2j+1}v_{2j+2}, i-2\right)\\
&+\cdots\\
&+m\left(S_k \cup S_{k-1} \cup \tilde{P}_{n-4j+3,k}, i-2j+3\right),
\end{aligned}$$

and

$$\begin{aligned}
&m\left(P_{n,k} - v_{2j-1}v_{2j}, i\right)\\
=&m\left(P_{n,k} - v_{2j-1}v_{2j} - v_{2j-3}v_{2j-2}, i\right)\\
&+m\left(P_{n,k} - v_{2j-1}v_{2j} - v_{2j-3} - v_{2j-2} - v_{2j}v_{2j+1}, i-1\right)\\
&+m\left(P_{n,k} - v_{2j-1}v_{2j} - v_{2j-3} - v_{2j-2} - v_{2j} - v_{2j+1} - v_{2j-5}v_{2j-4}, i-2\right)\\
&+\cdots\\
&+m\left(S_{k-1} \cup P_{n-4j+4,k}, i-2j+3\right).
\end{aligned}$$

So

$$\begin{aligned}
&m\left(P_{2j-2,k} \cup P_{n-2j+2,k}, i\right) - m\left(P_{2j,k} \cup P_{n-2j,k}, i\right)\\
=&m\left(S_k \cup S_{k-1} \cup \tilde{P}_{n-4j+3,k}, i-2j+3\right) - m\left(S_{k-1} \cup P_{n-4j+4,k}, i-2j+3\right).
\end{aligned}$$

Note that $m\left(P_{2j-2,k} \cup P_{n-2j+2,k}, i\right) \ge m\left(P_{2j,k} \cup P_{n-2j,k}, i\right)$ for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, and there exists $i = 2j-1$ such that $m\left(P_{2j-2,k} \cup P_{n-2j+2,k}, i\right) > m\left(P_{2j,k} \cup P_{n-2j,k}, i\right)$. The lemma holds. □

Let $\mathcal{G}$ be a k-uniform hypertree with size r. Let $\mathcal{G}_{s,t}$ be a k-uniform hypertree obtained from $\mathcal{G}$ by attaching two loose path of length s and t at the common vertex w of $\mathcal{G}$, as show in Fig. 3.1.

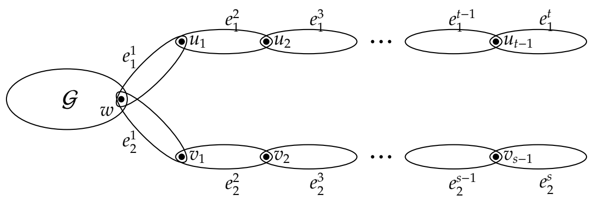


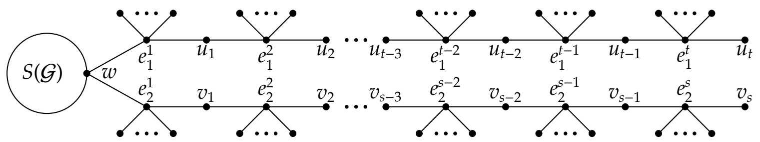
Fig. 3.1. The hypertree $\mathcal{G}_{s,t}$

Lemma 3.2. *If $s > t+1$, then $BE(\mathcal{G}_{s,t}) > BE(\mathcal{G}_{s-1,t+1})$.*

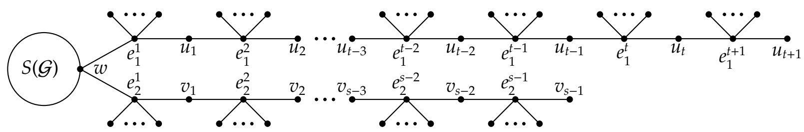
Proof. By Eq. (1), we have

$$BE(\mathcal{G}_{s,t}) = \frac{1}{2}E(S(\mathcal{G}_{s,t})), BE(\mathcal{G}_{s-1,t+1}) = \frac{1}{2}E(S(\mathcal{G}_{s-1,t+1})),$$

where $S(\mathcal{G}_{s,t})$ and $S(\mathcal{G}_{s-1,t+1})$ are shown in Fig. 3.2.



$S(\mathcal{G}_{s,t})$



$S(\mathcal{G}_{s-1,t+1})$

Fig. 3.2. The graphs $S(\mathcal{G}_{s,t})$ and $S(\mathcal{G}_{s-1,t+1})$

Since $S(\mathcal{G}_{s,t})$ and $S(\mathcal{G}_{s-1,t+1})$ are trees on $k(r+s+t)+1$ vertices, we only need to prove that $m(S(\mathcal{G}_{s,t}), i) \geq m(S(\mathcal{G}_{s-1,t+1}), i)$ for $1 \leq i \leq \left\lfloor \frac{k(r+s+t)+1}{2} \right\rfloor$ and at least one of the inequalities $m(S(\mathcal{G}_{s,t}), i) \geq m(S(\mathcal{G}_{s-1,t+1}), i)$ is strict.

We first consider $1 \leq i \leq 2t+1$. By repeatedly utilizing the Lemma 2.2, we have

$$\begin{aligned}
&m(S(\mathcal{G}_{s,t}), 1) - m(S(\mathcal{G}_{s-1,t+1}), 1) = 0,\\
&m(S(\mathcal{G}_{s,t}), 2) - m(S(\mathcal{G}_{s-1,t+1}), 2) = m\left(S(\mathcal{G}_{s,t}) - v_{s-1} - e_2^s, 1\right) - m\left(S(\mathcal{G}_{s-1,t+1}) - u_t - e_1^{t+1}, 1\right) = 0,\\
&m(S(\mathcal{G}_{s,t}), 3) - m(S(\mathcal{G}_{s-1,t+1}), 3)\\
=&m\left(S(\mathcal{G}_{s,t}) - v_{s-1} - e_2^s - e_1^t - u_t, 1\right) - m\left(S(\mathcal{G}_{s-1,t+1}) - u_t - e_1^{t+1} - e_2^{s-1} - v_{s-1}, 1\right) = 0.
\end{aligned}$$

Similarly, for $4 \leq i \leq 2t+1$, we may also get

$$m(S(\mathcal{G}_{s,t}), i) - m(S(\mathcal{G}_{s-1,t+1}), i) = 0.$$

For $2t+2 \leq i \leq \left\lfloor \frac{k(r+s+t)+1}{2} \right\rfloor$, letting A be the graph as shown in Fig. 3.3, by repeatedly utilizing the Lemma 2.2, it can be concluded that

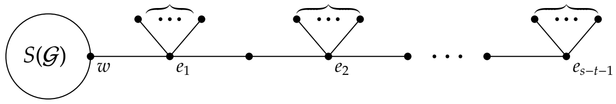


Fig. 3.3. The graph A

$$m(S(\mathcal{G}_{s,t}), i) - m(S(\mathcal{G}_{s-1,t+1}), i) = m(A, i-2t-1) - m\left((S(\mathcal{G}) - w) \cup P_{2(s-t-1),k}, i-2t-1\right).$$

Since $\left((S(\mathcal{G}) - w) \cup P_{2(s-t-1),k}\right) \subset A$, then $m(S(\mathcal{G}_{s,t}), i) - m(S(\mathcal{G}_{s-1,t+1}), i) > 0$. Thus $BE(\mathcal{G}_{s,t}) > BE(\mathcal{G}_{s-1,t+1})$. The lemma holds. □

Lemma 3.3. *Let $\mathcal{T}'$ be a k-uniform hypertree with size $r-a$ and exactly $x-1$ pendent edges. Let $\mathcal{T}$ be a k-uniform hypertree obtained from $\mathcal{T}'$ by adding a loose path of length a at a vertex u in $\mathcal{T}'$ such that there are x pendent edges in $\mathcal{T}$. Then*

$$BE(\mathcal{T}) \geq BE\left((\mathcal{S}^{a}_{r,x-1})^{k}\right)$$

with the equation holds if and only if $\mathcal{T} \cong (\mathcal{S}^{a}_{r,x-1})^{k}$.

Proof. Since $S(\mathcal{T})$ and $S\left((\mathcal{S}^{a}_{r,x-1})^{k}\right)$ are trees on $kr+1$ vertices, we only need to prove that for $1 \leq i \leq \left\lfloor \frac{kr+1}{2} \right\rfloor$, $m(S(\mathcal{T}), i) \geq m\left(S\left((\mathcal{S}^{a}_{r,x-1})^{k}\right), i\right)$ with all equalities hold if and only if $\mathcal{T} \cong (\mathcal{S}^{a}_{r,x-1})^{k}$. The result follows for $r \leq 4$. Suppose now that $r \geq 5$ and the result is true for the values less than r.

Let $N_{S(\mathcal{T})}(u) = \{u_1, u_2, \ldots, u_s\}$ and $S(\mathcal{T}) - uu_1 \cong S(\mathcal{T}') \cup P^{k}_{2a}$, where $N_{S(\mathcal{T})}(u)$ denotes the neighbor set of the vertex u in $S(\mathcal{T})$. Let $r-a = b(x-1)+y$, where $0 \leq y \leq x-2$. By Lemma 2.2 we have

$$m(S(\mathcal{T}), i) = m(S(\mathcal{T}') \cup P_{2a,k}, i) + m(S(\mathcal{T}) - u - u_1, i-1),$$

$$m\left(S\left((\mathcal{S}^{a}_{r,x-1})^{k}\right), i\right) = m\left(S\left((\mathcal{S}_{r-a,x-1})^{k}\right) \cup P_{2a,k}, i\right) + m\left(\tilde{P}_{2a-1,k} \cup yP_{2b+2,k} \cup (x-y-1)P_{2b,k}, i-1\right).$$

If $x=3$, then $(S(\mathcal{T}') \cup P_{2a,k}) \cong \left(S\left((\mathcal{S}_{r-a+1,x-1})^{k}\right) \cup P_{2a,k}\right)$. Suppose that $x \geq 4$ and Let $\mathcal{T}''$ be a k-uniform hypertree with size $r-a-c$ and exactly $x-2$ pendent edges such that $\mathcal{T}'$ can be regarded as the k-uniform hypertree obtained from $\mathcal{T}''$ by adding a loose path of length c at a vertex w in $\mathcal{T}''$. By induction hypothesis and Lemma 3.2, we have that for $1 \leq i \leq \left\lfloor \frac{kr+1}{2} \right\rfloor$,

$$m(S(\mathcal{T}') \cup P_{2a,k}, i) \geq m\left(S\left((\mathcal{S}^{c}_{r-a,x-2})^{k}\right) \cup P_{2a,k}, i\right) \geq m\left(S\left((\mathcal{S}_{r-a,x-1})^{k}\right) \cup P_{2a,k}, i\right).$$

In the following we will prove the inequality

$$m(S(\mathcal{T}) - u - u_1, i-1) \geq m\left(\tilde{P}_{2a-1,k} \cup yP_{2b+2,k} \cup (x-y-1)P_{2b,k}, i-1\right).$$

for $1 \leq i \leq \left\lfloor \frac{kr+1}{2} \right\rfloor$. There exists $x-s$ edges e_i $(i = 1, 2, \ldots x-s)$ in $S(\mathcal{T}')$ such that $S(\mathcal{T}) - u - u_1 - \bigcup_{i=1}^{x-s} e_i \cong \tilde{P}_{2a-1,k} \bigcup_{j=2}^{x} P_{n_j,k}$ and $\sum_{j=2}^{x} n_j = 2(r-a)$. Therefore $\tilde{P}_{2a-1,k} \bigcup_{j=2}^{x} P_{n_j,k}$ is a spanning subgraph of $S(\mathcal{T}) - u - u_1$. Since n_j is even, by Lemma 3.1 we have

$$m(S(\mathcal{T}) - u - u_1, i-1) \geq m\left(\tilde{P}_{2a-1,k} \bigcup_{j=2}^{x} P_{n_j,k}, i-1\right) \geq m\left(\tilde{P}_{2a-1,k} \cup yP_{2b+2,k} \cup (x-y-1)P_{2b,k}, i-1\right),$$

where the first equality holds if and only if $\mathcal{T}$ is a power graph of a starlike, and the second equality holds if and only if $\bigcup_{j=2}^{x} P_{n_j,k} \cong yP_{2b+2,k} \cup (x-y-1)P_{2b,k}$. These imply all equalities hold if and only if $\mathcal{T} \cong (\mathcal{S}^{a}_{r,x-1})^{k}$. □

Theorem 3.4. *Let $\mathcal{T}$ be a k-uniform hypertree with size r and exactly x pendent edges. Then*

$$BE(\mathcal{T}) \geq BE((\mathcal{S}_{r,x})^{k}),$$

the equality holds if and only if $\mathcal{T} \cong (\mathcal{S}_{r,x})^{k}$.

Proof. By Lemmas 3.2 and 3.3, the result follows. □

4. The minimum incidence energy of k-uniform unicyclic hypergraphs

Lemma 4.1. *Let $T_1(t, s_1, s_2, \ldots, s_t)$ and $T_2(t, s_1, s_2, \ldots, s_t)$ be two trees as shown in Fig. 4.1. If $t \geq 3$, then $m(T_1(t; s_1, s_2, \ldots, s_t), 1) = m(T_2(t; s_1, s_2, \ldots, s_t), 1)$ and $m(T_1(t; s_1, s_2, \ldots, s_t), i) > m(T_2(t; s_1, s_2, \ldots, s_t), i)$ for any $2 \leq i \leq \left\lfloor \frac{1+t+\sum_{i=1}^{t} s_i}{2} \right\rfloor$.*

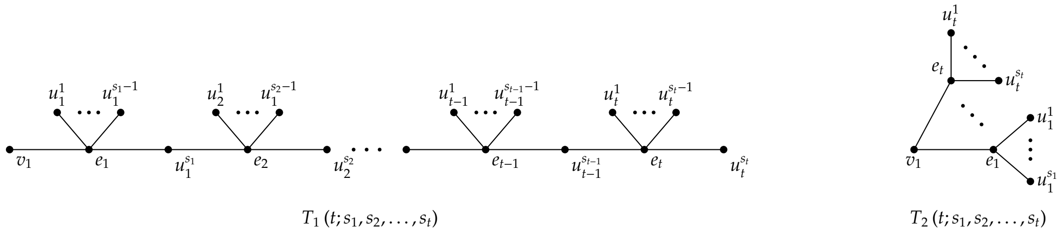


Fig. 4.1. The graphs $T_1(t; s_1, s_2, \ldots, s_t)$ and $T_2(t; s_1, s_2, \ldots, s_t)$

Proof. It is obvious to see that $m(T_1(t; s_1, s_2, \ldots, s_t), 1) = m(T_2(t; s_1, s_2, \ldots, s_t), 1)$. If $t \geq 3$, we have

$$\begin{aligned}
m(T_1, i) =& m(T_1 - e_1u_1^{s_1}, i) + m(T_1 - e_1 - u_1^{s_1}, i-1) \\
=& m(T_1 - e_1u_1^{s_1} - e_2u_2^{s_2}, i) + m(T_1 - e_1u_1^{s_1} - e_2 - u_2^{s_2}, i-1) + m(T_1 - e_1 - u_1^{s_1}, i-1) \\
=& \cdots \\
=& m(T_1 - e_1u_1^{s_1} - e_2u_2^{s_2} - \cdots - e_{t-1}u_{t-1}^{s_{t-1}}, i) \\
& + m(T_1 - e_1 - u_1^{s_1}, i-1) + m(T_1 - e_1u_1^{s_1} - e_2 - u_2^{s_2}, i-1) \\
& + \cdots \\
& + m(T_1 - e_1u_1^{s_1} - e_2u_2^{s_2} - \cdots - e_{t-2}u_{t-2}^{s_{t-2}} - e_{t-1} - u_{t-1}^{s_{t-1}}, i-1),
\end{aligned}$$

and

$$\begin{aligned}
m(T_2, i) =& m(T_2 - v_1e_1, i) + m(T_2 - v_1 - e_1, i-1) \\
=& m(T_2 - v_1e_1 - v_1e_2, i) + m(T_2 - v_1e_1 - v_1 - e_2, i-1) + m(T_2 - v_1 - e_1, i-1) \\
=& \cdots \\
=& m(T_2 - \bigcup_{i=1}^{t-1} v_1e_i, i) + (t-1)m(T_2 - v_1 - e_1, i-1).
\end{aligned}$$

It is obvious to see that

$$\begin{aligned}
&\left(T_2 - \bigcup_{i=1}^{t-1} v_1e_i\right) \cong \left(T_1 - \bigcup_{i=1}^{t-1} e_iu_i^{s_i}\right), \\
&(T_2 - v_1 - e_1) \subset \left(T_1 - e_1 - u_1^{s_1}\right), \\
&(T_2 - v_1 - e_1) \subset \left(T_1 - e_1u_1^{s_1} - e_2 - u_2^{s_2}\right), \\
&\qquad\qquad \vdots \\
&(T_2 - v_1 - e_1) \subset \left(T_1 - \bigcup_{i=1}^{t-3} e_iu_i^{s_i} - e_{t-2} - u_{t-2}^{s_{t-2}}\right), \\
&(T_2 - v_1 - e_1) \cong \left(T_1 - \bigcup_{i=1}^{t-2} e_iu_i^{s_i} - e_{t-1} - u_{t-1}^{s_{t-1}}\right).
\end{aligned}$$

Hence $m(T_1(t, s_1, s_2, \ldots, s_t), i) > m(T_2(t, s_1, s_2, \ldots, s_t), i)$ for any $2 \leq i \leq \left\lfloor \frac{1+t+\sum_{i=1}^{t} s_i}{2} \right\rfloor$. The lemma holds. □

Lemma 4.2. *Let G and G' be two connected graphs of order n as shown in Fig. 4.2, where G_1 is a subgraph of G and G' with $v_k \in G_1$. Then $m(G, 1) \geq m(G', 1)$ and $m(G, i) > m(G', i)$ for any $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

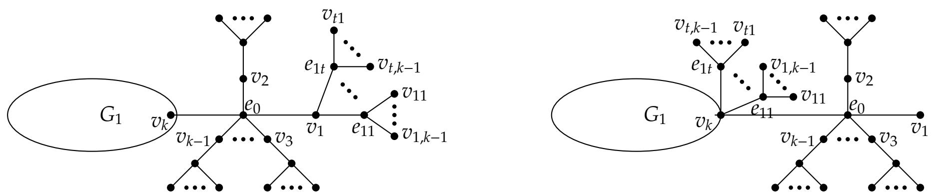


Fig. 4.2. The graphs G and G'

Proof. The proof method is similar to Lemma 2.1 in [13]. □

Lemma 4.3. *Let G and G' be two connected graphs of order n as shown in Fig. 4.3, where G_1 is a subgraph of G and G' with $v_k \in G_1$. Then $m(G, 1) \geq m(G', 1)$ and $m(G, i) > m(G', i)$ for any $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

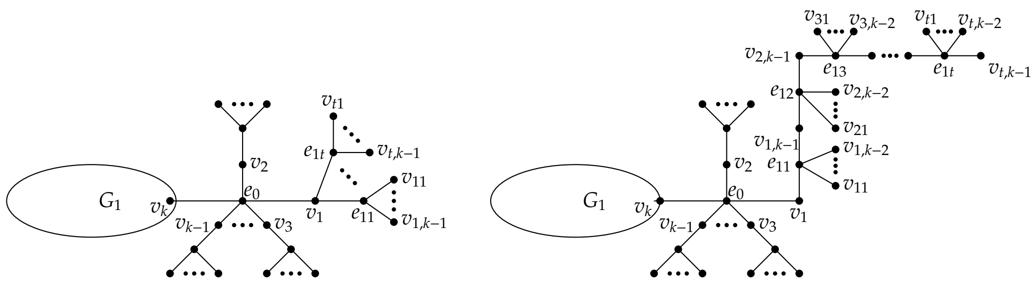


Fig. 4.3. The graphs G and G'

Proof. The proof method is similar to Lemma 2.3 in [13]. □

Let $\mathcal{G}$ be a k-uniform unicyclic hypergraph and $e_0 = \{v_1, v_2, \ldots, v_k\}$ be an edge which is not belonging to $\mathcal{G}$. Let $\mathcal{G}_1$ be the unicyclic hypergraph obtained by identifying v_k of e_0 and a vertex w of $\mathcal{G}$, denote the new vertex v_k. Let $\mathcal{H}_1$ be a unicyclic hypergraph obtained from $\mathcal{G}_1$ by attaching some pendent edges at some vertices of e_0. Let $e_{11}, \ldots, e_{1t}$ be the edges attaching at v_1, and let $\mathcal{H}_2$ be the unicyclic hypergraph obtained from $\mathcal{H}_1$ by moving the pendent edges attaching at v_1 to v_k, as shown in Fig. 4.4. Let $\mathcal{H}_3$ be the unicyclic hypergraph obtained from $\mathcal{H}_1$ by deleting the pendent edges attaching at v_1 and adding a loose path of length t at v_1, as shown in Fig. 4.5.

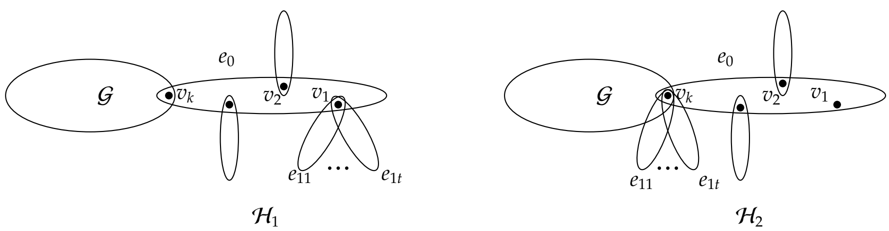


Fig. 4.4 The hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$

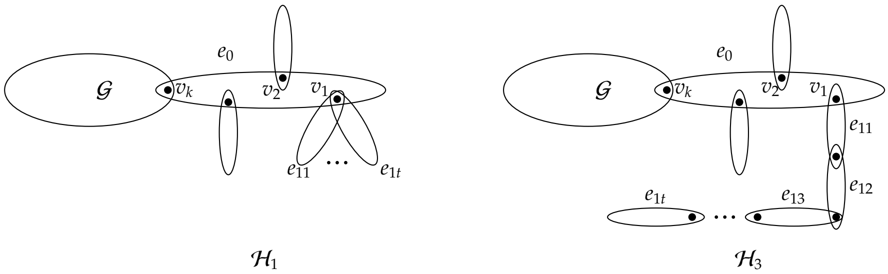


Fig. 4.5 The hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_3$

Lemma 4.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the k-uniform unicyclic hypergraphs of size r with unique cycle $C_{m,k}$ as shown in Fig. 4.4. Then $BE(\mathcal{H}_1) > BE(\mathcal{H}_2)$.*

Proof. By Eq. (1), $BE(\mathcal{H}_1) = \frac{1}{2}E(S(\mathcal{H}_1))$ and $BE(\mathcal{H}_2) = \frac{1}{2}E(S(\mathcal{H}_2))$. We only need to prove that $b_{2i}(S(\mathcal{H}_1)) \geq b_{2i}(S(\mathcal{H}_2))$ for $1 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$ and at least one of the inequalities $b_{2i}(S(\mathcal{H}_1)) \geq b_{2i}(S(\mathcal{H}_2))$ is strict. It is obvious to see that $b_2(S(\mathcal{H}_1)) = b_2(S(\mathcal{H}_2))$. For $2 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$, we consider two cases.

Case 1. m is odd.

By Eq. (2) we obtain

$$b_{2i}(S(\mathcal{H}_1)) = m(S(\mathcal{H}_1), i) + 2m(S(\mathcal{H}_1) - C_{2m}, i-m),$$
$$b_{2i}(S(\mathcal{H}_2)) = m(S(\mathcal{H}_2), i) + 2m(S(\mathcal{H}_2) - C_{2m}, i-m).$$

By Lemma 4.2, we have $m(S(\mathcal{H}_1), i) > m(S(\mathcal{H}_2), i)$ for $2 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$. If v_k is a vertex of C_{2m}, then $S(\mathcal{H}_2) - C_{2m}$ is a spanning subgraph of $S(\mathcal{H}_1) - C_{2m}$. So $m(S(\mathcal{H}_1) - C_{2m}, i-m) > m(S(\mathcal{H}_2) - C_{2m}, i-m)$ for $m+1 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$. If v_k is not a vertex of C_{2m}, By Lemma 4.2, we have $m(S(\mathcal{H}_1) - C_{2m}, i-m) \geq m(S(\mathcal{H}_2) - C_{2m}, i-m)$ for $m+1 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$. Hence $b_{2i}(S(\mathcal{H}_1)) > b_{2i}(S(\mathcal{H}_2))$ for $2 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$.

Case 2. m is even.

By Eq. (2) we obtain

$$b_{2i}(S(\mathcal{H}_1)) = m(S(\mathcal{H}_1), i) - 2m(S(\mathcal{H}_1) - C_{2m}, i-m) = m(S(\mathcal{H}_1), i) - m(C_{2m}, m) \times m(S(\mathcal{H}_1) - C_{2m}, i-m),$$
$$b_{2i}(S(\mathcal{H}_2)) = m(S(\mathcal{H}_2), i) - 2m(S(\mathcal{H}_2) - C_{2m}, i-m).$$

By Lemma 4.2, we have $b_{2i}(S(\mathcal{H}_1)) > b_{2i}(S(\mathcal{H}_2))$ for $2 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$. The lemma holds. □

Lemma 4.5. *Let $\mathcal{H}_1$ and $\mathcal{H}_3$ be the k-uniform unicyclic hypergraphs of size r with unique cycle $C_{m,k}$ as shown in Fig. 4.5. Then $BE(\mathcal{H}_3) > BE(\mathcal{H}_1)$.*

Proof. By Eq. (1), $BE(\mathcal{H}_1) = \frac{1}{2}E(S(\mathcal{H}_1))$ and $BE(\mathcal{H}_3) = \frac{1}{2}E(S(\mathcal{H}_3))$. We only need to prove that $b_{2i}(S(\mathcal{H}_3)) \geq b_{2i}(S(\mathcal{H}_1))$ for $1 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$ and at least one of the inequalities $b_{2i}(S(\mathcal{H}_3)) \geq b_{2i}(S(\mathcal{H}_1))$ is strict. It is obvious to see that $b_2(S(\mathcal{H}_1)) = b_2(S(\mathcal{H}_3))$. For $2 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$,

Case 1. m is odd.

By Eq. (2) we obtain

$$b_{2i}(S(\mathcal{H}_1)) = m(S(\mathcal{H}_1), i) + 2m(S(\mathcal{H}_1) - C_{2m}, i-m),$$
$$b_{2i}(S(\mathcal{H}_3)) = m(S(\mathcal{H}_3), i) + 2m(S(\mathcal{H}_3) - C_{2m}, i-m).$$

By Lemma 4.3 , we can obtain $m(S(\mathcal{H}_3), i) > m(S(\mathcal{H}_1), i)$ for $2 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$ and $m(S(\mathcal{H}_3) - C_{2m}, i-m) \geq m(S(\mathcal{H}_1) - C_{2m}, i-m)$ for $m+1 \leq i \leq \left\lfloor \frac{kr}{2} \right\rfloor$.

Case 2. m is even.
By Eq. (2) we obtain

$$b_{2i}\left(S(\mathcal{H}_1)\right) = m\left(S(\mathcal{H}_1), i\right) - 2m\left(S(\mathcal{H}_1) - C_{2m}, i - m\right) = m\left(S(\mathcal{H}_1), i\right) - m\left(C_{2m}, m\right) \times m\left(S(\mathcal{H}_1) - C_{2m}, i - m\right),$$
$$b_{2i}\left(S(\mathcal{H}_3)\right) = m\left(S(\mathcal{H}_3), i\right) - 2m\left(S(\mathcal{H}_3) - C_{2m}, i - m\right).$$

By Lemma 4.3, we have $b_{2i}\left(S(\mathcal{H}_3)\right) > b_{2i}\left(S(\mathcal{H}_1)\right)$ for $2 \le i \le \left\lfloor \frac{kr}{2} \right\rfloor$. The lemma holds. □

Theorem 4.6. *Let $\mathcal{G}$ be a k-uniform unicyclic hypergraph of size $r > 2$ with unique cycle $C_{m,k}$,Then*

$$BE(\mathcal{G}) \ge BE(\mathcal{U}_r^2),$$

the equality holds if and only if $\mathcal{G} \cong \mathcal{U}_r^2$.

Proof. By Lemmas 4.4 and 4.5, we can assume that all edges of $\mathcal{G}$ except those in $C_{m,k}$ are pendent edges attaching at some vertices in $C_{m,k}$. We only need to prove that $b_{2i}\left(S\left(\mathcal{G}\right)\right) \ge b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right)$ for any positive integer i, and all equalities hold if and only if $\mathcal{G} \cong \mathcal{U}_r^2$.

We use induction on r to prove it. If $r = m$, then $\mathcal{G} \cong C_{r,k}$. It is obvious to see that $b_2\left(S\left(C_{r,k}\right)\right) = b_2\left(S\left(\mathcal{U}_r^2\right)\right)$. Suppose now $2 \le i \le r$. Then by Lemma 2.3, we have

$$\begin{aligned} b_{2i}\left(S\left(C_{r,k}\right)\right) =& m\left(T_1\left(r; k-2, k-1, \ldots, k-1\right), i\right) + m\left(T_1\left(r-1; k-2, k-1, \ldots, k-1\right), i-1\right), \\ b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right) =& m\left(T_2\left(r; k-2, k-1, \ldots, k-1\right), i\right) + m\left(T_2\left(r-1; k-2, k-1, \ldots, k-1\right), i-1\right) \\ &- 2m\left(\left(r-2\right) S_k, i-2\right). \end{aligned}$$

By Lemma 4.1, we have

$$m\left(T_1\left(r; k-2, k-1, \ldots, k-1\right), i\right) > m\left(T_2\left(r; k-2, k-1, \ldots, k-1\right), i\right),$$

and

$$m\left(T_1\left(r-1; k-2, k-1, \ldots, k-1\right), i-1\right) \ge m\left(T_2\left(r-1; k-2, k-1, \ldots, k-1\right), i-1\right).$$

Thus $b_{2i}\left(S\left(\mathcal{G}\right)\right) > b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right)$ for $2 \le i \le r$. Hence the result is true for $r = m$. Suppose now $r \ge m + 1$. We consider two cases.

Case 1. There is at least one pendent edge attaching at $v_i \in V_0$ in $\mathcal{G}$.

Without loss of generality, we can assume there exist a pendent edge e_s attaching at $v_1 \in V_0$ in $\mathcal{G}$ and there exist a pendent edge e_t attaching at $v_1 \in V_0$ in $\mathcal{U}_r^2$. Let v_{e_s} be the vertex in $S\left(\mathcal{G}\right)$ which is adjacent to all vertices of e_s. Let v_{e_t} be the vertex in $S\left(\mathcal{U}_r^2\right)$ which is adjacent to all vertices of e_t. By Lemma 2.3, we have

$$\begin{aligned} b_{2i}\left(S\left(\mathcal{G}\right)\right) &= b_{2i}\left(S\left(\mathcal{G}\right) - v_1 v_{e_s}\right) + b_{2i-2}\left(S\left(\mathcal{G}\right) - v_1 - v_{e_s}\right) \\ &= b_{2i}\left(S\left(\mathcal{G} - e_s\right) \cup S_k\right) + m\left(S\left(\mathcal{G}\right) - v_1 - v_{e_s}, i-1\right), \\ b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right) &= b_{2i}\left(S\left(\mathcal{U}_r^2\right) - v_1 v_{e_t}\right) + b_{2i-2}\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right) \\ &= b_{2i}\left(S\left(\mathcal{U}_{r-1}^2\right) \cup S_k\right) + m\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}, i-1\right). \end{aligned}$$

By the induction hypothesis, $b_{2i}\left(S\left(\mathcal{G} - e_s\right) \cup S_k\right) \ge b_{2i}\left(S\left(\mathcal{U}_{r-1}^2\right) \cup S_k\right)$. Therefore

$$b_{2i}\left(S\left(\mathcal{G}\right)\right) - b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right) \ge m\left(S\left(\mathcal{G}\right) - v_1 - v_{e_s}, i-1\right) - m\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}, i-1\right).$$

Since $S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}$ is a spanning subgraph of $S\left(\mathcal{G}\right) - v_1 - v_{e_s}$, we have

$$m\left(S\left(\mathcal{G}\right) - v_1 - v_{e_s}, i-1\right) \ge m\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}, i-1\right),$$

that is, $b_{2i}(S(\mathcal{G})) \geq b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right)$. These equalities hold if and only if $(\mathcal{G} - e_s) \cong \left(\mathcal{U}_r^2 - e_t\right)$ and $\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right) \cong (S(\mathcal{G}) - v_1 - v_{e_s})$, that is, $\mathcal{G} \cong \mathcal{U}_r^2$.

Case 2. There are no pendent edges attaching at $v_i \in V_0$ in $\mathcal{G}$.

Without loss of generality, we can assume there exist a pendent edge e'_s attaching at $u_{11} \in e_1$ in $\mathcal{G}$ and there exist a pendent edge e_t attaching at $v_1 \in V_0$ in $\mathcal{U}_r^2$. Let $v_{e'_s}$ be the vertex in $S(\mathcal{G})$ which is adjacent to all vertices of e'_s. Let v_{e_t} be the vertex in $S\left(\mathcal{U}_r^2\right)$ which is adjacent to all vertices of e_t. It is obvious to see that $b_2(S(\mathcal{G})) = b_2\left(S\left(\mathcal{U}_r^2\right)\right)$. Suppose now $2 \leq i \leq r$. By Lemma 2.3, we have

$$\begin{aligned}
b_{2i}(S(\mathcal{G})) &= b_{2i}\left(S(\mathcal{G}) - u_{11}v_{e'_s}\right) + b_{2i-2}\left(S(\mathcal{G}) - u_{11} - v_{e'_s}\right) \\
&= b_{2i}\left(S(\mathcal{G} - e'_s) \cup S_k\right) + b_{2i-2}\left(S(\mathcal{G}) - u_{11} - v_{e'_s}\right), \\
b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right) &= b_{2i}\left(S\left(\mathcal{U}_r^2\right) - v_1 v_{e_t}\right) + b_{2i-2}\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right) \\
&= b_{2i}\left(S\left(\mathcal{U}_{r-1}^2\right) \cup S_k\right) + b_{2i-2}\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right).
\end{aligned}$$

By the induction hypothesis, $b_{2i}\left(S(\mathcal{G} - e'_s) \cup S_k\right) \geq b_{2i}\left(S\left(\mathcal{U}_{r-1}^2\right) \cup S_k\right)$. Therefore

$$b_{2i}(S(\mathcal{G})) - b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right) \geq b_{2i-2}\left(S(\mathcal{G}) - u_{11} - v_{e'_s}\right) - b_{2i-2}\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right).$$

If m is odd, by Eq. (2) we obtain

$$\begin{aligned}
b_{2i-2}\left(S(\mathcal{G}) - u_{11} - v_{e'_s}\right) &= m\left(S(\mathcal{G}) - u_{11} - v_{e'_s}, i-1\right) + 2m\left(S(\mathcal{G}) - v_1 - v_{e'_s} - C_{2m}, i-m-1\right), \\
b_{2i-2}\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right) &= m\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}, i-1\right).
\end{aligned}$$

Since $S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}$ is a spanning subgraph of $S(\mathcal{G}) - u_{11} - v_{e'_s}$, we have

$$m\left(S(\mathcal{G}) - u_{11} - v_{e'_s}, i-1\right) > m\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}, i-1\right),$$

that is, $b_{2i-2}\left(S(\mathcal{G}) - u_{11} - v_{e'_s}\right) > b_{2i-2}\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}\right)$.

If m is even, by Eq. (2), we obtain

$$\begin{aligned}
b_{2i-2}\left(S(\mathcal{G}) - u_{11} - v_{e'_s}\right) &= m\left(S(\mathcal{G}) - u_{11} - v_{e'_s}, i-1\right) - 2m\left(S(\mathcal{G}) - u_{11} - v_{e'_s} - C_{2m}, i-m-1\right) \\
&> m\left(S\left(\mathcal{U}_r^2\right) - v_1 - v_{e_t}, i-1\right).
\end{aligned}$$

Hence $b_{2i}(S(\mathcal{G})) > b_{2i}\left(S\left(\mathcal{U}_r^2\right)\right)$.

The theorem now holds. □

Acknowledgment

The authors would like to express their sincere gratitude to the anonymous referee for careful reading and insightful comments on this paper. This work was supported by Natural Science Foundation of Shanxi Province of China (No. 202303021211154).

References

[1] A. Banerjee, *On the spectrum of hypergraphs*, Linear Algebra Appl. **614** (2021) 82–110.
[2] K. Cardoso, R. Del-Vecchio, L. Portugal, V. Trevisan, *Adjacency energy of hypergraphs*, Linear Algebra Appl. **648** (2022) 181–204.
[3] K. Cardoso, V. Trevisan, *Energies of hypergraphs*, Electron. J. Linear Algebra **36** (2020) 293–308.
[4] L. Duttweiler, N. Reff, *Spectra of cycle and path families of oriented hypergraphs*, Linear Algebra Appl. **578** (2019) 251–271.
[5] Y. Fu, Y. Gao, *Incidence energy of k-uniform hypertrees*, MATCH Commun. Math. Comput. Chem. **92** (2024) 133–150.

[6] I. Gutman, *Acyclic systems with extremal Hückel π-electron energy*, Theor. Chim. Acta. **45** (1997) 79–87.
[7] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.
[8] X. Li, J. Zhang, *On bicyclic graphs with maximal energy*, Linear Algebra Appl. **427** (2007) 87-98.
[9] H. Lin, B. Zhou, *Spectral radius of uniform hypergraphs*, Linear Algebra Appl. **527** (2017) 32–52.
[10] V. Nikiforov, *The energy of graphs and matrices*, J. Math. Anal. Appl. **326(2)** (2007) 1472–1475.
[11] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symb. Comput. **40** (2005) 1302–1324.
[12] Y. Wang, B. Zhou, *On distance spectral radius of hypergraphs*, Linear Multilinear Algebra **66(11)** (2018) 2232–2246.
[13] Q. Zhu, *Extremal k-uniform hypertrees on incidence energy*, Int. J. Quantum Chem. **121(9)** (2021) e26592.