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On approximation operators involving Tricomi function of ν^{th}-order

Nusrat Raza[a], **Manoj Kumar**[b], **M. Mursaleen**[c,d,*]

[a] *Mathematics section, Women's College, Aligarh Muslim University, Aligarh, Inida-202002*
[b] *Department of Mathematics, Aligarh Muslim University, Aligarh, India-202002*
[c] *Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India*
[d] *Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*

Abstract. The primary objective of this research article is to introduce the Szász-operators [22] by initiating the "ν^{th}-order Tricomi function" and study the approximation properties of these operators by leveraging Korovkin's theorem alongside a classical method based on the modulus of continuity, we explore the convergence properties of this operator within Lipschitz-type spaces. We rigorously demonstrate the rate of convergence and establish key properties of the proposed operator. Additionally, we provide a detailed illustration of the convergence behavior through various graphical representations and error-estimation tables based on numerical examples, offering a comprehensive understanding of the operator's effectiveness.

1. Introduction

Approximation theory has been essential in the development of computer methods by bridging the gap between theoretical and applied mathematics. The main objective is to optimize the approximation of functions by employing simpler or more accessible functions and procedures that rely on contemporary approximation techniques. Positive approximation approaches are crucial in this theory and naturally emerge in diverse issues related to the approximation of continuous functions. By providing tools and methods for approximating complex functions, approximation theory helps bridge the gap between theoretical mathematics and practical problem-solving, making it an essential area of study in both pure and applied mathematics.

Approximation theory has progressed from the initial geometric approximations made by ancient Greeks to the polynomial techniques developed by Newton and Lagrange in the 17th and 18th centuries. It further advanced with Fourier's fundamental contributions to the Fourier series in the 19th century. Significant progress in the 19th century involved Chebyshev's invention of orthogonal polynomials and Weierstrass's demonstration that polynomials can approximate continuous functions. Hilbert and Banach developed functional analysis in the 20th century, which offered a rigorous foundation for approximation in spaces

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* Corresponding author: M. Mursaleen
Email addresses: nraza.maths@gmail.com (Nusrat Raza), mkumar5@myamu.ac.in (Manoj Kumar), mursaleenm@gmail.com (M. Mursaleen)

of infinite dimensions. Korovkin's theorem, developed by P. P. Korovkin in the 1950s, was a major breakthrough that extended the understanding of convergence in approximation operators and offered valuable tools for studying the convergence of different approximation approaches. This advancement connected abstract ideas with real-world uses, and current investigations are further building upon these fundamental principles in areas like machine learning and data science. Numerous researchers have explored the convergence rates and properties of Korovkin-type approximation, as evidenced by studies such as [9, 10, 21, 24]. Important progress in approximation theories can be attributed to publications such as [1, 3, 5–7, 12–18, 20].

The field of approximation theory greatly benefits from the utilization of Szász-operators [22], which expand Bernstein operators to encompass infinite intervals. Szász introduced a series of positive linear operators:

$$\tilde{\mathcal{S}}_\eta(\hat{g}, y) = e^{-\eta y}\sum_{r=0}^{\infty}\frac{(\eta y)^r}{r!}\hat{g}\left(\frac{r}{\eta}\right), \tag{1}$$

where $y \in [0,\infty)$ and $\hat{g} \in C[0,\infty)$, ensuring the series in equation (1) converges. Current research in approximation theory has significantly expanded the study of Szász-operators through the use of specialized polynomials. Szász-operators, originally designed for function approximation, have been generalized in various ways to enhance their applicability and effectiveness. These generalizations involve incorporating specialized polynomials, which provide new sequences of operators with improved or tailored properties.

It is advantageous to establish precise terminology and clearly emphasize certain outcomes within this framework.

Definition 1.1. *[11] The modulus of continuity, represented as $\omega(f;\sigma)$, is defined for any function f that is uniformly continuous on the interval $[0,\infty)$ and for any positive value of σ*

$$\omega(f;\sigma) := \sup_{\substack{s,t\in[0,\infty)\\ |s-t|\le\sigma}} |f(s) - f(t)|. \tag{2}$$

Observe that for any $\sigma > 0$ and for every $s, t \in [0,\infty)$, we can express it as follows

$$|f(s) - f(t)| \le \omega(f;\sigma)\left(\frac{|s-t|}{\sigma} + 1\right). \tag{3}$$

Definition 1.2. *[23] Let f be any function within the space of real-valued bounded and uniformly continuous functions $C_B[0,\infty)$. The second modulus of continuity for f is defined as:*

$$\omega_2(f;\sigma) := \sup_{0<x\le\sigma} \|f(.+2x) - 2f(.+x) + f(.)\|, \tag{4}$$

with the associated norm

$$\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(y)|. \tag{5}$$

Lemma 1.3. *[4] Let $\{\mathfrak{L}_n\}_{n\ge0}$ be a sequence of linear positive operators, with the property $\mathfrak{L}_n(1;y) = 1$ and $\tilde{g} \in C^2[0,a]$. Then:*

$$|\mathfrak{L}_n(\tilde{g};y) - \tilde{g}(y)| \le \|\tilde{g}'\|\sqrt{\mathfrak{L}_n\left((\chi-y)^2;y\right)} + \frac{1}{2}\|\tilde{g}''\|\mathfrak{L}_n\left((\chi-y)^2;y\right). \tag{6}$$

Definition 1.4. *[25] For $g \in C[a,b]$, the second-order Steklov function of g is defined by*

$$g_h(y) := \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right)g(h;y+t)dt, \quad x \in [a,b], \tag{7}$$

where $g(h;.) : [a-h, b+h] \to \mathbb{R}, h > 0$ is given by

$$g(h;y) = \begin{cases} L_-(y), & a-h \le y \le a \\ g(y), & a \le y \le b \\ L_+(y), & b \le y \le b+h \end{cases} \tag{8}$$

and L_-, L_+ are the linear best approximations to f on the mentioned intervals.

Lemma 1.5. *[25] Let f_h denote the second-order Steklov function of f, where $f \in C[a,b]$ and h is in the interval $\left(0, \frac{b-a}{2}\right)$. The following inequalities are then valid:*

$$\|f_h - f\| \le \frac{3}{4}\omega_2(f;h), \tag{9}$$

$$\|f_h''\| \le \frac{3}{2h^2}\omega_2(f;h). \tag{10}$$

Additionally, the Landau inequality is expressed as:

$$\|f_h'\| \le \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\|. \tag{11}$$

Special functions play a pivotal role in applied mathematics, serving as crucial tools for modeling and solving complex problems across diverse disciplines. Among these, the Tricomi function is particularly noteworthy for its applications in approximation theory. The Tricomi function, which arises in the context of differential equations and special function theory, offers valuable insights into the behavior of solutions and approximations. In approximation theory, it facilitates the construction of operators that approximate functions with desired properties, leveraging its unique mathematical characteristics to achieve more accurate and efficient approximations. The study and application of special functions like the Tricomi function thus enhance our ability to solve intricate problems and improve approximation methods in various scientific and engineering fields.

The generating function and explicit representation for defining the ν^{th}-order Tricomi functions $C_\nu(y)$ is given by [19]:

$$\sum_{n=-\infty}^{\infty} C_\nu(y)t^n = \exp\left(t - \frac{y}{t}\right), \qquad t \ne 0 \tag{12}$$

and

$$C_\nu(y) = \sum_{r=0}^{\infty} \frac{(-1)^r y^r}{r!(\nu+r)!}. \tag{13}$$

respectively.

In this article, we develop positive linear approximation operators using the Tricomi function of ν^{th}-order on the interval $\mathbb{R}^+ \cup \{0\}$. Section 2 introduces the definition of the operators and derives their central moments. In Section 3, we establish the transformation properties and prove a global Korovkin's theorem for our operators. We then determine the rate of convergence using the modulus of continuity and examine convergence within various Lipschitz-type spaces. In Section 4, we apply our operators to approximate two functions, providing graphical comparisons between the approximated and actual functions. We also compute the absolute error for different values of η, presenting the results in both tabular and graphical forms. The programming codes for these computations were executed in WOLFRAM MATHEMATICA v13.3.1 on a MacOS X 13.2.1 x86(64-bit) processor. Finally, Section 5 offers concluding remarks and discusses potential applications.

2. Formulation of Operators

Driven by the applications discussed earlier, this section begins by constructing positive linear operators. We then derive several equalities that are used to analyze the convergence properties of these operators. To develop the positive linear operators, we alter $C_\nu(y)$ by substituting y with $-y$, yielding:

$$\mathcal{T}_\nu(yt) := C_\nu(-yt) = \sum_{r=0}^{\infty} \frac{(yt)^r}{r!(r+\nu)!}. \tag{14}$$

Utilizing the above equation, we can define the positive linear operators for $\nu, \eta \in \mathbb{N}$ as follows:

$$\mathfrak{I}_\eta^\nu(\hat{g}; y) = \frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(\nu+r)!} \hat{g}\left(\frac{r(\nu+r)}{\eta}\right), \tag{15}$$

where $\hat{g} \in E := \left\{\hat{g} \in C_B[0,\infty) : \lim_{y\to\infty} \frac{\hat{g}(y)}{1+y^2} \text{ is finite}\right\}$ and $C_B[0,\infty)$ represents the space of continuous and bounded functions defined on $[0,\infty)$. It is important to highlight that the Banach lattice E is equipped with the norm

$$\|\hat{g}\|_* := \sup_{y\in[0,\infty)} \frac{|\hat{g}(y)|}{1+y^2}. \tag{16}$$

The subsequent outcomes are demonstrated to be essential for establishing our primary conclusion:

Lemma 2.1. *The operators specified by equation* (15) *form a linear and positive sequence.*

Lemma 2.2. *The sequence of operators $\mathfrak{I}_\eta^\nu(\hat{g}; y)$ satisfies the following inequalities for $y \in [0,\infty)$:*

$$\mathfrak{I}_\eta^\nu(1; y) = 1, \tag{17}$$

$$\mathfrak{I}_\eta^\nu(t; y) = y, \tag{18}$$

$$\mathfrak{I}_\eta^\nu(t^2; y) = y^2 + 2y^2 \frac{\mathcal{T}_{\nu+1}(\eta y)}{\mathcal{T}_\nu(\eta y)} + \frac{(\nu+1)y}{\eta}, \qquad \forall \eta \in \mathbb{N}. \tag{19}$$

Proof. In view of equation (14), It is obvious that $\mathfrak{I}_\eta^\nu(1; y) = 1$.

Now, for $\hat{g}(t) = t$, equation (15) gives

$$\mathfrak{I}_\eta^\nu(t; y) = \frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(\nu+r)!} \frac{r(\nu+r)}{\eta} \tag{20}$$

which gives

$$\mathfrak{I}_\eta^\nu(t; y) = \frac{y}{\mathcal{T}_\nu(\eta y)} \sum_{r=1}^{\infty} \frac{(\eta y)^{r-1}}{(r-1)!(\nu+r-1)!}. \tag{21}$$

Using equation (14), we get $\mathfrak{I}_\eta^\nu(t; y) = y$.

Finally, for $\hat{g}(t) = t^2$

$$\mathfrak{I}_\eta^\nu(t^2; y) = \frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(\nu+r)!} \frac{r^2(\nu+r)^2}{\eta^2}. \tag{22}$$

We consider,

$$\begin{aligned} C(yt) &= \sum_{r=0}^{\infty} \frac{(yt)^r}{r!(\nu+r)!} \frac{r^2(\nu+r)^2}{\eta^2} \\ &= \frac{1}{\eta^2} \sum_{r=0}^{\infty} \frac{(yt)^r}{(r-1)!)(\nu+r-1)!} r(\nu+r) \\ &= \frac{yt}{\eta^2} \sum_{r=1}^{\infty} \frac{(yt)^{r-1}}{(r-1)!)(\nu+r-1)!} r(\nu+r) \end{aligned} \tag{23}$$

which on simplification, gives

$$C(yt) = \frac{yt}{\eta^2} \left\{ \sum_{r=0}^{\infty} \frac{(yt)^r}{r!(\nu+r)!} r(\nu+r) + 2 \sum_{r=0}^{\infty} \frac{(yt)^r}{r!(\nu+r)!} r + (\nu+1)\mathcal{T}_\nu(yt) \right\} \tag{24}$$

which, on further simplifying by using equation (14) and taking $t = 1$ and replacing y by ηy in equation (24), we arrive at

$$C(\eta y) = \frac{\eta y}{\eta^2} \left\{ (\eta y)\mathcal{T}_\nu(\eta y) + 2(\eta y)\mathcal{T}_{\nu+1}(\eta y) + (\nu+1)\mathcal{T}_\nu(\eta y) \right\}, \tag{25}$$

which on using in equation (22), we get assertion (19). □

Remark 2.3. *With the aid of Wolfram Mathematica, we can observe that*

$$\lim_{\eta\to\infty} \frac{\mathcal{T}_{\nu+1}(\eta y)}{\mathcal{T}_\nu(\eta y)} = 0. \tag{26}$$

We determine the central moments of the Tricomi operators as follows:

Lemma 2.4. *If $\Lambda_m^y(t) = (t-y)^m$. Then for $m = 2$, we have*

$$\mathfrak{I}_\eta^\nu(\Lambda_2^y(t); y) = 2y^2 \frac{\mathcal{T}_{\nu+1}(\eta y)}{\mathcal{T}_\nu(\eta y)} + \frac{(\nu+1)y}{\eta}, \qquad \forall \eta \in \mathbb{N}. \tag{27}$$

Proof. By using the linearity of the operators defined in equation (15), we have

$$\mathfrak{I}_\eta^\nu(\Lambda_2^y(t); y) = \mathfrak{I}_\eta^\nu(t^2; y) - 2y\mathfrak{I}_\eta^\nu(t; y) + y^2\mathfrak{I}_\eta^\nu(1; y). \tag{28}$$

By applying equations (17), (18), and (19), we obtain the result stated in assertion (27). □

3. Rate of Convergence

In this section, we will determine the convergence rate of the operators $\mathfrak{I}_\eta^\nu(\hat{g}; y)$ using the modulus of continuity. We will also calculate the convergence rate in various Lipschitz-type spaces.

We now establish uniform convergence using the universal Korovkin theorem as follows:

Theorem 3.1. *Let $\hat{g} \in E := \left\{ \hat{g} \in C[0,\infty) : \lim_{y\to\infty} \frac{\hat{g}(y)}{1+y^2} \text{ is finite} \right\}$ and $C[0,\infty)$ denotes the space of continuous functions defined on $[0,\infty)$. Then the sequence of operators defined in (15) converges uniformly on the compact subsets of the interval $[0,\infty)$, i.e.,*

$$\lim_{\eta\to\infty} \mathfrak{I}_\eta^\nu(\hat{g}; y) = \hat{g}(y). \tag{29}$$

Proof. Using Lemma 2.2 and Remark 2.3, we obtain

$$\lim_{n\to\infty} \mathfrak{I}^{\nu}_{\eta}(t^s; y) = y^s, \quad \text{for} \quad s = 0, 1, 2, \tag{30}$$

uniformly on the compact subsets of the interval $[0, \infty)$. Therefore, by applying the universal Korovkin theorem [2], we derive the result in assertion (29). □

From this point forward, we will present the results of the approximation.

Theorem 3.2. *Assume that $\hat{g}$ is uniformly continuous on the interval $[0, \infty)$. Then, for the sequence of operators $\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y)$, the following inequality holds:*

$$\left|\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y) - \hat{g}(y)\right| \leq 2\omega\left(\hat{g}; \sqrt{\mathfrak{I}^{\nu}_{\eta}(\Lambda^{y}_{2}(t); y)}\right), \tag{31}$$

where ω is the modulus of continuity of the function as defined in (2).

Proof. Consider

$$\left|\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y) - \hat{g}(y)\right| = \left|\frac{1}{\mathcal{T}_{\nu}(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(\nu + r)!}\left[\hat{g}\left(\frac{r(r+\nu)}{\eta}\right) - \hat{g}(y)\right]\right| \tag{32}$$

using the triangle inequality gives

$$\left|\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y) - \hat{g}(y)\right| \leq \frac{1}{\mathcal{T}_{\nu}(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left|\hat{g}\left(\frac{r(r+\nu)}{\eta}\right) - \hat{g}(y)\right|. \tag{33}$$

By applying the definition of the modulus of continuity and referring to inequality (3), we obtain:

$$\left|\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y) - \hat{g}(y)\right| \leq \frac{1}{\mathcal{T}_{\nu}(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left[1 + \frac{1}{\sigma}\left|\frac{r(r+\nu)}{\eta} - y\right|\right]\omega(\hat{g}; \sigma) \tag{34}$$

$$\left|\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y) - \hat{g}(y)\right| \leq \left[1 + \frac{1}{\sigma}\frac{1}{\mathcal{T}_{\nu}(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left|\frac{r(r+\nu)}{\eta} - y\right|\right]\omega(\hat{g}; \sigma). \tag{35}$$

In view of Cauchy-Schwartz inequality, we find

$$\sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left|\frac{r(r+\nu)}{\eta} - y\right| = \sum_{r=0}^{\infty} \sqrt{\frac{(\eta y)^r}{r!(r+\nu)!}}\sqrt{\frac{(\eta y)^r}{r!(r+\nu)!}}\left|\frac{r(r+\nu)}{\eta} - y\right| \tag{36}$$

$$\sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left|\frac{r(r+\nu)}{\eta} - y\right| \leq \left\{\sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\right\}^{\frac{1}{2}}\left\{\sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left(\frac{r(r+\nu)}{\eta} - y\right)^2\right\}^{\frac{1}{2}}. \tag{37}$$

By multiplying and dividing the right-hand side of inequality (37) by $\sqrt{\mathcal{T}_{\nu}(\eta y)}$, and considering equation (14), we obtain:

$$\sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\left|\frac{r(r+\nu)}{\eta} - y\right| \leq \mathcal{T}_{\nu}(\eta y)\left\{\mathfrak{I}^{\nu}_{\eta}((t-y)^2; y)\right\}^{\frac{1}{2}} \tag{38}$$

using above inequality in inequality (35), we find

$$\left|\mathfrak{I}^{\nu}_{\eta}(\hat{g}; y) - \hat{g}(y)\right| \leq \left[1 + \frac{1}{\sigma}\sqrt{\mathfrak{I}^{\nu}_{\eta}(\Lambda^{y}_{2}(t); y)}\right]\omega(\hat{g}; \sigma) \tag{39}$$

by choosing

$$\sigma = \sqrt{\mathfrak{I}_\eta^\nu(\Lambda_2^y(t); y)}$$

we get assertion (31). $\square$

Now, for $0 < \alpha \leq 1$ and $x_1, x_2 \in [0, \infty)$, Let's introduce the following class of the functions:

$$Lip_{\mathcal{M}}^{(\alpha)} := \{\phi : |\phi(x_1) - \phi(x_2)| \leq \mathcal{M}|x_1 - x_2|^\alpha\}. \tag{40}$$

Theorem 3.3. *Suppose $\phi \in Lip_{\mathcal{M}}^{(\alpha)}$. In that case,*

$$\left|\mathfrak{I}_\eta^\nu(\phi; y) - \phi(y)\right| \leq \mathcal{M}\left[\mathfrak{I}_\eta^\nu(\Lambda_2^y(t); y)\right]^{\frac{\alpha}{2}}. \tag{41}$$

Proof. Since, $\mathfrak{I}_\eta^\nu(\hat{g}; y)$ is positive linear operator and $\phi \in Lip_{\mathcal{M}}^{(\alpha)}$, therefore in view of equation (40), we obtain

$$\left|\mathfrak{I}_\eta^\nu(\phi; y) - \phi(y)\right| = \left|\mathfrak{I}_\eta^\nu(\phi(t) - \phi(y); y)\right| \leq \mathfrak{I}_\eta^\nu(|\phi(t) - \phi(y)|; y) \leq \mathcal{M}\mathfrak{I}_\eta^\nu(|t - y|^\alpha; y) \tag{42}$$

which applying Hölder's inequality to the right-hand side yields

$$\mathfrak{I}_\eta^\nu(|t-y|^\alpha; y) = \frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!} \left|\frac{r(r+\nu)}{\eta} - y\right|^\alpha \tag{43}$$

$$\mathfrak{I}_\eta^\nu(|t-y|^\alpha; y) = \frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \left\{\frac{(\eta y)^r}{r!(r+\nu)!}\right\}^{\frac{2-\alpha}{2}} \left\{\frac{(\eta y)^r}{r!(r+\nu)!}\right\}^{\frac{\alpha}{2}} \left|\frac{r(r+\nu)}{\eta} - y\right|^\alpha \tag{44}$$

$$\mathfrak{I}_\eta^\nu(|t-y|^\alpha; y) \leq \frac{1}{\mathcal{T}_\nu(\eta y)} \times [\mathcal{T}_\nu(\eta y)]^{\frac{2-\alpha}{2}} \left\{\frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!}\right\}^{\frac{2-\alpha}{2}} \times [\mathcal{T}_\nu(\eta y)]^{\frac{\alpha}{2}} \left\{\frac{1}{\mathcal{T}_\nu(\eta y)} \sum_{r=0}^{\infty} \frac{(\eta y)^r}{r!(r+\nu)!} \left(\frac{r(r+\nu)}{\eta} - y\right)^2\right\}^{\frac{\alpha}{2}}. \tag{45}$$

In view of equations (15) and (27), the above inequality gives

$$\mathfrak{I}_\eta^\nu(|t-y|^\alpha; y) \leq [\mathfrak{I}_\eta^\nu(1; y)]^{\frac{2-\alpha}{2}} \left[\mathfrak{I}_\eta^\nu(\Lambda_2^y(t); y)\right]^{\frac{\alpha}{2}}, \tag{46}$$

which, by applying inequalities (42) and (46), leads to the result in assertion (41). $\square$

Theorem 3.4. *Assume that ζ is a continuous function on $[0, \infty)$. Then the following holds:*

$$\left|\mathfrak{I}_\eta^\nu(\zeta; y) - \zeta(y)\right| \leq \frac{3}{2}\left(1 + \frac{a}{2} + \frac{h^2}{2}\right)\omega_2(\zeta, h) + \frac{2h^2}{a}\|\zeta\|, \tag{47}$$

where

$$h := \Omega_n(y) = \left\{\mathfrak{I}_\eta^\nu(\Lambda_2^y(t); y)\right\}^{\frac{1}{4}}$$

and ω_2 is the second modulus of the continuity of the function ζ defined in (4).

Proof. Let f_h denote the second-order Steklov function of the function ζ. Then, we have

$$\left|\Im_\eta^\nu(\zeta;y)-\zeta(y)\right| \le \left|\Im_\eta^\nu(|\zeta-f_h|;y)\right| + \left|\Im_\eta^\nu(f_h;y)-f_h(y)\right| + \left|f_h(y)-\zeta(y)\right| \tag{48}$$

using equation (17), we have

$$\left|\Im_\eta^\nu(\zeta;y)-\zeta(y)\right| \le 2\|\zeta-f_h\| + \left|\Im_\eta^\nu(f_h;y)-f_h(y)\right| \tag{49}$$

in view of inequality (9), above inequality becomes

$$\left|\Im_\eta^\nu(\zeta;y)-\zeta(y)\right| \le \frac{3}{2}\omega_2(\zeta,h) + \left|\Im_\eta^\nu(f_h;y)-f_h(y)\right|. \tag{50}$$

Considering that $f_h \in C^2[0,a]$, it follows from inequality (6) that:

$$\left|\Im_\eta^\nu(f_h;y)-f_h(y)\right| \le \|f_h'\|\sqrt{\Im_\eta^\nu(\Lambda_2^y(t);y)} + \frac{1}{2}\|f_h''\|\Im_\eta^\nu(\Lambda_2^y(t);y), \tag{51}$$

now using inequality (10) in above inequality, we find

$$\left|\Im_\eta^\nu(f_h;y)-f_h(y)\right| \le \|f_h'\|\sqrt{\Im_\eta^\nu(\Lambda_2^y(t);y)} + \frac{3}{4h^2}\omega_2(f,h)\Im_\eta^\nu(\Lambda_2^y(t);y). \tag{52}$$

Further, in view of inequalities (10) and (11), we have

$$\left|\Im_\eta^\nu(f_h;y)-f_h(y)\right| \le \left(\frac{2}{a}\|\zeta\| + \frac{3a}{4h^2}\omega_2(\zeta,h)\right)\sqrt{\Im_\eta^\nu(\Lambda_2^y(t);y)} + \frac{3}{4h^2}\omega_2(\zeta,h)\Im_\eta^\nu(\Lambda_2^y(t);y). \tag{53}$$

Thus, based on equations (50) and (53), we obtain assertion (47) by selecting

$$h := \Omega_\eta(y) = \left\{\Im_\eta^\nu(\Lambda_2^y(t);y)\right\}^{\frac{1}{4}}. \tag{54}$$

□

Remark 3.5. *In Theorem 3.4,* $\Omega_\eta(y) \to 0$ *as* $\eta \to \infty$.

4. Numerical Examples

In this section, we demonstrate the convergence of our operators using numerical examples. We present graphical comparisons between the approximated expressions and the actual functions. Additionally, we include graphs showing the absolute error and provide tables for various values of y and η.

Example 4.1. *For* $\eta = 60, 70, 80, 90, 100$, *and* $\nu = 2$ *the convergence rate of the operators* $\Im_\eta^\nu(\hat{g};y)$ *to the function* $\hat{g}(y) = \frac{y^2}{2} - \frac{1}{1+y^2+\pi y}$ *is shown in Figure 1. Additionally, Table 1 presents the estimated absolute error* $E_\eta = \left|\Im_\eta^\nu(\hat{g};y)-\hat{g}(y)\right|$ *for various values of* η, *with a corresponding error graph depicting the convergence in Figure 2. It is evident from Figure 1, Figure 2, and Table 1 that as* η *increases, the proposed operator* (15) *converges to* $\hat{g}(y)$.

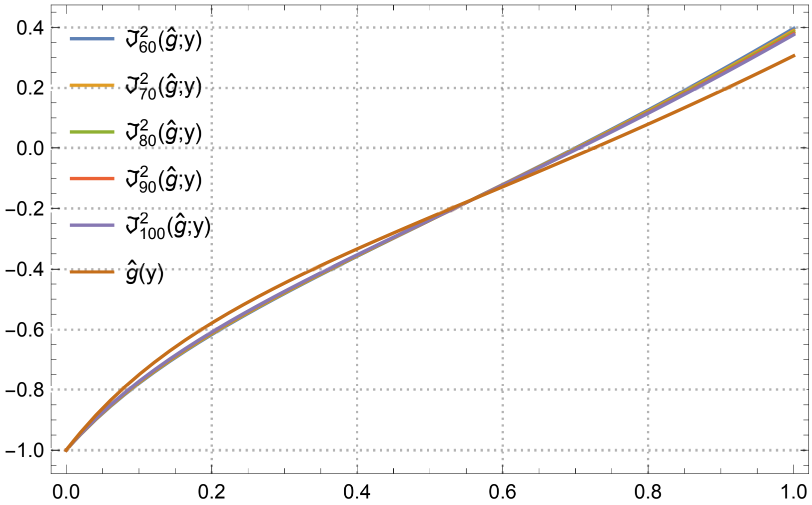

Figure 1: The Convergence of operators $\mathfrak{I}^{\nu}_{\eta}(\hat{g};y)$ to $\hat{g}(y) = \frac{y^2}{2} - \frac{1}{1+y^2+\pi y}$

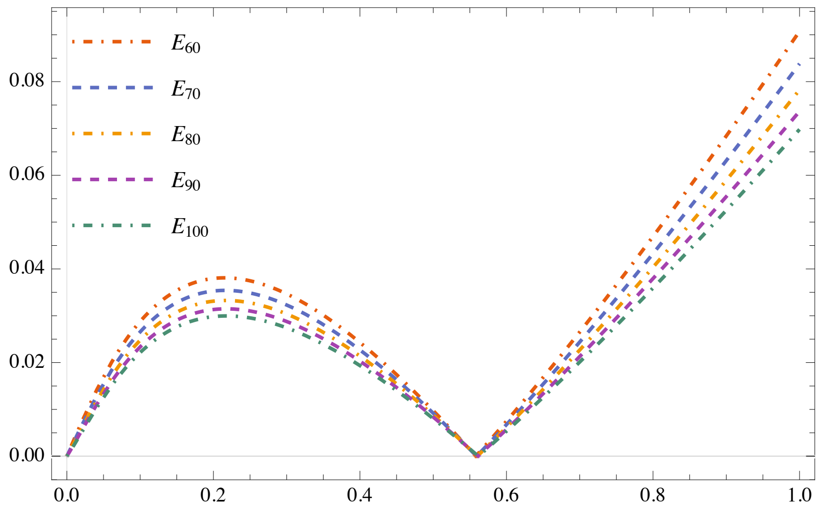

Figure 2: Absolute error of operators $\mathfrak{I}^{\nu}_{\eta}(\hat{g};y)$ to $\hat{g}(y) = \frac{y^2}{2} - \frac{1}{1+y^2+\pi y}$

Example 4.2. *For $\eta = 60, 70, 80, 90, 100,$ and $\nu = 2$ the convergence rate of the operators $\mathfrak{I}^{\nu}_{\eta}(\hat{g};y)$ towards the function $\hat{g}(y) = \frac{1}{y^3+\pi+y^2} + \frac{y}{2} - \frac{\sqrt{y}}{3}$ is depicted in Figure 3. Additionally, Table 2 provides the estimated absolute error $E_{\eta} = \left|\mathfrak{I}^{\nu}_{\eta}(\hat{g};y) - \hat{g}(y)\right|$ for various values of η, with the corresponding error graph illustrating the convergence in Figure 4. It is clearly shown in Figure 3, Figure 4, and Table 2 that as η increases, the proposed operator* (15) *converges to $\hat{g}(y)$.*

Table 1: Error of approximation process for $\hat{g}(y) = \frac{y^2}{2} - \frac{1}{1+y^2+\pi y}$

y	$\mathbf{E_{60}}$	$\mathbf{E_{70}}$	$\mathbf{E_{80}}$	$\mathbf{E_{90}}$	$\mathbf{E_{100}}$
0.1	0.02877	0.02655	0.02479	0.02335	0.02214
0.2	0.03797	0.03528	0.03312	0.03133	0.02981
0.3	0.03467	0.03230	0.03038	0.02878	0.02742
0.4	0.02431	0.02271	0.02140	0.02031	0.01938
0.5	0.00974	0.00921	0.00876	0.00838	0.00804
0.6	0.00748	0.00675	0.00618	0.00572	0.00535
0.7	0.02653	0.02439	0.02269	0.02129	0.02012
0.8	0.04693	0.04327	0.04035	0.03794	0.03592
0.9	0.06839	0.06313	0.05892	0.05545	0.05252
1.0	0.09076	0.08382	0.07826	0.07367	0.06980
1.1	0.11392	0.10524	0.09827	0.09252	0.08768
1.2	0.13781	0.12733	0.11891	0.11197	0.10611
1.3	0.16239	0.15005	0.14014	0.13197	0.12507
1.4	0.18763	0.17338	0.16194	0.15250	0.14453
1.5	0.21350	0.19730	0.18429	0.17354	0.16447
1.6	0.24000	0.22179	0.20717	0.19509	0.18490
1.7	0.26711	0.24685	0.23058	0.21714	0.20580
1.8	0.29482	0.27247	0.25451	0.23968	0.22716
1.9	0.32312	0.29863	0.27895	0.26270	0.24898
2.0	0.35201	0.32533	0.30390	0.28620	0.27126

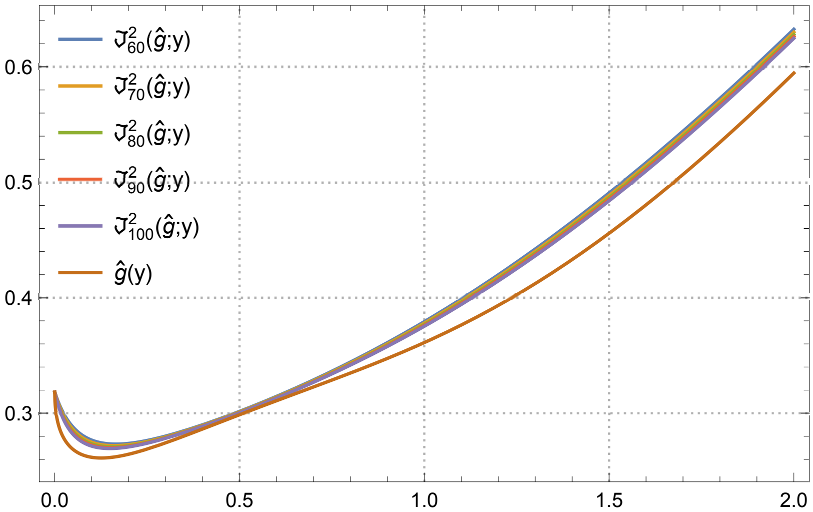

Figure 3: The Convergence of operators $\mathfrak{J}^{\nu}_{\eta}(\hat{g};y)$ to $\hat{g}(y) = \frac{1}{y^3+\pi+y^2} + \frac{y}{2} - \frac{\sqrt{y}}{3}$

Example 4.3. *For $\eta = 60, 70, 80, 90, 100$, and $\nu = 2$ the convergence rate of the operators $\mathfrak{J}^{\nu}_{\eta}(\hat{g};y)$ towards the function $\hat{g}(y) = \frac{(\pi y)^2}{3y+e} + \sin\left(\frac{y}{\pi}\right)$ is depicted in Figure 5. Additionally, Table 3 provides the estimated absolute error $E_\eta = \left|\mathfrak{J}^{\nu}_{\eta}(\hat{g};y) - \hat{g}(y)\right|$ for various values of η, with the corresponding error graph illustrating the convergence in Figure 6. It is clearly shown in Figure 5, Figure 6, and Table 3 that as η increases, the proposed operator* (15) *converges to $\hat{g}(y)$.*

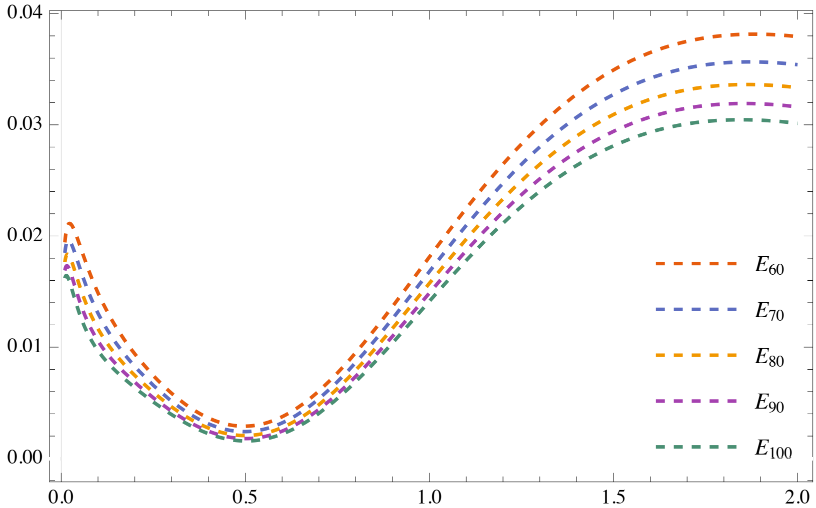

Figure 4: Absolute error of operators $\mathfrak{I}_{\eta}^{\nu}(\hat{g};y)$ to $\hat{g}(y)=\frac{1}{y^3+\pi+y^2}+\frac{y}{2}-\frac{\sqrt{y}}{3}$

Table 2: Error of approximation process for $\hat{g}(y)=\frac{1}{y^3+\pi+y^2}+\frac{y}{2}-\frac{\sqrt{y}}{3}$

y	$\mathcal{E}_{60}$	$\mathcal{E}_{70}$	$\mathcal{E}_{80}$	$\mathcal{E}_{90}$	$\mathcal{E}_{100}$
0.1	0.01488	0.01305	0.01165	0.01055	0.00968
0.2	0.00938	0.00827	0.00745	0.00683	0.00635
0.3	0.00585	0.00514	0.00462	0.00424	0.00393
0.4	0.00363	0.00310	0.00272	0.00244	0.00221
0.5	0.00289	0.00240	0.00205	0.00178	0.00156
0.6	0.00374	0.00318	0.00277	0.00244	0.00218
0.7	0.00605	0.00535	0.00482	0.00439	0.00404
0.8	0.00950	0.00862	0.00792	0.00736	0.00689
0.9	0.01366	0.01257	0.01170	0.01098	0.01038
1.0	0.01811	0.01680	0.01574	0.01487	0.01413
1.1	0.02247	0.02095	0.01971	0.01868	0.01780
1.2	0.02647	0.02474	0.02334	0.02216	0.02115
1.3	0.02993	0.02802	0.02646	0.02515	0.02403
1.4	0.03276	0.03069	0.02900	0.02757	0.02636
1.5	0.03494	0.03274	0.03093	0.02942	0.02812
1.6	0.03650	0.03419	0.03230	0.03071	0.02934
1.7	0.03751	0.03511	0.03315	0.03150	0.03009
1.8	0.03803	0.03557	0.03356	0.03187	0.03042
1.9	0.03815	0.03565	0.03360	0.03188	0.03041
2.0	0.03794	0.03541	0.03334	0.03161	0.03014

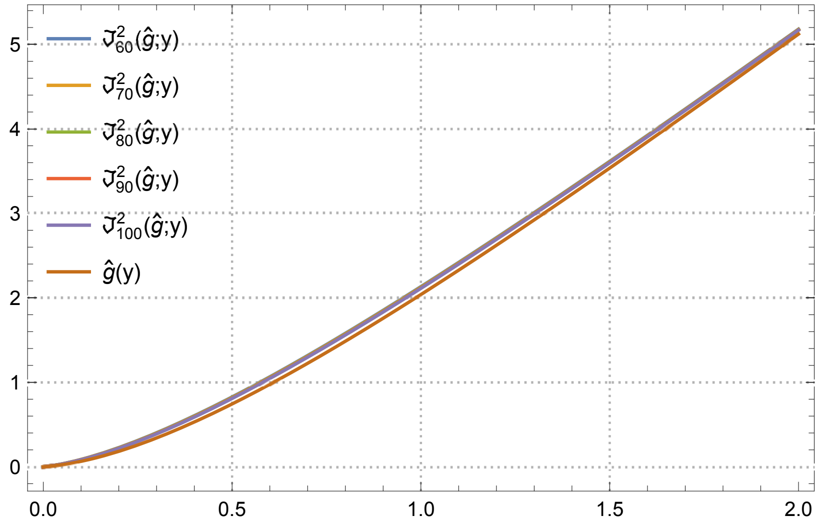

Figure 5: The Convergence of operators $\mathfrak{I}^{\nu}_{\eta}(\hat{g};y)$ to $\hat{g}(y) = \frac{(\pi y)^2}{3y+e} + \sin\left(\frac{y}{\pi}\right)$

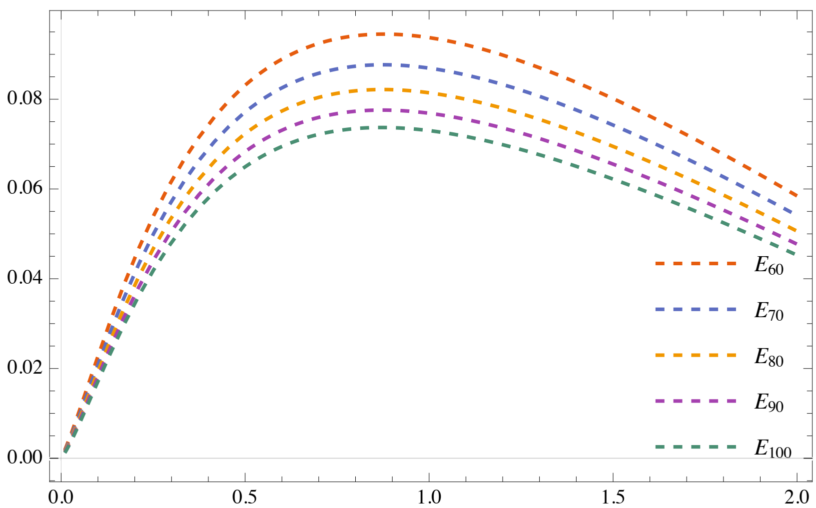

Figure 6: Absolute error of operators $\mathfrak{I}^{\nu}_{\eta}(\hat{g};y)$ to $\hat{g}(y) = \frac{(\pi y)^2}{3y+e} + \sin\left(\frac{y}{\pi}\right)$

Table 3: Error of approximation process for $\hat{g}(y) = \frac{(\pi y)^2}{3y+e} + \sin\left(\frac{y}{\pi}\right)$

y	$\mathbf{E}_{60}$	$\mathbf{E}_{70}$	$\mathbf{E}_{80}$	$\mathbf{E}_{90}$	$\mathbf{E}_{100}$
0.1	0.01488	0.01305	0.01165	0.01055	0.00968
0.2	0.00938	0.00827	0.00745	0.00683	0.00635
0.3	0.00585	0.00514	0.00462	0.00424	0.00393
0.4	0.00363	0.00310	0.00272	0.00244	0.00221
0.5	0.00289	0.00240	0.00205	0.00178	0.00156
0.6	0.00374	0.00318	0.00277	0.00244	0.00218
0.7	0.00605	0.00535	0.00482	0.00439	0.00404
0.8	0.00950	0.00862	0.00792	0.00736	0.00689
0.9	0.01366	0.01257	0.01170	0.01098	0.01038
1.0	0.01811	0.01680	0.01574	0.01487	0.01413
1.1	0.02247	0.02095	0.01971	0.01868	0.01780
1.2	0.02647	0.02474	0.02334	0.02216	0.02115
1.3	0.02993	0.02802	0.02646	0.02515	0.02403
1.4	0.03276	0.03069	0.02900	0.02757	0.02636
1.5	0.03494	0.03274	0.03093	0.02942	0.02812
1.6	0.03650	0.03419	0.03230	0.03071	0.02934
1.7	0.03751	0.03511	0.03315	0.03150	0.03009
1.8	0.03803	0.03557	0.03356	0.03187	0.03042
1.9	0.03815	0.03565	0.03360	0.03188	0.03041
2.0	0.03794	0.03541	0.03334	0.03161	0.03014

5. Conclusions

In this paper, we introduce a sequence of new operators utilizing the ν^{th}-order Tricomi function. We explore the approximation properties and convergence characteristics of the positive linear operator sequence in (15). Numerical examples are conducted using Wolfram Mathematica, and we also examine the approximation error, providing graphical representations of both the approximated function $\hat{g}$ and the associated error.

In future research, scholars can explore a new sequence of operators that extends those presented in (15). For example, modifications or generalizations of these operators could be pursued to achieve more accurate approximations. Additionally, the results obtained from this study are valuable and applicable in areas such as mathematical analysis and mathematical physics. Moreover, the proposed operators could be applied to study and manage real-world issues, such as analyzing the daily average global surface air temperature [8]. This sequence of operators holds significant potential for influencing various scientific fields.

Declarations

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