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On subbalancing numbers involving the third balancing number in the Diophantine equation

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Abstract. In this paper, we consider subbalancing numbers formed by using the third balancing number in the Diophantine equation. We obtain some algebraic identities on these numbers which are also known as B_3-subbalancing numbers. We provide a variety of sum formulas and some divisibility properties associated with these numbers. We also give several Pythagorean triples obtained by using B_3-subbalancing numbers. Furthermore, we derive some functions that takes B_3-subbalancing and balancing numbers values for B_3-subbalancing number arguments.

1. Introduction

In [6], the sequence $\{u_n\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$u_{n+2} = ru_{n+1} + su_n \qquad (n \geq 0) \tag{1}$$

is called a binary recurrence sequence such that r and s are two non-zero integers and $r^2 + 4s \neq 0$. In fact, (1) is a linear homogeneous recurrence relation and is also commonly known as binary recurrence. The characteristic equation for this sequence is

$$x^2 - rx - s = 0 \tag{2}$$

It is obvious that the equation (1) has two different roots as α and β. Thus, there are fix coefficients a and b such that

$$u_n = a\alpha^n + b\beta^n \qquad (n \geq 0) \tag{3}$$

Let u_0 and u_1 be the initial terms of the sequence u_n. From (3), we get

$$a = \frac{u_0\beta - u_1}{\beta - \alpha} \quad \text{and} \quad b = \frac{u_1 - u_0\alpha}{\beta - \alpha} \tag{4}$$

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Thus, from (3) and (4)

$$u_n = \frac{(u_0\beta - u_1)\alpha^n + (u_1 - u_0\alpha)\beta^n}{\beta - \alpha} \tag{5}$$

is obtained which is the Binet formula for the sequence u_n.

Two of the most well-known binary recurrence sequences that have been studied are Fibonacci and Lucas sequences. The recurrence relations of Fibonacci and Lucas sequences which is obtained by taking $r = s = 1$ in (1) are the same. The initial terms of these sequences $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively (for further details see [7, 8, 12, 30]).

Another binary recurrence sequences are the sequences of balancing and Lucas-balancing numbers. In [1], the sequence of balancing numbers is defined as the sequence of n that satisfy the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{6}$$

for some positive integers r, which is called the balancer of n. The n^{th} balancing number is denoted by B_n.

If the Diophantine equation (6) is rearranged, we get

$$n^2 = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \tag{7}$$

From (7), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square. In [20], the square root of $8B_n^2 + 1$ is called the nth Lucas-balancing number and denoted by $C_n = \sqrt{8B_n^2 + 1}$. The recurrence relations of the sequences of balancing and Lucas-balancing numbers, obtained by taking $r = 6$ and $s = -1$ in (1), are as follows:

$$\begin{aligned} B_{n+1} &= 6B_n - B_{n-1} \qquad (n \geq 2) \\ C_{n+1} &= 6C_n - C_{n-1} \qquad (n \geq 2) \end{aligned}$$

where $B_1 = 1, B_2 = 6$ and $C_1 = 3, C_2 = 17$.

From (2), the roots of the characteristic equation for the sequence of balancing and Lucas-balancing numbers are $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. By taking $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$ and by using (5), the Binet formulas for balancing and Lucas-balancing numbers are obtained as

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \quad \text{and} \quad C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2}.$$

In [22], the sequence of cobalancing numbers is defined as the sequence of n that satisfy the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) \tag{8}$$

for some positive integers r, which is called the cobalancer of n. Cobalancing numbers are also called the balancers of balancing numbers. The n^{th} cobalancing number is denoted by b_n.

If the Diophantine equation (8) is rearranged, we get

$$n(n+1) = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1) + \sqrt{8n^2+8n+1}}{2} \tag{9}$$

From (9), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. In [23], the square root of $8b_n^2 + 8b_n + 1$ is called the n^{th} Lucas-cobalancing number and denoted by $c_n = \sqrt{8b_n^2 + 8b_n + 1}$. The recurrence relations of the sequence of cobalancing and Lucas-cobalancing numbers are as follows:

$$\begin{aligned} b_{n+1} &= 6b_n - b_{n-1} + 2 \qquad (n \geq 2) \\ c_{n+1} &= 6c_n - c_{n-1} \qquad (n \geq 2) \end{aligned}$$

where $b_1 = 0, b_2 = 2$ and $c_1 = 1, c_2 = 7$.

As it is seen, while Lucas-cobalancing numbers satisfy the linear homogeneous recurrence relation, cobalancing numbers satisfy the linear non-homogeneous recurrence relation.

The Binet formula for cobalancing numbers is $b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$, where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$. Furthermore by using (5), the Binet formula for Lucas-cobalancing numbers is obtained as $c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}$, where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

Since the first research on balancing numbers was made, balancing numbers have attracted the attention of many researchers. In [14], (a, b)-balancing numbers were presented as an extended concept of balancing numbers. Later in [21], Panda and Panda introduced almost balancing numbers and called these two types of almost balancing numbers as A_1-balancing and A_2-balancing numbers. Furthermore in [27], Tekcan obtained several algebraic relations on A_1-balancing and A_2-balancing numbers (for further details on balancing numbers, see also [2, 9, 10, 15-19, 28]).

In addition to these, in [3] balancing numbers were generalized to t-balancing numbers, with the definition given as follows:

A positive integer n is called a t-balancing number if

$$1 + 2 + \cdots + n = (n + 1 + t) + (n + 2 + t) + \cdots + (n + r + t)$$

for some positive integers r, which is called the t-balancer of n. The n^{th} t-balancing number is denoted by B_n^t (for further details, see [4, 29]).

In [5], Davala and Panda defined D-subbalancing numbers as n numbers that satisfy the following Diophantine equation for some positive integer r and fixed positive integer D.

$$1 + 2 + \cdots + (n - 1) + D = (n + 1) + (n + 2) + \cdots + (n + r) \tag{10}$$

It follows from (10) that

$$r = \frac{-(2n + 1) + \sqrt{8n^2 + 8D + 1}}{2} \tag{11}$$

It is obvious from (11) that n is a D-subbalancing number if and only if $8n^2 + 8D + 1$ is a perfect square.

In [5], it was deduced that D-subbalancing numbers cannot be constructed for every D. In order to determine the values of D such that D-subbalancing numbers can be obtained, they examined the case where D values are taken as the terms of the sequence of cobalancing numbers. They showed that if D is taken as the terms of the sequence of cobalancing numbers, the values of n satisfying the Diophantine equation (10) can be found. Thus, the concept of b_k-subbalancing numbers was introduced and b_3-subbalancing and b_5-subbalancing numbers were examined. Later, Sarı and Karadeniz-Gözeri [25] dealt with b_3-subbalancing numbers and obtained various new identities related to these numbers. Furthermore, they introduced b_3-Lucas subbalancing numbers and derived various algebraic identities between b_3-Lucas subbalancing and

b_3-subbalancing numbers.

In [24], it was shown that if $D = T_k$ $(k \geq 1)$, the values of n satisfying the Diophantine equation (10) can be found. Thus, the concept of T_k-subbalancing numbers was introduced and some algebraic relations on these numbers were obtained. In addition to these, T_k-Lucas subbalancing numbers were introduced and several algebraic identities on these numbers were given. Later, Sarı and Karadeniz-Gözeri [26] showed that if $D = B_m$, the values of n satisfying the Diophantine equation (10) can be found. Thus, they introduced the concept of B_m-subbalancing numbers and proved that there exist at least two solution classes of the Diophantine equation of B_m-subbalancing numbers. They obtained these solution classes as $(c_m+2)B_k+b_mC_k$ and $(c_m+2)B_{k+1} - b_mC_{k+1}$, for $k \geq 0$.

In the present work, we deal with the solutions of the Diophantine equation of subbalancing numbers related to the third balancing number. By using the third balancing number, the following Diophantine equation is obtained:

$$1 + 2 + \cdots + (n-1) + B_3 = (n+1) + (n+2) + \cdots + (n+r)$$

Thus, it can be seen that the solutions of this Diophantine equation form an integer sequence called the sequence of B_3-subbalancing numbers. We provide some new algebraic relations and several sum formulas for this sequence. Besides these, we give some Pythagorean triples obtained by using the terms of the sequence of B_3-subbalancing numbers. Furthermore, we present a variety of new results on the functions related to the terms of the sequence of B_3-subbalancing numbers.

2. Preliminiaries

In order to prove the theorems in the part of the main results, we need to following theorems and corollaries which are included in [26].

Theorem 2.1. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and B_m denote the m^{th} balancing number . Then*

$$(SB_3)_{2m} = 34B_m - (SB_3)_{2m-1}$$

for $m \geq 1$.

Theorem 2.2. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and B_m denote the m^{th} balancing number . Then*

$$(SB_3)_{2m+1} = 14(SB_3)_{2m} - 195B_m$$

for $m \geq 0$.

Theorem 2.3. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and c_m denote the m^{th} Lucas-cobalancing number. Then*

$$(SB_3)_{2m+1} = 15c_{m+1} - (SB_3)_{2m}$$

for $m \geq 0$.

Theorem 2.4. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number. Then*

$$(SB_3)_m^2 = (SB_3)_{m-2}(SB_3)_{m+2} + 281$$

for $m \geq 2$.

Corollary 2.1. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number, let B_m denote the m^{th} balancing number and let C_m denote the m^{th} Lucas-balancing number. Then*

$$\begin{aligned}(SB_3)_{2m} &= 17B_m + C_m\\ (SB_3)_{2m+1} &= 17B_{m+1} - C_{m+1}\end{aligned}$$

for $m \geq 0$.

Corollary 2.2. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and let B_m denote the m^{th} balancing number. Then*

$$\begin{aligned}(SB_3)_{2m} &= B_{m+1} + 14B_m\\ (SB_3)_{2m+1} &= 14B_{m+1} + B_m\end{aligned}$$

for $m \geq 0$.

3. Main Results

This section consists of four subsection. While the first, second and third subsections include sum formulas, divisibility properties and Pythagorean triples regarding B_3-subbalancing numbers, the last subsection includes the functions generating balancing and B_3-subbalancing numbers.

3.1. Sum Formulas

In this subsection, we obtain several sum formulas related to B_3-subbalancing numbers by using balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers.

Theorem 3.1. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number, b_m denote the m^{th} cobalancing number and B_m denote the m^{th} balancing number. Then*

$$\sum_{i=0}^{m}(SB_3)_{2i} = \frac{15}{2}b_{m+1} + B_{m+1}$$

and

$$\sum_{i=0}^{m}(SB_3)_{2i+1} = \frac{15}{2}b_{m+1} + 14B_{m+1}.$$

Proof. By using Corollary 2.1, we obtain

$$\begin{aligned}\sum_{i=0}^{m}(SB_3)_{2i} &= 17(B_0 + B_1 + B_2 + B_3 + \cdots + B_m) + (C_0 + C_1 + C_2 + \cdots + C_m)\\ &= 17(B_0 + B_1 + B_2 + B_3 + \cdots + B_m) + B_{m+1} - 2(B_0 + B_1 + B_2 + B_3 + \cdots + B_m)\\ &= 15\sum_{i=0}^{m} B_i + B_{m+1}\\ &= \frac{15}{2}b_{m+1} + B_{m+1}.\end{aligned}$$

Similarly, we obtain

$$\sum_{i=0}^{m}(SB_3)_{2i+1} = 17(B_1 + B_2 + B_3 + \cdots + B_{m+1}) - (C_1 + C_2 + C_3 + \cdots + C_{m+1})$$

$$
\begin{aligned}
&= \left(17\sum_{i=1}^{m} B_i + 17B_{m+1}\right) - \left(3B_{m+1} + 2\sum_{i=1}^{m} B_i\right)\\
&= 15\sum_{i=1}^{m} B_i + 14B_{m+1}\\
&= \frac{15}{2}b_{m+1} + 14B_{m+1}.
\end{aligned}
$$

Corollary 3.1. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number, b_m denote the m^{th} cobalancing number and B_m denote the m^{th} balancing number. Then*

$$\sum_{i=1}^{2m}(SB_3)_i = 17b_{m+1}$$

and

$$\sum_{i=0}^{2m+1}(SB_3)_i = 15(B_{m+1} + b_{m+1}).$$

Proof. By using Theorem 3.1, we obtain

$$
\begin{aligned}
\sum_{i=1}^{2m}(SB_3)_i &= [(SB_3)_2 + (SB_3)_4 + \cdots + (SB_3)_{2m}] + [(SB_3)_1 + (SB_3)_3 + \cdots + (SB_3)_{2m-1}]\\
&= \left(\frac{15b_{m+1}}{2} + B_{m+1} - 1\right) + \left(\frac{15b_m}{2} + 14B_m\right)\\
&= \left(\frac{15b_{m+1}}{2} + \frac{6b_{m+1} - b_m + 2 - b_{m+1}}{2} - 1\right) + \left(\frac{15b_m}{2} + 7b_{m+1} - 7b_m\right)\\
&= \left(10b_{m+1} - \frac{b_m}{2}\right) + \left(\frac{15b_m}{2} + 7b_{m+1} - 7b_m\right)\\
&= 17b_{m+1}.
\end{aligned}
$$

The other case can be proved similarly.

Theorem 3.2. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number, c_m denote the m^{th} Lucas-cobalancing number and B_m denote the m^{th} balancing number . Then*

$$2\sum_{m=0}^{\infty}(SB_3)_m = 34\sum_{m=1}^{\infty} B_m + 15\sum_{m=1}^{\infty} c_{m+1} + 1.$$

Proof. By using Theorem 2.1 and Theorem 2.3, we get

$$\sum_{m=1}^{\infty}[(SB_3)_{2m-1} + (SB_3)_{2m}] = 34\sum_{m=1}^{\infty} B_m \tag{12}$$

and

$$\sum_{m=0}^{\infty}[(SB_3)_{2m} + (SB_3)_{2m+1}] = 15\sum_{m=1}^{\infty} c_{m+1} \tag{13}$$

Since

$$(SB_3)_0 + \sum_{m=1}^{\infty}[(SB_3)_{2m-1} + (SB_3)_{2m}] = (SB_3)_0 + [(SB_3)_1 + (SB_3)_2] + [(SB_3)_3 + (SB_3)_4] + \cdots ,$$

we obtain

$$\begin{aligned}(SB_3)_0 + \sum_{m=1}^{\infty}[(SB_3)_{2m-1} + (SB_3)_{2m}] &= [(SB_3)_0 + (SB_3)_1] + [(SB_3)_2] + (SB_3)_3] + \cdots . \\ &= \sum_{m=0}^{\infty}[(SB_3)_{2m} + (SB_3)_{2m+1}].\end{aligned}$$

Thus, we get

$$(SB_3)_0 + \sum_{m=1}^{\infty}[(SB_3)_{2m-1} + (SB_3)_{2m}] = \sum_{m=0}^{\infty}[(SB_3)_{2m} + (SB_3)_{2m+1}].$$

From this equation and (12), we obtain

$$\sum_{m=0}^{\infty}[(SB_3)_{2m} + (SB_3)_{2m+1}] = 1 + \sum_{m=1}^{\infty} 34B_m. \tag{14}$$

Further from (13) and (14), we obtain

$$2\left(\sum_{m=0}^{\infty}(SB_3)_{2m} + (SB_3)_{2m+1}\right) = 34\sum_{m=1}^{\infty} B_m + 15\sum_{m=1}^{\infty} c_{m+1} + 1.$$

Thus, we get

$$2\sum_{m=0}^{\infty}(SB_3)_m = 34\sum_{m=1}^{\infty} B_m + 15\sum_{m=1}^{\infty} c_{m+1} + 1.$$

Theorem 3.3. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and c_m denote the m^{th} Lucas-cobalancing number. Then*

$$\sum_{i=1}^{m}[(SB_3)_{2i} - (SB_3)_{2i-1}] = c_{m+1} - 1.$$

Proof. From Corollary 2.2, we obtain

$$\begin{aligned}\sum_{i=1}^{m}[(SB_3)_{2i} - (SB_3)_{2i-1}] &= \sum_{i=1}^{m}(B_{i+1} - B_{i-1}) \\ &= (B_2 - B_0) + (B_3 - B_1) + (B_4 - B_2) + \cdots + (B_{m+1} - B_{m-1}) \\ &= B_{m+1} + B_m - 1 \\ &= B_{m+1} + (c_{m+1} - B_{m+1}) - 1 \\ &= c_{m+1} - 1.\end{aligned}$$

3.2. Divisibility Properties

In this subsection, we deal with some divisibility properties regarding B_3-subbalancing numbers.

Theorem 3.4. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and b_m denote the m^{th} cobalancing number. Then*

$$2b_{m+1} + 1 | (SB_3)_{2m+1} - (SB_3)_{2m}$$

for $m \geq 0$.

Proof. From Corollary 2.2, we obtain

$$\begin{aligned}(SB_3)_{2m+1}-(SB_3)_{2m} &= (14B_{m+1}+B_m)-(14B_m+B_{m+1})\\ &= 13(B_{m+1}-B_m)\\ &= 13\left(\frac{b_{m+2}-b_{m+1}-b_{m+1}+b_m}{2}\right)\\ &= 13\left(\frac{6b_{m+1}-b_m+2-2b_{m+1}+b_m}{2}\right)\\ &= 13(2b_{m+1}+1).\end{aligned}$$

Thus, we obtain

$$2b_{m+1}+1|(SB_3)_{2m+1}-(SB_3)_{2m}.$$

Theorem 3.5. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and C_m denote the m^{th} Lucas-balancing number. Then*

$$C_m|(SB_3)_{2m}-(SB_3)_{2m-1}$$

for $m \geq 1$.

Proof. From Corollary 2.1, we obtain

$$\begin{aligned}(SB_3)_{2m}-(SB_3)_{2m-1} &= (17B_m+C_m)-(17B_m-C_m)\\ &= 2C_m.\end{aligned}$$

Thus, we get

$$C_m|(SB_3)_{2m}-(SB_3)_{2m-1}.$$

Corollary 3.2. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and B_m denote the m^{th} balancing number. Then*

$$B_{2m}|(SB_3)_{2m}-(SB_3)_{2m-1}$$

for $m \geq 1$.

Proof. From Theorem 3.5 and the relation between balancing and Lucas-balancing numbers, we get

$$\begin{aligned}B_{2m} &= 2B_mC_m\\ &= B_m[(SB_3)_{2m}-(SB_3)_{2m-1}].\end{aligned}$$

Thus, we obtain

$$B_{2m}|(SB_3)_{2m}-(SB_3)_{2m-1}.$$

Theorem 3.6. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number and B_m denote the m^{th} balancing number. Then*

$$B_{2m}|(SB_3)^2_{2m}-(SB_3)^2_{2m-1} \qquad (m \geq 1)$$

and

$$B_{2m+1}|(SB_3)^2_{2m+1}-(SB_3)^2_{2m} \qquad (m \geq 0).$$

Proof. By using Theorem 2.1 and Theorem 3.5, we obtain

$$\begin{aligned}(SB_3)^2_{2m}-(SB_3)^2_{2m-1} &= [(SB_3)_{2m}-(SB_3)_{2m-1}][(SB_3)_{2m}+(SB_3)_{2m-1}]\\ &= 68B_mC_m\\ &= 34B_{2m}.\end{aligned}$$

Thus, we get

$$B_{2m}|(SB_3)^2_{2m}-(SB_3)^2_{2m-1}.$$

The other case can be proved similarly.

3.3. Pythagorean Triples

In this subsection, we give several Pythagorean triples obtained by using B_3-subbalancing numbers. We obtain the following theorems by using some techniques included in [11] and [13].

Theorem 3.7. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number . Then*

$$\left((SB_3)_{4m-4}, \frac{(SB_3)^2_{4m-4} - 1}{2}, \frac{(SB_3)^2_{4m-4} + 1}{2} \right) \quad \text{and} \quad \left((SB_3)_{4m-1}, \frac{(SB_3)^2_{4m-1} - 1}{2}, \frac{(SB_3)^2_{4m-1} + 1}{2} \right)$$

are Pythagorean triples.

Proof. We first show that $(SB_3)_{4m-4}$ is odd. For this purpose, we assume that $(SB_3)_{4m-4}$ is even.

From Corollary 2.2, we get

$$(SB_3)_{4m-4} = 14B_{2m-2} + B_{2m-1} \tag{15}$$

It follows from (15) that B_{2m-1} is even. Since every odd term of the sequence of balancing numbers is odd, we get a contradiction. Thus, $(SB_3)_{4m-4}$ is odd and $\left(\frac{(SB_3)^2_{4m-4} \pm 1}{2}\right)$ are integers.

Since

$$\begin{aligned}(SB_3)^2_{4m-4} + \left(\frac{(SB_3)^2_{4m-4} - 1}{2}\right)^2 &= \frac{(SB_3)^4_{4m-4} + 2(SB_3)^2_{4m-4} + 1}{4} \\ &= \left(\frac{(SB_3)^2_{4m-4} + 1}{2}\right)^2,\end{aligned}$$

we obtain that

$$\left((SB_3)_{4m-4}, \frac{(SB_3)^2_{4m-4} - 1}{2}, \frac{(SB_3)^2_{4m-4} + 1}{2} \right)$$

is a Pythagorean triple.

The other case can be proved similarly.

Theorem 3.8. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number . Then*

$$\left((SB_3)_{4m-2}, \left(\frac{(SB_3)_{4m-2}}{2}\right)^2 - 1, \left(\frac{(SB_3)_{4m-2}}{2}\right)^2 + 1\right) \quad \text{and} \quad \left((SB_3)_{4m-3}, \left(\frac{(SB_3)_{4m-3}}{2}\right)^2 - 1, \left(\frac{(SB_3)_{4m-3}}{2}\right)^2 + 1\right)$$

are Pythagorean triples.

Proof. We first show that $(SB_3)_{4m-2}$ is even. For this purpose, we assume that $(SB_3)_{4m-2}$ is odd.

From Corollary 2.2, we get

$$(SB_3)_{4m-2} = 14B_{2m-1} + B_{2m} \tag{16}$$

It follows from (16) that B_{2m} is odd. Since every even term of the sequence of balancing numbers is even, we get a contradiction. Thus, $(SB_3)_{4m-2}$ is even and $\left(\frac{(SB_3)_{4m-2}}{2}\right)^2 \pm 1$ are integers.

Since

$$\begin{aligned}(SB_3)^2_{4m-2}+\left[\left(\frac{(SB_3)_{4m-2}}{2}\right)^2-1\right]^2 &= \frac{(SB_3)^4_{4m-2}+8(SB_3)^2_{4m-2}+16}{16}\\ &= \left[\left(\frac{(SB_3)_{4m-2}}{2}\right)^2+1\right]^2,\end{aligned}$$

we obtain that

$$\left((SB_3)_{4m-2},\left(\frac{(SB_3)_{4m-2}}{2}\right)^2-1,\left(\frac{(SB_3)_{4m-2}}{2}\right)^2+1\right)$$

is a Pythagorean triple.

The other case can be proved similarly.

Theorem 3.9. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number . Then*

$$\left((SB_3)^3_{2m},\left(\frac{(SB_3)^4_{2m}-(SB_3)^2_{2m}}{2}\right),\left(\frac{[(SB_3)_{4m}+281B^2_m][(SB_3)_{4m}+281B^2_m+1]}{2}\right)\right)$$

is a Pythagorean triple.

Proof. From Corollary 2.2 and the relations between the terms of the sequence of balancing number, we get

$$\begin{aligned}[(SB_3)^3_{2m}]^2+\left[\frac{(SB_3)^4_{2m}-(SB_3)^2_{2m}}{2}\right]^2 &= (SB_3)^6_{2m}+\frac{(SB_3)^8_{2m}-2(SB_3)^6_{2m}+(SB_3)^4_{2m}}{4}\\ &= \frac{(SB_3)^8_{2m}+2(SB_3)^6_{2m}+(SB_3)^4_{2m}}{4}\\ &= (SB_3)^4_{2m}\left[\frac{(SB_3)^4_{2m}+2(SB_3)^2_{2m}+1}{4}\right]\\ &= (SB_3)^4_{2m}\left[\frac{(SB_3)^2_{2m}+1}{2}\right]^2\\ &= \left[\frac{[(SB_3)_{4m}+281B^2_m][(SB_3)_{4m}+281B^2_m+1]}{2}\right]^2.\end{aligned}$$

Thus, we obtain that

$$\left((SB_3)^3_{2m},\left(\frac{(SB_3)^4_{2m}-(SB_3)^2_{2m}}{2}\right),\left(\frac{[(SB_3)_{4m}+281B^2_m][(SB_3)_{4m}+281B^2_m+1]}{2}\right)\right)$$

is a Pythagorean triple.

Theorem 3.10. *Let $(SB_3)_m$ denote the m^{th} B_3-subbalancing number . Then*

$$\left((SB_3)^3_{2m+1},\left(\frac{(SB_3)^4_{2m+1}-(SB_3)^2_{2m+1}}{2}\right),\left(\frac{[281B^2_{m+1}-(SB_3)_{4m+3}][281B^2_{m+1}-(SB_3)_{4m+3}+1]}{2}\right)\right)$$

is a Pythagorean triple.

Proof. From Corollary 2.2 and the relations between the terms of the sequence of balancing number, we get

$$\begin{aligned}
[(SB_3)^3_{2m+1}]^2+\left[\frac{(SB_3)^4_{2m+1}-(SB_3)^2_{2m+1}}{2}\right]^2 &= (SB_3)^6_{2m+1}+\frac{(SB_3)^8_{2m+1}-2(SB_3)^6_{2m+1}+(SB_3)^4_{2m+1}}{4}\\
&= \frac{(SB_3)^8_{2m+1}+2(SB_3)^6_{2m+1}+(SB_3)^4_{2m+1}}{4}\\
&= (SB_3)^4_{2m+1}\left[\frac{(SB_3)^4_{2m+1}+2(SB_3)^2_{2m+1}+1}{4}\right]\\
&= (SB_3)^4_{2m+1}\left[\frac{(SB_3)^2_{2m+1}+1}{2}\right]^2\\
&= \left[\frac{[281B^2_{m+1}-(SB_3)_{4m+3}][281B^2_{m+1}-(SB_3)_{4m+3}+1]}{2}\right]^2.
\end{aligned}$$

Thus, we obtain that

$$\left((SB_3)^3_{2m+1},\left(\frac{(SB_3)^4_{2m+1}-(SB_3)^2_{2m+1}}{2}\right),\left(\frac{[281B^2_{m+1}-(SB_3)_{4m+3}][281B^2_{m+1}-(SB_3)_{4m+3}+1]}{2}\right)\right)$$

is a Pythagorean triple.

3.4. *Functions Generating Balancing and B_3-Subbalancing Numbers*

In this subsection, we obtain some functions that takes B_3-subbalancing and balancing numbers values for B_3-subbalancing number arguments.

Theorem 3.11. *Let $f(x)=\frac{17x-\sqrt{8x^2+281}}{281}$ and $g(y)=\frac{17y+\sqrt{8y^2+281}}{281}$. If x is an even term and y is an odd term of the sequence of B_3-subbalancing numbers, then $f(x)$ and $g(y)$ are balancing numbers.*

Proof. Let x be an even term of the sequence of B_3-subbalancing numbers. Then there exist a positive integer m such that $x=(SB_3)_{2m}$.

From Corollary 2.2, Theorem 2.4 and the recurrence relation of B_3-subbalancing numbers, we obtain

$$\begin{aligned}
8(SB_3)^2_{2m}+281 &= 8(SB_3)^2_{2m}+(SB_3)^2_{2m}-(SB_3)_{2m-2}(SB_3)_{2m+2}\\
&= 9(SB_3)^2_{2m}-[6(SB_3)_{2m}-(SB_3)_{2m+2}](SB_3)_{2m+2}\\
&= 9(SB_3)^2_{2m}-6(SB_3)_{2m}(SB_3)_{2m+2}+(SB_3)^2_{2m+2}\\
&= 9(14B_m+B_{m+1})^2-6(14B_m+B_{m+1})(14B_{m+1}+B_{m+2})+(14B_{m+1}+B_{m+2})^2\\
&= 9(14B_m+B_{m+1})^2-6(14B_m+B_{m+1})(20B_{m+1}-B_m)+(20B_{m+1}-B_m)^2\\
&= 1849B^2_m-1462B_{m+1}B_m+289B^2_{m+1}\\
&= 289(B^2_{m+1}+28B_{m+1}B_m+196B^2_m)-9554B_m(B_{m+1}+14B_m)+78961B^2_m\\
&= [17(B_{m+1}+14B_m)-281B_m]^2\\
&= [17(SB_3)_{2m}-281B_m]^2.
\end{aligned}$$

Thus, we deduce that

$$\sqrt{8(SB_3)^2_{2m}+281} = 17(SB_3)_{2m}-281B_m \tag{17}$$

If we take $x=(SB_3)_{2m}$, we get

$$f((SB_3)_{2m}) = \frac{17(SB_3)_{2m}-\sqrt{8(SB_3)^2_{2m}+281}}{281}.$$

Thus, it is obvious from (17) that $f(x) = B_m$.

Similarly, we get

$$\sqrt{8(SB_3)_{2m+1}^2 + 281} = 281B_{m+1} - 17(SB_3)_{2m+1}. \tag{18}$$

If we take $y = (SB_3)_{2m+1}$, we get

$$g((SB_3)_{2m+1}) = \frac{17(SB_3)_{2m+1} + \sqrt{8(SB_3)_{2m+1}^2 + 281}}{281}.$$

Thus, it is obvious from (18) that $g(y) = B_{m+1}$.

Theorem 3.12. *Let* $g(x) = \frac{297x - 34\sqrt{8x^2+281}}{281}$ *and* $\tilde{g}(x) = \frac{619x + 195\sqrt{8x^2+281}}{281}$. *If x is an even term of the sequence of B_3-subbalancing numbers, then $g(x)$ is the B_3-subbalancing number just prior to it and $\tilde{g}(x)$ is the B_3-subbalancing number next to it.*

Proof. Let x be an even term of the sequence of B_3-subbalancing numbers. Then there exist a positive integer m such that $x = (SB_3)_{2m}$.
From Theorem 2.1 and Theorem 3.11, we obtain

$$\begin{aligned} g((SB_3)_{2m}) &= \frac{297(SB_3)_{2m} - 34\sqrt{8(SB_3)_{2m}^2 + 281}}{281} \\ &= 34\left(\frac{17(SB_3)_{2m} - \sqrt{8(SB_3)_{2m}^2 + 281}}{281}\right) - (SB_3)_{2m} \\ &= (SB_3)_{2m-1}. \end{aligned}$$

Thus, we get $g((SB_3)_{2m})) = (SB_3)_{2m-1}$.

By using Theorem 2.2 and Theorem 3.11, we get

$$\begin{aligned} \tilde{g}((SB_3)_{2m}) &= \frac{619(SB_3)_{2m} + 195\sqrt{8(SB_3)_{2m}^2 + 281}}{281} \\ &= 14(SB_3)_{2m} - 195\left(\frac{17(SB_3)_{2m} - \sqrt{8(SB_3)_{2m}^2 + 281}}{281}\right) \\ &= (SB_3)_{2m+1}. \end{aligned}$$

Thus, we get $\tilde{g}((SB_3)_{2m}) = (SB_3)_{2m+1}$.

Theorem 3.13. *Let* $g(x) = \frac{619x - 195\sqrt{8x^2+281}}{281}$ *and* $\tilde{g}(x) = \frac{297x + 34\sqrt{8x^2+281}}{281}$. *If x is an odd term of the sequence of B_3-subbalancing numbers, then $g(x)$ is the B_3-subbalancing number just prior to it and $\tilde{g}(x)$ is the B_3-subbalancing number next to it.*

Proof. Let x be an odd term of the sequence of B_3-subbalancing numbers. Then there exist a positive integer m such that $x = (SB_3)_{2m+1}$.
From Corollary 2.2 and Theorem 3.11, we get

$$B_m = (SB_3)_{2m+1} - 14\left(\frac{17(SB_3)_{2m+1} + \sqrt{8(SB_3)_{2m+1}^2 + 281}}{281}\right)$$

$$= \frac{43(SB_3)_{2m+1} - 14\sqrt{8(SB_3)^2_{2m+1} + 281}}{281} \tag{19}$$

It follows from (19), Corollary 2.2 and Theorem 3.11 that

$$\begin{aligned} g((SB_3)_{2m+1}) &= \frac{619(SB_3)_{2m+1} - 195\sqrt{8(SB_3)^2_{2m+1} + 281}}{281} \\ &= 14\left(\frac{43(SB_3)_{2m+1} - 14\sqrt{8(SB_3)^2_{2m+1} + 281}}{281}\right) + \left(\frac{17(SB_3)_{2m+1} + \sqrt{8(SB_3)^2_{2m+1} + 281}}{281}\right) \\ &= 14B_m + B_{m+1} \\ &= (SB_3)_{2m}. \end{aligned}$$

Thus, we get $g((SB_3)_{2m+1}) = (SB_3)_{2m}$.

By using Theorem 2.1 and Theorem 3.11, we get

$$\begin{aligned} \tilde{g}((SB_3)_{2m+1}) &= \frac{297(SB_3)_{2m+1} + 34\sqrt{8(SB_3)^2_{2m+1} + 281}}{281} \\ &= 34\left(\frac{17(SB_3)_{2m+1} + \sqrt{8(SB_3)^2_{2m+1} + 281}}{281}\right) - (SB_3)_{2m+1} \\ &= 34B_{m+1} - (SB_3)_{2m+1} \\ &= (SB_3)_{2m+2}. \end{aligned}$$

Thus, we get $\tilde{g}((SB_3)_{2m+1}) = (SB_3)_{2m+2}$.

Theorem 3.14. *Let $f(x) = 3x + \sqrt{8x^2 + 281}$ and $\tilde{f}(x) = 3x - \sqrt{8x^2 + 281}$. If x is a B_3-subbalancing number, then $f(x)$ and $\tilde{f}(x)$ are also B_3-subbalancing numbers.*

Proof. Let x be a B_3-subbalancing number. Then there exist a positive integer m such that $x = (SB_3)_m$.

By using Theorem 2.4 and the recurrence relation of B_3-subbalancing numbers, we get

$$\begin{aligned} 8(SB_3)^2_m + 281 &= 9(SB_3)^2_m - (SB_3)_{m-2}(SB_3)_{m+2} \\ &= 9(SB_3)^2_m - 6(SB_3)_m(SB_3)_{m-2} + (SB_3)^2_{m-2} \\ &= [3(SB_3)_m - (SB_3)_{m-2}]^2. \end{aligned}$$

Thus, we deduce that

$$\sqrt{8(SB_3)^2_m + 281} = 3(SB_3)_m - (SB_3)_{m-2} \tag{20}$$

From (20) and the recurrence relation of B_3-subbalancing numbers, we obtain

$$\begin{aligned} f(SB_3)_m) &= 3(SB_3)_m + \sqrt{8(SB_3)^2_m + 281} \\ &= 6(SB_3)_m - (SB_3)_{m-2} \\ &= (SB_3)_{m+2}. \end{aligned}$$

Thus, we get $f(x) = (SB_3)_{m+2}$.

Similarly, by using (20), we get

$$\begin{aligned} \tilde{f}((SB_3)_m) &= 3(SB_3)_m - \sqrt{8(SB_3)_m^2 + 281} \\ &= (SB_3)_{m-2}. \end{aligned}$$

Thus, we get $\tilde{f}(x) = (SB_3)_{m-2}$.

Consequently, $f(x)$ is equal to the $(m+2)^{th}$ B_3-subbalancing number and $\tilde{f}(x)$ is equal to the $(m-2)^{th}$ B_3-subbalancing number where $x = (SB_3)_m$.

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References

[1] A. Behera, G.K. Panda, *On the square roots of triangular numbers*, Fibonacci Quart. **37**(2) (1999), 98-105.
[2] P. Catarino, H. Campos, P. Vasco, *On some identities for balancing and cobalancing numbers*, Ann. Math. Inform. **45** (2015), 11-24.
[3] K. K. Dash, R. S. Ota, S. Dash, *t-balancing numbers*, Int. J. Contemp. Math. Sciences **7**(41) (2012), 1999-2012.
[4] K. K. Dash, R. S. Ota, S. Dash, *Sequence t-balancing numbers*, Int. J. Contemp. Math. Sciences **7**(47) (2012), 2305-2310.
[5] R.K. Davala, G.K. Panda, *Subbalancing numbers*, Matematika **34**(1) (2018), 163-172.
[6] G. Grossman, F. Luca, *Sums of factorials in binary recurrence sequences*, J. Number Theory **93**(2) (2002), 87-107.
[7] V.E. Hoggatt, *Fibonacci and Lucas numbers*, Houghton Mifflin Company, Santa Clara, 1969.
[8] A.F. Horadam, *A generalized Fibonacci sequence*, The American Mathematical Monthly **68**(5) (1961), 455-459.
[9] G. Karadeniz-Gözeri, *On Pell, Pell-Lucas, and balancing numbers*, J. Inequal. Appl. **3** (2018).
[10] G. Karadeniz-Gözeri, A. Özkoç, A. Tekcan, *Some algebraic relations on balancing numbers*, Util. Math. **103** (2017), 217-236.
[11] M. Keith, *More 666 Curiosa*, J. Recreational Mathematics **31** (2002-2003), 47-49.
[12] T. Koshy, *Fibonacci and Lucas numbers with applications*, John Wiley, New York, 2001.
[13] T. Koshy, *Elemantary number theory with applications*, (2nd edition), Academic Press, Burlington, MA, 2007.
[14] T. Kovács, K. Liptai, P. Olajos, *On (a,b)-balancing numbers*, Publ. Math. Debrecen **77**(3-4) (2010), 485-498.
[15] K. Liptai, *Fibonacci balancing numbers*, Fibonacci Quart. **42**(4) (2004), 330-340.
[16] K. Liptai, *Lucas balancing numbers*, Acta Math. Univ. Ostrav. **14**(1) (2006), 43-47.
[17] K. Liptai, F. Luca, A. Pinter, L. Szalay, *Generalized balancing numbers*, Indag. Math. (N.S.) **20**(1) (2009), 87-100.
[18] P. Olajos, *Properties of balancing, cobalancing and generalized balancing numbers*, Ann. Math. Inform. **37** (2010), 125-138.
[19] A. Özkoç, *Tridiagonal matrices via k-balancing number*, British Journal of Mathematics and Computer Science **10**(4) (2015), 1-11.
[20] G.K. Panda, *Some fascinating properties of balancing numbers*, Congr. Numer. **194** (2009), 185-189.
[21] G.K. Panda, A.K. Panda, *Almost balancing numbers*, J. Indian Math. Soc. **82**(3-4) (2015), 147-156.
[22] G.K. Panda, P.K. Ray, *Cobalancing numbers and cobalancers*, Int. J. Math. Math. Sci. **8** (2005), 1189-2000.
[23] G.K. Panda, P.K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bull. Inst. Math. Acad. Sin. (N.S.) **6**(1) (2011), 41-72.
[24] S.G Rayaguru, G.K Panda, *A generalization to almost balancing and cobalancing numbers using triangular numbers*, Notes on Number Theory and Discrete Mathematics **26**(3) (2020), 135-148.
[25] S. Sarı, G. Karadeniz-Gözeri, *b_3-subbalancing and b_3-Lucas subbalancing numbers*, Filomat **37**(22) (2023), 7623-7639.
[26] S. Sarı, G. Karadeniz-Gözeri, *On the existence of solutions of Diophantine equations related to subbalancing numbers*, Submitted.
[27] A. Tekcan, *Almost balancing, triangular and square triangular numbers*, Notes on Number Theory and Discrete Mathematics **25**(1) (2019), 108-121.
[28] A. Tekcan, *Sums and spectral norms of all almost balancing numbers*, Creat. Math. Inform. **28**(2) (2019), 203-214.
[29] A. Tekcan, M. Tayat, M.E. Ozbek, *The Diophantine equation* $8x^2 - y^2 + 8x(1+t) + (2t+1)^2 = 0$ *and t-balancing numbers*, Hindawi Publishing Corporation ISR Combinatorics **2014** (2014).
[30] S. Vajda, *Fibonacci & Lucas numbers, and the golden section: Theory and applications*, Ellis Horwood Limited Publ., England, 1989.