Filomat 38:30 (2024), 10687–10694
https://doi.org/10.2298/FIL2430687B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On independent $[1, k]$-set in graphs

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Abstract. For an integer $k \geq 1$, a subset $S \subseteq V$ in a graph $G = (V, E)$ is an independent $[1, k]$-set of G if S is independent and every vertex in $V - S$ is adjacent to one but no more than k vertices in S. The upper $[1, k]$-independence number noted $\alpha_{[1,k]}(G)$ is the maximum cardinality of an independent $[1, k]$-set of G. In this paper, we provide a constructive characterization of graphs having an independent $[1, k]$-set, while for split graphs, a necessary and sufficient condition is given for those having an independent $[1, k]$-set. Moreover, some upper bounds on $\alpha_{[1,k]}(G)$ are established for graphs having an independent $[1, k]$-set. We also establish a Nordhaus-Gaddum type result for the upper $[1, k]$-independence number, where in addition, a characterization of extremal graphs attaining each bound is provided. Finally, we show that the decision problem corresponding to the problem of computing the upper $[1, k]$-independence number is $\mathcal{NP}$-complete for bipartite and chordal graphs.

1. Introduction

We consider simple graphs $G = (V(G), E(G))$ of order $|V(G)| = n(G)$ and size $|E(G)| = m(G)$. The *neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$, while the *degree* of v is $d_G(v) = |N_G(v)|$. The *minimum* and *maximum degrees* of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. When no ambiguity on G is possible, we simply write $V, E, n, \delta, \Delta, N(v)$ and $d(v)$. The *neighborhood* of a set $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, and let $G[S]$ denote the subgraph induced by S in G. For a set S and a vertex x, we denote by $N_S(x)$ the set of vertices in S that are adjacent to x, and let $d_S(x) = |N_S(x)|$. For disjoint subsets A and B of vertices in a graph G, we denote by $m(A, B)$ the number of edges having one endvertex in A and the other in B.

The *path* (*cycle*, *clique*, *star*, respectively) of order n is denoted by P_n (C_n, K_n, $K_{1,n-1}$, respectively). We say that G is *regular* if all vertices have the same degree. Moreover, if every vertex of G has degree r, then G is called r-regular. Let H be a graph. A graph G is said to be *H-free* if it has no induced subgraph isomorphic to H. A *tree* is an acyclic connected graph. Also let $S_{p,q}$ denote the *double star* of order $p + q + 2$. A graph is *bipartite* if its vertex set can be partitioned in two independent sets, while it is a *split graph* if its vertex set can be partitioned into an independent set and a clique. The *corona* of graphs G and G', denoted $G \circ G'$, is the graph formed from one copy of G and $|V(G)|$ copies of G', where the ith vertex in $V(G)$ is adjacent to every vertex in the ith copy of G'.

2020 *Mathematics Subject Classification*. Primary 05C69.
Keywords. Independent $[1, k]$-sets, upper $[1, k]$-independence number.
Received: 11 May 2024; Accepted: 10 August 2024
Communicated by Paola Bonacini
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In [6], Chellali et al. defined a subset $S \subseteq V$ in a graph $G = (V, E)$ to be a $[j, k]$-*set* if for every vertex $v \in V - S$, $j \leq |N(v) \cap S| \leq k$, that is, every vertex in $V - S$ is adjacent to at least j vertices but not more than k vertices in S. Further results on $[j, k]$-sets can be found in [1–7, 10].

In [5], Chellali et al. considered $[j, k]$-sets that are also independent and called them *independent* $[j, k]$-*sets*. However, these sets were only studied when $j = 1$. The authors [5] noted that not all graphs have an independent $[1, k]$-set for some integer $k \geq 1$. But for $k = \Delta$, every graph has an independent $[1, \Delta]$-set. This let to defining k_0 as the smallest integer k such that G admits an independent $[1, k]$-set. Since every independent $[1, k_0]$-set is an independent $[1, k]$-set for each $k \geq k_0$, we can therefore define the *lower* $[1, k]$-*independence number* $i_{[1,k]}(G)$ to be the minimum cardinality of an independent $[1, k]$-set of G, and the *upper* $[1, k]$-*independence number* $\alpha_{[1,k]}(G)$ as the maximum cardinality of an independent $[1, k]$-set of G. Note that for $k = \Delta$, $i_{[1,\Delta]}(G)$ is the independent domination number $i(G)$ and $\alpha_{[1,\Delta]}(G)$ is the independence number $\alpha(G)$. Moreover, the following chain of inequality, that can be found in [5], relating the $[1, k]$-independence numbers holds for any graph.

$$i = i_{[1,\Delta]} \leq i_{[1,\Delta-1]} \leq \ldots \leq i_{[1,k_0]} \leq \alpha_{[1,k_0]} \leq \ldots \leq \alpha_{[1,\Delta-1]} \leq \alpha_{[1,\Delta]} = \alpha.$$

Upper bounds for $i_{[1,k]}(G)$ and $\alpha_{[1,k]}(G)$ have been established in [5] for graphs G having independent $[1, k]$-sets, and a characterization of trees having sets such sets has been provided. In addition, it has been shown that the problem of determining $i_{[1,k]}(G)$ is NP-complete for each $k \geq 1$. It is worth noting that, to our knowledge, with the exception of the few upper bounds established in [5], no other work has yet been done on the upper $[1, k]$-independence number, and this is the main motivation for our study of this parameter.

In this paper, we are interested in continuing the study of independent $[1, k]$-sets. We start by giving a constructive characterization of graphs having such sets. In particular for split graphs, a necessary and sufficient condition for the existence of such sets is given, leading to the determination of the exact values of the upper $[1, k]$-independence number. Moreover, upper bounds on $\alpha_{[1,k]}(G)$ are established, and a characterization of extremal graphs of a Nordhaus-Gaddum bound for $\alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G})$ is provided, where $\overline{G}$ is the complement graph of G. Finally, we show that the problem of computing $\alpha_{[1,k]}(G)$ is NP-complete for bipartite and chordal graphs G for all k with $1 < k < \Delta$.

2. Graphs having independent [1, k]-sets

Bange et al. [1] gave a constructive characterization of trees having an independent $[1, 1]$-set as well as a linear time algorithm to find such a set. Restricted again the class of trees, Chellali et al. [5] gave a constructive characterization of trees having independent $[1, k]$-sets for $k \geq 2$. But in this section, we consider any graphs by characterizing those which have independent $[1, k]$-sets.

We say that a graph G is a *pk-connected bipartite graph,* abbreviated *pk-CBG*, if every vertex in one of its partite sets has degree at most k, and we call such a partite set a *pk-set*. Clearly, if G is a connected bipartite graph with partite sets X and Y such that Y is a *pk*-set, then X is an independent $[1, k]$-set of G.

We now define the family $\mathcal{F}_k$ of graphs G obtained first from the union of t *pk*-CBGs $G_1, G_2, ..., G_t$, where for each $i \in \{1, ..., t\}$, the partite sets of G_i are A_i and B_i with B_i as a *pk*-set of G_i, and then adding possibly edges between vertices of B_i's.

Theorem 2.1. *For any integer $k \geq 1$, a graph G has an independent $[1, k]$-set if and only if $G \in \mathcal{F}_k$.*

Proof. Assume that $G \in \mathcal{F}_k$ for some k. If $t = 1$, then $G = G_1$ and since B_1 is a *pk*-set, A_1 is an independent $[1, k]$-set. Hence let $t \geq 2$. By definition, G is obtained from the union of $G_1, G_2, ..., G_t$, where each G_i is a *pk*-CBG having partite sets A_i and B_i, with B_i as a *pk*-set, by adding edges (possibly none) joining vertices of *pk*-sets. Let $A = \cup_i A_i$ and $B = \cup_i B_i$. Then A is an independent set and since for every y in B, $1 \leq \left|N(y) \cap A\right| \leq k$, the set A is an independent $[1, k]$-set of G.

Conversely, assume that G is a graph having an independent $[1, k]$-set S for some integer $k \geq 1$. If $V(G) - S$ is independent, then each component of G is a *pk*-CBG, and so $G \in \mathcal{F}_k$. Hence, we may assume that $V(G) - S$ contains at least one edge, and let F be the set of edges in the subgraph induced by $V(G) - S$.

Then removing all edges of F from G produces a graph H in which each component is a pk-CBG. Let G_i be a component of H, $A_i = V(G_i) \cap S$ and $B_i = V(G_i) \cap (V(G) - S)$. Since A_i is an independent $[1,k]$-set of G_i and B_i is independent, we deduce that A_i and B_i are the partite sets of G_i, and thus G_i is pk-CBG. Now, since G_i is an arbitrary component of H and all endvertices of the removed edges belong to $V(G) - S$, by construction $G \in \mathcal{F}_k$. □

Restricted to split graphs, we provide a necessary and sufficient condition for the existence of independent $[1,k]$-sets, and as a consequence we determine the exact values of the upper $[1,k]$-independence number.

Theorem 2.2. *Let G be a connected split graph whose vertex set is partitioned into a clique Q and an independent set I such that Q is minimal and $|Q| = q$. Then G has an independent $[1,k]$-set if and only if one of the following conditions holds:*

1. $\Delta \leq q + k - 1$.
2. *There exists a vertex v in Q such that* $\left|N_G(y) \cap (Q \cup (I - N_G(v)))\right| \leq k + q - 2$ *for all* $y \in Q - \{v\}$.

Proof. Assume that G has an independent $[1,k]$-set S. Clearly, $|Q \cap S| \leq 1$ and a vertex of maximum degree belongs to Q. Now if $Q \cap S = \emptyset$, then $S = I$, and thus $1 \leq \left|N_G(y) \cap I\right| \leq k$ for all $y \in Q$. Since each vertex of Q has exactly $q - 1$ neighbors in Q, $d_G(y) \leq q + k - 1$ for all $y \in Q$, and thus $\Delta \leq q + k - 1$. Assume now that $|Q \cap S| = 1$ and let $v \in Q \cap S$. Hence $S = (I - N_G(v)) \cup \{v\}$ and $V - S = (Q - \{v\}) \cup (N_G(v) \cap I) = N_G(v)$. Since $\left|N_G(y) \cap S\right| \leq k$ for all $y \in N_G(v)$, we deduce that $\left|N_G(y) \cap (Q \cup (I - N_G(v)))\right| \leq k + q - 2$ for all $y \in Q - \{v\}$.

Conversely, if $\Delta \leq q + k - 1$, then $d_G(y) \leq q + k - 1$ for all $y \in Q$ and thus $\left|N_G(y) \cap I\right| \leq k$ for all $y \in Q$. Since Q is minimal, $\left|N_G(y) \cap I\right| \geq 1$ for all $y \in Q$, the set I is an independent $[1,k]$-set. Now assume there exists a vertex v in Q such that $\left|N_G(y) \cap (Q \cup (I - N_G(v)))\right| \leq k + q - 2$ for all $y \in Q - \{v\}$. We only need to show that $I^* = (I - N_G(v)) \cup \{v\}$ is a $[1,k]$-set, since I^* is independent. Every vertex in $N_G(v) \cap I$ has exactly one neighbor in I^* which is v, and every vertex in $Q - \{v\}$ is adjacent to v and has at most k neighbors in I^*. Therefore I^* is an independent $[1,k]$-set, and the proof is complete. □

As a consequence of Theorem 2.2, the exact value of $\alpha_{[1,k]}(G)$ is determined according to which of the two conditions given in Theorem 2.2 will be fulfilled,

Corollary 2.3. *Let G be a connected split graph of order $n \geq 2$ whose vertex set is partitioned into a clique Q and an independent set I such that $|Q| = q$ and $|I| = p$. Then*

1. *If* $\Delta \leq q + k - 1$, *then* $\alpha_{[1,k]}(G) = p$.
2. *If* $\Delta > q + k - 1$ *and there exists a subset Q' of Q such that every vertex $v \in Q'$ satisfies* $\left|N_G(y) \cap (Q \cup (I - N_G(v)))\right| \leq k + q - 2$ *for all* $y \in Q - \{v\}$, *then* $\alpha_{[1,k]}(G) = n - \min_{v \in Q'} d_G(v)$.

Through the following example of a split graph, we can see the different cases that arise for Corollary 2.3. Consider the split graph G obtained from a complete bipartite graph $K_{4,4}$ minus a perfect matching, by adding all edges between the vertices belonging to the same partite set, see Figure 1.

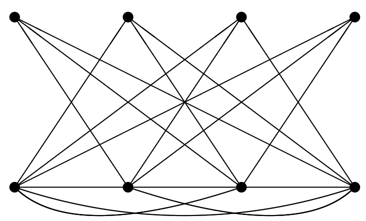

Figure 1: A split graph G.

Clearly, $\Delta = 6$ and G has a clique Q and an independent set I each of size 4. Moreover, for $k \geq 3$, Condition (1) of Corollary 2.3 is fulfilled and thus $\alpha_{[1,k]}(G) = 4$, while for $k = 2$, Condition (2) is fulfilled and thus $\alpha_{[1,k]}(G) = 2$. But when $k = 1$, G has no an independent $[1,1]$-set.

3. Bounds on the upper $[1,k]$-independence number

In [5], Chellali et al. determined upper bounds on the independence $[1,k]$-domination number for graphs having independent $[1,k]$-sets. Here we give two bounds in terms of n, m, Δ, δ and k.

Let $G=(X,Y,E)$ be a bipartite graph X and Y, and let p,q two positive integers. Then G is said to be *q-semiregular* if the degree of every vertex in one partite set of G is q, while G is said to be *(p,q)-biregular* if every vertex in X has degree p and every vertex in Y has degree q. Let $\mathcal{H}_q(X,Y)$ be the set of all q-semiregular bipartite graphs $G=(X,Y,E)$ and let $\mathcal{H}_q^p(X,Y)$ be the set of all (p,q)-biregular graphs. Clearly, $\mathcal{H}_q^p(X,Y)$ is a subclass of $\mathcal{H}_q(X,Y)$.

Theorem 3.1 ([5]). *Let G be a graph of order n with minimum degree $\delta \geq 1$. If G has an independent $[1,k]$-set, then*

$$\alpha_{[1,k]}(G) \leq \frac{kn}{k+\delta},$$

with equality if and only if G is obtained from a graph of $\mathcal{H}_k^\delta(X,Y)$ by adding possibly edges between vertices of Y in such a way that $d_G(v) \geq \delta$ for all $v \in Y$.

Corollary 3.2 ([5]). *If G is a graph of order n with minimum degree $\delta = 1$, and G has an independent $[1,k]$-set, then $\alpha_{[1,k]}(G) = \dfrac{kn}{k+1}$ if and only if G is the corona $H \circ \overline{K_k}$, where H is any graph.*

Theorem 3.3 ([5]). *Let G be a graph of order n and size m. If G has an independent $[1,k]$-set, then*

$$\alpha_{[1,k]}(G) \leq \frac{2n+2k-1-\sqrt{8m(G)+(2k-1)^2}}{2},$$

with equality if and only if G is obtained from a split graph, whose vertex set is partitioned into a clique Q and an independent set I such that $|N_G(v) \cap I| = k$ for all $v \in Q$.

Corollary 3.4. *If G is a connected bipartite graph of order n and size m, and G has an independent $[1,k]$-set, then $\alpha_{[1,k]}(G) = \frac{2n+2k-1-\sqrt{8m+(2k-1)^2}}{2}$ if and only if $G \cong K_{1,k}$ or $S_{k,k}$.*

Theorem 3.5. *Let G be a graph of order n and minimum degree δ, and let k be a positive integer. If G has an independent $[1,k]$-set, then*

$$\alpha_{[1,k]}(G) \leq n-\delta+k-1,$$

with equality if and only if $k=1$ and $G \cong \overline{K_n}$ or $G \cong K_n$.

Proof. If $k \geq \delta+1$, then $n-\delta+k-1 \geq n$, and obviously $\alpha_{[1,k]}(G) \leq n \leq n-\delta+k-1$. If further $\alpha_{[1,k]}(G) = n-\delta+k-1$, then $\alpha_{[1,k]}(G) = n$ and thus $\delta = 0$, $k=1$ and $G = \overline{K_n}$. In the following we can assume that $k \leq \delta$. Let S be a maximum independent $[1,k]$-set in G, and let v be any vertex in $V-S$. Using the facts $d_S(v)+d_{V-S}(v) = d_G(v)$, $d_S(v) \leq k$ and $d_{V-S}(v) \leq |V-S|-1$, we deduce that $d_G(v)-k \leq n-|S|-1$. Moreover, since $\delta \leq d_G(v)$ and $\alpha_{[1,k]}(G) = |S|$, $\delta-k \leq n-\alpha_{[1,k]}(G)-1$ leading the desired upper bound. If further, $\alpha_{[1,k]}(G) = n-\delta(G)+k-1$, then we have equality throughout the previous inequality chain. In particular, for each vertex $v \in V-S$ we have $d_S(v) = k$, $d_{V-S}(v) = |V-S|-1$ and $d_G(v) = \delta$. It follows that $V-S$ induces a complete subgraph. Also, since for every vertex u in S, $d_S(u) = 0$, we deduce that $\delta \leq d_G(u) \leq |V-S| = \delta-k+1$, and thus $k \leq 1$. Consequently, $k=1$ and $d_G(u) = |V-S| = \delta$. Clearly if S contains more than one vertex, then the degree of every vertex in $V-S$ will be greater than δ, leading to a contradiction. Hence $|S|=1$ and G is a complete graph of order n.

The converse is obvious. □

The following corollary is immediate from Theorem 3.5.

Corollary 3.6. *Let G be a graph of order n and minimum degree δ, and let k be a positive integer. If G has an independent $[1,k]$-set, then*

$$\alpha_{[1,k]}(G) \leq n - \delta + k - 2,$$

whenever $G \notin \{\overline{K_n}, K_n\}$ or $k \geq 2$.

Theorem 3.7. *Let G be a graph of order n, size m and maximum degree Δ, and let k be a positive integer with $k \leq \Delta$. If G has an independent $[1,k]$-set, then*

$$\alpha_{[1,k]}(G) \leq n - \frac{2m}{\Delta + k},$$

with equality if and only if G is obtained from a graph of $\mathcal{H}_k(X,Y)$ by adding possibly edges between vertices of Y in such a way that each vertex in Y has degree Δ.

Proof. Let S be an $\alpha_{[1,k]}(G)$-set. Then $m(G[S]) = 0$, $m(V-S,S) = \sum_{y\in V-S} d_S(y)$ and $m(G[V-S]) = \frac{1}{2}\sum_{y\in V-S} d_{V-S}(y)$. Therefore

$$\begin{aligned} m &= m(V-S,S) + m(G[V-S]) \\ &= \sum_{y\in V-S} d_S(y) + \frac{1}{2}\sum_{y\in V-S} d_{V-S}(y) \\ &= \frac{1}{2}\sum_{y\in V-S}(2d_S(y) + d_{V-S}(y)) = \frac{1}{2}\sum_{y\in V-S}(d_S(y) + d_G(y)). \end{aligned}$$

Since $d_S(y) \leq k$ and $d_G(y) \leq \Delta$ for all $y \in V-S$, we obtain that

$$\begin{aligned} 2m &\leq \sum_{y\in V-S}(k+\Delta) \\ &\leq (k+\Delta)(n-|S|), \end{aligned}$$

that is $|S| \leq n - \frac{2m}{\Delta+k}$.

If further, $\alpha_{[1,k]}(G) = n - \frac{2m}{\Delta+k}$, then we have equality throughout the previous inequality chain. In particular, $d_S(y) = k$ and $d_G(y) = \Delta$ for all $y \in V-S$. Since S is independent, by posing $S = X$ and $Y = V-S$, we deduce that G is obtained from a graph of $\mathcal{H}_k(X,Y)$ by adding possibly edges between vertices of Y in such a way that each vertex in Y has degree Δ.

Conversely, assume that G is obtained from a graph of $\mathcal{H}_k(X,Y)$ by adding possibly edges between vertices of Y in such a way that each vertex in Y has degree Δ. It follows from the definition that X is an independent $[1,k]$-set of G and thus $\alpha_{[1,k]}(G) \geq |X|$. On the other hand, since each vertex in Y has maximum degree, $G[Y]$ is regular of degree $\Delta - k$. Therefore

$$\begin{aligned} m &= m(Y,X) + m(G[Y]) \\ &= k|Y| + \frac{(\Delta-k)|Y|}{2} = \frac{(\Delta+k)|Y|}{2} \\ &= \frac{(\Delta+k)(n-|X|)}{2}, \end{aligned}$$

and thus $|X| = n - \frac{2m}{\Delta+k}$. Consequently, $\alpha_{[1,k]}(G) \geq |X| = n - \frac{2m}{\Delta+k}$, and the equality follows. □

We note that since the bounds in Theorems 3.1 and 3.7 are the same for regular graphs, extremal regular graphs attaining these bounds are the same too. Further, by Theorem 3.7, we derive the following corollaries.

Corollary 3.8. *Let G be a connected graph of order n, size m and maximum degree Δ. Then $\alpha_{[1,\Delta]}(G) = \alpha(G) = n - \frac{m}{\Delta}$ if and only if $G \in \mathcal{H}_\Delta(X,Y)$.*

Corollary 3.9. *Let T be a tree of order n and maximum degree Δ, and let k be positive integer with $k \leq \Delta$. If T has an independent $[1,k]$-set, then $\alpha_{[1,k]}(T) = n - \frac{2m}{\Delta+k}$ if and only if either $k = \Delta$ and $T \in \mathcal{H}_k(X,Y)$ or $k = \Delta - 1$ and T is obtained from a forest $F \in \mathcal{H}_k(X,Y)$ by adding edges between vertices of Y so that Y induces a 1-regular graph.*

4. Nordhaus-Gaddum type inequality

In this section, we present a relation Nordhaus-Gaddum type inequality for the upper $[1,k]$-independence number.

Theorem 4.1. *Let G be a graph of order $n \geq 2$, and let k be positive integer. If G and $\overline{G}$ have independent $[1,k]$-sets, then*

$$3 \leq \alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G}) \leq n + 1.$$

Moreover, the lower bound is sharp if and only if G or $\overline{G}$ has a support vertex of degree $n-1$, while the upper bound is sharp if and only if G is a split graph, whose vertex set is partitioned into a clique Q and an independent set I such that

1. *every vertex in Q has degree at least $|Q|$ and at most $|Q| + k - 1$.*
2. *I has a vertex v of degree $|Q|$ and every vertex in I has degree at least $|Q| - k + 1$.*

Proof. To prove the lower bound, we only need to show that if $\alpha_{[1,k]}(G) = 1$, then $\alpha_{[1,k]}(\overline{G}) \geq 2$. Suppose $\alpha_{[1,k]}(G) = 1$, and let $S = \{x\}$ be a $\alpha_{[1,k]}(G)$-set. Since every vertex in $V - S$ has a neighbor in S, $d_G(x) = n - 1$, and thus x is isolated in $\overline{G}$. Hence x belongs to every independent $[1,k]$-set of $\overline{G}$, and since $n \geq 2$, we deduce that $\alpha_{[1,k]}(\overline{G}) \geq 2$.

Now assume that $\alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G}) = 3$. Then, without lost of generality, $\alpha_{[1,k]}(G) = 1$ and $\alpha_{[1,k]}(\overline{G}) = 2$. As seen above, G has a vertex of degree $n-1$, namely x. Moreover, since x is in every independent $[1,k]$-set of $\overline{G}$, we conclude that the subgraph induced by $V - \{x\}$ in $\overline{G}$ has a maximum $[1,k]$-set of size one, and thus some vertex y in $\overline{G}$ is adjacent to all vertices of $V(\overline{G})$ but x, that is y is a leaf in G whose support vertex is certainly x. The converse is obvious.

To prove the upper bound, let S and S' be maximum independent $[1,k]$-sets, respectively, in G and $\overline{G}$. Clearly, S' induces a clique in G, and thus $|S \cap S'| \leq 1$. It follows that

$$\begin{aligned} n &= |S| + |S'| - |S' \cap S| + |V - (S' \cup S)| \\ &\geq |S| + |S'| - 1 \geq \alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G}) - 1, \end{aligned}$$

and the upper bound follows.

Assume now that $\alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G}) = n + 1$. Then, according to the previous inequality chain, $V - (S' \cup S) = \emptyset$ and $|S \cap S'| = 1$. Hence $V = S' \cup S$ and $S \cap S'$ consists of unique vertex, say v. Since S' induces a clique in G, we therefore have G a split graph, whose vertex set is partitioned into an independent set $I = S$ and a clique $Q = S' - \{v\}$ such that v is adjacent to every vertex in Q. Since I is a $[1,k]$-set of G, every vertex in Q has degree at least $|Q|$ and at most $|Q| + k - 1$ in G. Moreover, since $Q \cup \{v\}$ is a $[1,k]$-set of $\overline{G}$, every vertex in $I - \{v\}$ has degree at least $|I| - 1$ and at most $|I| + k - 2$ in $\overline{G}$. Therefore, every vertex in $I - \{v\}$ has degree at most $|Q|$ and at least $|Q| - k + 1$ in G.

Conversely, suppose that G is a split graph with clique Q and independent set I fulfilling conditions (1) and (2). Clearly, from (1) every vertex in Q has at least one neighbor and at most k neighbors in I in G, and from (2) every vertex in $I - \{v\}$ is adjacent to v and has at least one and at most k neighbors in Q in $\overline{G}$, meaning that I and $Q \cup \{v\}$ are independent $[1,k]$-sets, respectively, of G and $\overline{G}$. Hence $\alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G}) \geq |I| + |Q \cup \{v\}| = n + 1$, and the equality follows from the upper bound established earlier. □

From Theorem 4.1, we have the following immediate corollaries, for the cases $k \geq \max\left\{\Delta(G), \Delta\left(\overline{G}\right)\right\}$ and $k = 1$.

Corollary 4.2. *If G is a graph of order $n \geq 2$ such that G and $\overline{G}$ have independent $[1,1]$-sets, then $\alpha_{[1,1]}(G) + \alpha_{[1,1]}(\overline{G}) = n + 1$ if and only if $G = K_n$.*

Corollary 4.3. *If G is a graph of order $n \geq 2$, and $k \geq \max\left\{\Delta(G), \Delta\left(\overline{G}\right)\right\}$ an integer, then $\alpha_{[1,k]}(G) + \alpha_{[1,k]}(\overline{G}) = n + 1$ if and only if G is a split graph, whose vertex set is partitioned into a clique Q and an independent set I, such that I has a vertex with degree $|Q|$.*

5. Complexity results

In [5], Chellali et al. showed that the decision problem corresponding to the problem of computing the lower $[1,k]$-independence number is $\mathcal{NP}$-complete for an arbitrary graph for each integer $k \geq 1$. We recall that for $k = 1$ (respectively, $k \geq \Delta$), independent $[1,k]$-sets coincide with efficient dominating sets (respectively, independent dominating sets) for which the corresponding decision problems are $\mathcal{NP}$-complete even for bipartite graphs (see [11] and [8]). Furthermore, since all efficient dominating sets have the same cardinality, it is no longer interesting to consider the case $k = 1$, as this has already been done.

MAXIMUM INDEPENDENT $[1,k]$-SET
Instance: A graph $G = (V, E)$ and a positive integer $p \leq |V|$.
Question: Does G have an independent $[1,k]$-set of cardinality at least p?

We will show that MAXIMUM INDEPENDENT $[1,k]$-SET remains $\mathcal{NP}$-complete for bipartite graphs and chordal graphs by reducing to it the special case of Exact Cover by 3-sets (X3C), to which we refer as X3C3. Note that the $\mathcal{NP}$-completeness of X3C3 was proven in 2008 by Hickey et al. [9].

X3C3
Instance: A set of elements X with $|X| = 3q$, and a collection C of $3q$ 3-element subsets of X, such that each element occurs in exactly 3 members of C.
Question: Does C contain an exact cover for X, i.e. does there exist a subcollection $C' \subset C$ such that every element of X occurs in exactly one member of C'?

Theorem 5.1. *The* MAXIMUM INDEPENDENT $[1,k]$-SET *is $\mathcal{NP}$-complete for bipartite graphs for all $k \geq 2$.*

Proof. It is easy to verify a "yes" instance of *MAXIMUM INDEPENDENT* $[1,k]$-*SET* in polynomial time, that is, for a graph G, a positive integer p and a set S of G with $|S| \geq p$, by checking that S is independent and every vertex in $V - S$ is adjacent to at least one vertex and not more than k vertices in S.

Next, we show how to construct a bipartite graph G and a positive integer p from any instance X and C of X3C3 so that C has a solution if and only if G has an independent $[1,k]$-set of cardinality at least p. Let $X = \left\{x_1, x_2, ..., x_{3q}\right\}$ and $C = \left\{C_1, C_2, ..., C_{3q}\right\}$ be an arbitrary instance of X3C3, where $p = (2k+1)q$. The construction of the bipartite graph G is as follows: for each $C_j \in C$ we create a star $K_{1,k}$ with center vertex labeled c_j, and let $Y = \{c_1, c_2, ..., c_{3q}\}$. For each element $x_i \in X$ we create a vertex x_i. To complete the construction, we add edges $x_i c_r$ if $x_i \in C_r$, see Figure 2.

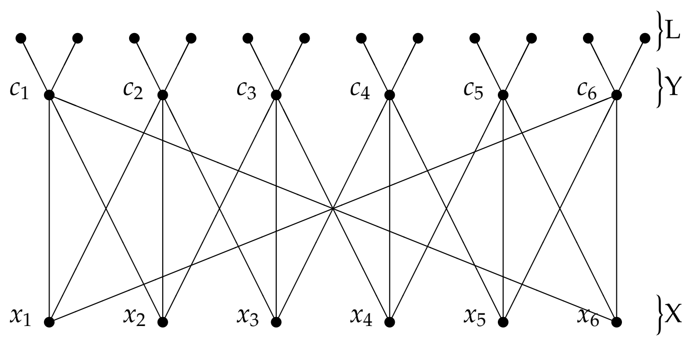

Figure 2: A construction of bipartite graph G for $q = 2$ and $k = 2$.

Suppose that the instance X and C of X3C3 has a solution C'. We construct an independent $[1,k]$-set I as follows: for each $C_i \in C'$, put c_i in I and for each $C_i \notin C'$, we put in I all leaf neighbors of c_i. Observe that since C' is a solution for X3C3, each x_i has exactly one neighbor in I and exactly q vertices of Y are in I. Moreover, for every i, either c_i belongs to I or its k leaf neighbors. Therefore, I is an independent $[1,k]$-set for G of cardinality $q + 2qk = p$.

Conversely, assume that G has an independent $[1,k]$-set, say I, of cardinality at least $p = q\,(2k+1)$. First of all, note that I does not contain any vertices of X. Indeed, suppose that I contains some vertex x_i and let c_j be a neighbor of x_i. Then $c_j \notin I$ and thus all k leaf neighbors of c_j belong to I leading that $\left|N(c_j) \cap I\right| \geq k+1$, contradicting the fact that I is an independent $[1,k]$-set. Hence $X \cap I = \emptyset$, and thus $Y \cap I \neq \emptyset$ (to dominate the x_i's). Let $A = Y \cap I$ and observe that $|A| \geq q$ (since every vertex of Y has exactly 3 neighbors in X). On the other hand, I contains all leaf neighbors of any c_i which is not in I, and thus $|I| = k(3q - |A|) + |A|$. Combining this with the fact that $|I| \geq p = q(2k+1)$ we deduce that $|A| \leq q$. It follows that $|A| = q$, and hence every vertex of X is adjacent to exactly one vertex of A. Therefore X3C3 has a solution $C' = \{C_j : c_j \in A\}$. □

Looking at the above proof we can observe that it also works for chordal graphs by adding edges between the x_i's so that X induces a complete graph. Therefore we can state the following.

Corollary 5.2. *The* INDEPENDENT $[1,k]$-*SET is* $\mathcal{NP}$*-complete for chordal graphs for all* $k \geq 2$.

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