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Non-self-adjoint singular matrix Sturm–Liouville operators with general boundary conditions

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Abstract. In the Hilbert space $\mathfrak{L}_A^2(\mathcal{I};E)$ ($\mathcal{I} := [a,b)$, $-\infty < a < b \leq +\infty$, $\dim E = m < +\infty$, $A > 0$), the maximal dissipative singular matrix-valued Sturm–Liouville operators that the extensions of a minimal symmetric operator with maximal deficiency indices $(2m, 2m)$ (in limit-circle case at singular endpoint b) are studied. The maximal dissipative operators with general (for example coupled or separated) boundary conditions are investigated. A self-adjoint dilation is constructed for dissipative operator and its incoming and outgoing spectral representations, which make it possible to determine the scattering matrix of the dilation. We also construct a functional model of the dissipative operator and determine its characteristic function in terms of the scattering matrix of the dilation (or in terms of the Weyl function of self-adjoint operator). Moreover a theorem on completeness of the system of eigenvectors and associated vectors (or root vectors) of the dissipative operators proved.

1. Introduction

The contour integration method is one of the main methods to investigate the spectral analysis of non-self-adjoint (dissipative) operators. To use the contour integration methods one requires a sharp estimate of the resolvent on expanding contours separating the spectrum. Moreover weak perturbations of self-adjoint operators with sparse discrete spectrum are needed. Since for wide classes of singular differential equations there are no asymptotics of the solutions, the method cannot be applied to them.

Theorems on representation of linear relations turned out to be useful for the description of various classes of extensions of symmetric operators. The first result of this type is due to Rofe-Beketov [30]. Bruk [11] and Kochubei [19] independently introduced the term "space of boundary values" and in terms of this described all maximal dissipative (accumulative), self-adjoint, etc. extensions of symmetric operators.

To investigate the spectral properties of non-self-adjoint (dissipative) and nonunitary operators acting on a Hilbert space, functional model theory plays a prominent role. The rich and comprehensive theory has been developed since pioneering works of M. Brodskiĭ, M. Livŝiç, B. Sz.-Nagy, C. Foiaş, L. de Branges, and J. Rovnyak, see [24, 26] and references therein. It should be noted that the first work of B. S. Pavlov that contains an application of functional model (see [27-29]) rooted in the scattering theory of Lax and Phillps

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[21]. Functional model generated by dilations represents a new trend in spectral theory of dissipative (contractive) operators (see [24, 26-29]). Characteristic function carries the complete information on the spectral properties of the dissipative operators and plays a main role in this theory. To be more precise, the dissipative operators in the incoming spectral representation of the dilation become the model. The information about the completeness of the system of eigenvectors and associated vectors may be obtained in terms of the factorization of the characteristic function. self-adjoint dilation of the corresponding scattering problem is important to compute the characteristic functions of dissipative operators, in which the characteristic function is realized as the scattering matrix (see [21]). The adequacy of this approach to dissipative singular differential and difference operators has been demonstrated, for example, in [1-6, 8, 18, 27-29].

Let $\mathfrak{L}_A^2(\mathcal{I};E)$ ($\mathcal{I} := [a,b), -\infty < a < b \leq +\infty$, $\dim E = m < +\infty$, $A > 0$) denote a Hilbert space in which the minimal symmetric singular matrix-valued Sturm–Liouville operator $\mathfrak{L}_{\min}$ with maximal deficiency indices $(2m, 2m)$ (in limit-circle case at singular endpoint b) acts on it. In this paper, the maximal dissipative operators that the extensions of a minimal symmetric operator $\mathfrak{L}_{\min}$ are studied. We investigate maximal dissipative operators with general (for example coupled or separated) boundary conditions. In particular, if we consider separated boundary conditions, the at a and at b non-self-adjoint (dissipative) boundary conditions are prescribed simultaneously. We construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations, and determine the scattering matrix of the dilation, using to the scheme of Lax and Phillips [21]. With the help of the incoming spectral representation, we then construct a functional model of the maximal dissipative operator and define its characteristic function in terms of the scattering matrix of the dilation (or in terms of the Weyl function of self-adjoint operator). Finally, using these results, we prove the theorem on completeness of the system of eigenvectors and associated vectors (or root vectors) of the maximal dissipative operators.

2. Preliminaries and extensions of a symmetric operator

The matrix-valued Sturm–Liouville differential equation with singular endpoint b is considered as follows:

$$L_1(y) := -((P(x)y'(x))' + Q(x)y(x) = \lambda A(x)\, y(x),\ x \in \mathcal{I}. \tag{2.1}$$

Here λ is a spectral parameter, $P^*(x) = P(x)$, $\det P(x) \neq 0$, $Q^*(x) = Q(x)$, $A^*(x) = A(x) > 0$ (for almost all $x \in \mathcal{I}$) on E (E is the m ($m < \infty$) dimensional Euclidean space), and the entries of the $m \times m$ matrices $P^{-1}(x)$, $Q(x)$ and $A(x)$ are Lebesgue measurable and locally integrable functions on $\mathcal{I}$ (see [3, 13, 25, 34]). Note that these conditions for P, Q and A are minimal and there is no sign restriction on the coefficient P.

Let $L(y) := A^{-1}L_1(y)$ ($x \in \mathcal{I}$) denote the differential expression and $\mathfrak{H} := \mathfrak{L}_A^2(\mathcal{I};E)$ denote the Hilbert space consisting of all the vector-valued functions y such that

$$\int_a^b (A(x)y(x), y(x))_E dx < +\infty$$

with the inner product $(y,z) := \int_a^b (A(x)y(x), z(x))_E dx$. Let $\mathfrak{D}_{\max}$ be the linear set of the vectors $y \in \mathfrak{H}$ such that y and Py' are locally absolutely continuous vector-valued functions on $\mathcal{I}$, and $L(y) \in \mathfrak{H}$. We define the operator $\mathfrak{L}_{\max}$ on $\mathfrak{D}_{\max}$ by the equality $\mathfrak{L}_{\max} y = L(y)$.

For two arbitrary vectors $y, z \in \mathfrak{D}_{\max}$, we have Green's formula

$$\int_a^x ((L_1(y))(\xi), z(\xi))_E d\xi - \int_a^x (y(\xi), (L_1(z))(\xi))_E d\xi = [y,z](x) - [y,z](a), \tag{2.2}$$

where $[y,z](x) := (y(x), (Pz')(x))_E - ((Py')(x), z(x))_E$ ($x \in \mathcal{I}$). (2.2) implies that limit $[y,z](b) := \lim_{x\to b^-} [y,z](x)$ exists and is finite for all $y, z \in \mathfrak{D}_{\max}$. For any vector-valued function $y \in \mathfrak{D}_{\max}$, $y(a)$ and $(py')(a)$ can be defined by $y(a) := \lim_{x\to a^+} y(x)$ and $(Py')(a) := \lim_{x\to a^+} (Py')(x)$. These limits exist and are finite (since y and

Py' are absolutely continuous vector-valued functions on $[a,c]$, $\forall c \in (a,b)$). Therefore, passing to the limit as $x \to b^-$ in (2.2), we get that for arbitrary $y,z \in \mathfrak{D}_{\max}$

$$(\mathfrak{L}_{\max}y,z) - (y,\mathfrak{L}_{\max}z) = [y,z](b) - [y,z](a). \tag{2.3}$$

Let $\mathfrak{D}_{\min}$ be a dense set in $\mathfrak{H}$ consisting of smooth, compactly supported vector-valued functions. Denote by $\mathfrak{L}'_0$ the restriction of the operator $\mathfrak{L}_{\max}$ to $\mathfrak{D}'_{\min}$. It follows from (2.3) that $\mathfrak{L}'_{\min}$ is symmetric operator. Consequentially, it admits closure which is denoted by $\mathfrak{L}_{\min}$. The domain of $\mathfrak{L}_{\min}$ consist of vectors $y \in \mathfrak{D}_{\max}$ satisfying the conditions

$$y(a) = (Py')(a) = 0,\ [y,z](b) = 0,\ \forall z \in \mathfrak{D}_{\max}. \tag{2.4}$$

The operator $\mathfrak{L}_{\min}$ is a symmetric operator with deficiency indices (k_-,k_+), where $0 \le k_\pm \le 2m$, and satisfying $\mathfrak{L}_{\max} = \mathfrak{L}^*_{\min}$ [3, 13, 25, 34]. The operators $\mathfrak{L}_{\min}$ and $\mathfrak{L}_{\max}$ are called, the *minimal* and *maximal operators*, respectively.

Further, we assume that the symmetric operator $\mathfrak{L}_{\min}$ has maximal deficiency indices $(2m,2m)$, so that the Weyl limit-circle case holds for $\mathfrak{L}_{\min}$ or L. There are several sufficient conditions that guarantee limit-circle case (see [7, 9, 10, 13-15, 20, 22, 23, 25, 30, 33, 34]).

Let $V_1(x)$ and $V_2(x)$ denote the matrix (operator)-valued solution of the equation

$$L(y) = O\ (x \in \mathcal{I}) \tag{2.5}$$

satisfying the initial conditions

$$V_1(a) = I,\ (PV_1')(a) = O,\ V_2(a) = O,\ (PV_2')(a) = I. \tag{2.6}$$

Here O (respectively I) is the zero (respectively identity) operator in E. We note that the Wronskian of the two matrix-valued solutions $V_1(x)$ and $V_2(x)$ does not depend on x:

$$\begin{aligned} W_x(V_1,V_2) &= (V_2'^*P)(x)V_1(x) - V_2^*(x)(PV_1')(x) \\ &= W_a(V_1,V_2) = I\ (x \in \mathcal{I}). \end{aligned} \tag{2.7}$$

Construct the matrix

$$\mathbf{U}(x) := \begin{pmatrix} V_1(x) & V_2(x) \\ (PV_1')(x) & (PV_2')(x) \end{pmatrix}.$$

Using the formula (2.7) one can show that

$$J^{-1}\mathbf{U}^*(x)J = \mathbf{U}^{-1}(x),\ \mathbf{U}^{-1}(x) = \begin{pmatrix} (V_2'^*P)(x) & -V_2^*(x) \\ -(V_1'^*P)(x) & V_1^*(x) \end{pmatrix},$$

where

$$J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix},$$

$J^{-1} = J^* = -J$, $J^2 = -I_{E\oplus E}$ and $I_{E\oplus E}$ is the identity operator in $E \oplus E$.

Let us adopt the following:

$$\begin{aligned} (Sy)(x) \quad : \quad &= \begin{pmatrix} (S_1y)(x) \\ (S_2y)(x) \end{pmatrix} := \mathbf{U}^{-1}(x)\begin{pmatrix} y(x) \\ (Py')(x) \end{pmatrix} \\ &= \begin{pmatrix} (V_2'^*P)(x)y(x) - V_2^*(x)(Py')(x) \\ -(V_1'^*P)(x)y(x) + V_1^*(x)(Py')(x) \end{pmatrix}. \end{aligned}$$

It can be seen that for all vectors $f \in D_{\max}$ the limit $\lim_{x\to b^-}(Sf)(x) = (Sf)(b)$ exists and finite ([3]). The domain $D_{\min}$ of the operator $L_{\min}$ consists of vectors $f \in D_{\max}$, satisfying the following boundary conditions ([3])

$$f(a) = (Pf')(a) = 0,\ (S_1f)(b) = (S_2f)(b) = 0.$$

For arbitrary vectors $f, g \in D_{\max}$, we have the equality ([3])

$$[f,g](x) = ((S_1f)(x),(S_2g)(x))_E - ((S_2f)(x),(S_1g)(x))_E,\ a \le x \le b. \tag{2.8}$$

It is better to remind that a linear operator $\mathbf{S}$ (with dense domain $\mathfrak{D}(\mathbf{S})$) acting on some Hilbert space $\mathbf{H}$ is called *dissipative (accumulative)* if $\Im(\mathbf{S}f,f) \ge 0$ ($\Im(\mathbf{S}f,f) \le 0$) for all $f \in \mathfrak{D}(\mathbf{S})$ and *maximal dissipative (maximal accumulative)* if it does not have a proper dissipative (accumulative) extension ([17]).

The space of boundary values of the symmetric operator stays in the centre of the extension theory. The triplet $(\mathbf{H},\Phi_1,\Phi_2)$, where $\mathbf{H}$ is a Hilbert space and Φ_1 and Φ_2 are linear mappings of $\mathfrak{D}(\mathcal{A}^*)$ into $\mathbf{H}$, is called (see [11, 17, p. 152, 19]) a *space of boundary values* of a closed symmetric operator $\mathcal{A}$ acting in a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficiency indices if

(i) $(\mathcal{A}^*h,g)_{\mathcal{H}} - (h,\mathcal{A}^*g)_{\mathcal{H}} = (\Phi_1h,\Phi_2g)_{\mathbf{H}} - (\Phi_2h,\Phi_1g)_{\mathbf{H}},\ \forall h,g \in \mathfrak{D}(\mathcal{A}^*)$, and

(ii) for every $G_1, G_2 \in \mathbf{H}$, there exists a vector $g \in \mathfrak{D}(\mathcal{A}^*)$ such that $\Phi_1g = G_1$ and $\Phi_2g = G_2$.

We consider the following linear maps $\mathfrak{D}_{\max}$ into $E \oplus E$

$$F_1f = \begin{pmatrix} -f(a) \\ (S_1f)(b) \end{pmatrix},\ F_2f = \begin{pmatrix} (Pf')(a) \\ (S_2f)(b) \end{pmatrix}\ (f \in \mathfrak{D}_{\max}). \tag{2.9}$$

Then we have ([3])

Theorem 2.1. *The triplet $(E \oplus E, F_1, F_2)$ defined by (2.9) is a space of boundary values of the symmetric operator $\mathfrak{L}_{\min}$. For any contraction T in $E \oplus E$ the restriction of operator $\mathfrak{L}_{\max}$ to set of vectors $y \in \mathfrak{D}_{\max}$ satisfying the boundary conditions*

$$(T - I)F_1y + i(T + I)F_2y = 0 \tag{2.10}$$

or

$$(T - I)F_1y - i(T + I)F_2y = 0 \tag{2.11}$$

is, respectively, a maximal dissipative or a maximal accumulative extensions of the symmetric operator $\mathfrak{L}_{\min}$. Conversely, any maximal dissipative (maximal accumulative) extensions of $\mathfrak{L}_{\min}$ is the restriction of $\mathfrak{L}_{\max}$ to a set of vectors $y \in \mathfrak{D}_{\max}$ satisfying (2.10) ((2.11)), and the contraction T is uniquely determined by the extension. These conditions define a self-adjoint extension of $\mathfrak{L}_{\min}$ if and only if T is unitary. In the latter case, (2.10) and (2.11) are equivalent to the condition $(\cos S)F_1y - (\sin S)F_2y = 0$, where S is a self-adjoint operator in $E \oplus E$. The general form of dissipative and accumulative extensions of the operator $\mathfrak{L}_{\min}$ is given by the conditions

$$T(F_1y + iF_2y) = F_1y - iF_2y,\ F_1y + iF_2y \in \mathfrak{D}(T), \tag{2.12}$$

$$T(F_1y - iF_2y) = F_1y + iF_2y,\ F_1y - iF_2y \in \mathfrak{D}(T), \tag{2.13}$$

respectively, where T is a linear operator with $\|Tf\| \le \|f\|$, $f \in \mathfrak{D}(T)$. The general form of symmetric extensions is given by the formulae (2.12) and (2.13), where T is an isometric operator in $E \oplus E$.

Construct the matrix

$$T = \begin{pmatrix} T_1 & O \\ O & T_2 \end{pmatrix}.$$

Then by the Theorem 2.1 we have

Corollary 2.2. *The boundary conditions* ($y \in \mathfrak{D}_{\max}$)

$$(T_1 - I)y(a) \mp i(T_1 + I)y(a) = 0, \tag{2.13}$$

$$(T_2 - I)(S_1 y)(b) \pm i(T_2 + I)(S_2 y)(b) = 0, \tag{2.14}$$

where T_1 and T_2 are contractions in E, describe all the maximal dissipative and maximal accumulative extensions with separated boundary conditions of the symmetric operator $\mathfrak{L}_{\min}$. These conditions define a self-adjoint extensions of $\mathfrak{L}_{\min}$ if T_1 and T_2 are unitary.

Throughout the sequel we shall study the maximal dissipative operator $\mathfrak{L}_T$, where $\|T\|_{E\oplus E} < 1$ (i.e., a strict contraction in $E \oplus E$), generated by the expression L and boundary condition (2.10). It is obvious that the boundary condition, generally speaking, may be nonseparated. In particular, if we consider separated boundary conditions (2.13) and (2.14), then at end points a and b there are simultaneously non-self-adjoint (dissipative) boundary conditions.

One can infer that the boundary condition (2.10) is equivalent to the condition

$$F_2 y + BF_1 y = 0, \tag{2.15}$$

where $B = -i(T + I)^{-1}(T - I)$, $\Im B > 0$, and $-T$ is the Cayley transform of the dissipative operator B. This result comes from the fact that T is a strict contraction and therefore the operator $T + I$ must be invertible. We denote by $\widetilde{\mathfrak{L}}_B$ (= $\mathfrak{L}_T$) the maximal dissipative operator generated by the expression L and the boundary condition (2.15)

3. Self-adjoint dilation of the dissipative operator

The Hilbert space $\mathbb{H}$ is constructed as adding the 'incoming' and 'outgoing' channels $\mathfrak{L}^2(\mathbb{R}_-; E \oplus E)$ ($\mathbb{R}_- := (-\infty, 0]$) and $\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ ($\mathbb{R}_+ := [0, \infty)$) to the Hilbert space $\mathfrak{H}$. In fact, we form the orthogonal sum $\mathbb{H} = \mathfrak{L}^2(\mathbb{R}_-; E \oplus E) \oplus \mathfrak{H} \oplus \mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$. The Hilbert space $\mathbb{H}$ is called the *main Hilbert space of the dilation*. Clearly, the elements of $\mathbb{H}$ are three-component vector-valued functions $h = \langle \phi_-, y, \phi_+ \rangle$. In the main Hilbert space $\mathbb{H}$, let us consider the operator $\mathcal{L}_B$ generated by the expression

$$\mathcal{L}\langle \phi_-, y, \phi_+ \rangle = \langle i\frac{d\phi_-}{d\xi}, L(y), i\frac{d\phi_+}{d\zeta} \rangle \tag{3.1}$$

on the set of elements $\mathfrak{D}(\mathcal{L}_B)$ satisfying the conditions: $\phi_- \in W_2^1(\mathbb{R}_-; E \oplus E)$, $\phi_+ \in W_2^1(\mathbb{R}_+; E \oplus E)$, $y \in \mathfrak{D}_{\max}$ and

$$F_2 y + BF_1 y = K\phi_-(0), \; F_2 y + B^* F_1 y = K\phi_+(0), \tag{3.2}$$

where $K^2 := 2\Im B$, $K > 0$, and $W_2^1(\mathbb{R}_\mp; E \oplus E)$ is the Sobolev space. Then we have the following theorem.

Theorem 3.1. *The operator $\mathcal{L}_B$ is self-adjoint in $\mathbb{H}$ and it is a self-adjoint dilation of the maximal dissipative operator $\widetilde{\mathfrak{L}}_B$ (= $\mathfrak{L}_T$).*

Proof. For the vector-valued functions $h = \langle \phi_-, y, \phi_+ \rangle$, $g = \langle \vartheta_-, z, \vartheta_+ \rangle \in \mathfrak{D}(\mathcal{L}_B)$ we obtain that

$$(\mathcal{L}_B h, g)_{\mathbb{H}} - (h, \mathcal{L}_B g)_{\mathbb{H}} = i\left(\phi_-(0), \vartheta_-(0)\right)_{E\oplus E}$$

$$-i\left(\phi_+(0), \vartheta_+(0)\right)_{E\oplus E} + [y, z](b) - [y, z](a). \tag{3.3}$$

Using the boundary conditions (3.2) and (2.8), with a direct calculation we obtain that $i\left(\phi_-(0), \vartheta_-(0)\right)_{E\oplus E}$ $-i\left(\phi_+(0), \vartheta_+(0)\right)_{E\oplus E} + [y, z](b) - [y, z](a) = 0$. Thus, the operator $\mathcal{L}_B$ is symmetric, and $\mathfrak{D}(\mathcal{L}_B) \subseteq \mathfrak{D}(\mathcal{L}_B^*)$

It is easy to check that $\mathcal{L}_B$ and $\mathcal{L}_B^*$ are generated by the same expression (3.1). Let us describe the domain of $\mathcal{L}_B^*$. We shall compute the terms outside the integral sign, which are obtained by integration by parts in bilinear form $(\mathcal{L}_B h, g)_{\mathbb{H}}$, $h \in \mathfrak{D}(\mathcal{L}_B)$, $g \in \mathfrak{D}\left(\mathcal{L}_B^*\right)$. Their sum is equal to zero:

$$[y,z](b) - [y,z](a) + i\left(\phi_-(0), \vartheta_-(0)\right)_{E\oplus E} - i\left(\phi_+(0), \vartheta_+(0)\right)_{E\oplus E} = 0. \tag{3.4}$$

Moreover, solving the boundary conditions (3.2) for $F_1 y$ and $F_2 y$, we find that

$$F_1 y = -iK^{-1}\left(\phi_-(0) - \phi_+(0)\right),\ F_2 y = K\phi_-(0) + iBK^{-1}\left(\phi_-(0) - \phi_+(0)\right).$$

Hence, using (2.8), one may see that (3.4) is equivalent to the following

$$i\left(\phi_+(0), \vartheta_+(0)\right)_{E\oplus E} - i\left(\phi_-(0), \vartheta_-(0)\right)_{E\oplus E} = [y,z](b) - [y,z](a)$$

$$= (F_1 y, F_2 z)_{E\oplus E} - (F_2 y, F_1 z)_{E\oplus E} = -i\left(K^{-1}\left(\phi_-(0) - \phi_+(0)\right), F_2 z\right)_{E\oplus E}$$

$$-\left(K\phi_-(0), F_1 z\right)_{E\oplus E} - i\left(BK^{-1}\left(\phi_-(0) - \phi_+(0)\right), F_1 z\right)_{E\oplus E}.$$

Since the values $\phi_\pm(0)$ can be arbitrary vectors, a comparison of the coefficients of $\phi_{i\pm}(0)$ $(i = 1, 2, ..., 2m)$ on the left and right of this equality gives that the vector $g = \langle \vartheta_-, z, \vartheta_+ \rangle$ satisfies the boundary conditions (3.2): $F_2 z + BF_1 z = K\vartheta_-(0)$, $F_2 z + B^* F_1 z = K\vartheta_+(0)$. This implies that $\mathfrak{D}(\mathcal{L}_B^*) \subseteq \mathfrak{D}(\mathcal{L}_B)$, and consequently $\mathcal{L}_B = \mathcal{L}_B^*$.

It is known that the self-adjoint operator $\mathcal{L}_B$ generates the unitary group $\mathbb{U}(s) = \exp[i\mathcal{L}_B s]$ $(s \in \mathbb{R} := (-\infty, \infty))$ on $\mathbb{H}$. Denote by $\mathcal{P} : \mathbb{H} \to \mathfrak{H}$ and $\mathcal{P}_1 : \mathfrak{H} \to \mathbb{H}$ being the mappings defined as $\mathcal{P} : \langle \phi_-, y, \phi_+ \rangle \to y$ and $\mathcal{P}_1 : y \to \langle 0, y, 0 \rangle$, respectively. Let us construct a strongly continuous semigroup of completely nonunitary contractions on $\mathfrak{H}$ as $\mathcal{V}(s) = \mathcal{P}\mathbb{U}(s)\mathcal{P}_1$, $s \geq 0$. Denote by $\mathbf{A}_B$ the generator of this semigroup, i.e., $\mathbf{A}_B y = \lim_{s\to+0}(is)^{-1}(\mathcal{V}(s)y - y)$. The domain of $\mathbf{A}_B$ consists of all the vectors for which the limit exists. It is known that the operator $\mathbf{A}_B$ is a maximal dissipative and the operator $\mathcal{L}_B$ is called the *self-adjoint dilation* of $\mathbf{A}_B$ [24, 26]. It should be shown that $\widetilde{\mathfrak{L}}_B = \mathbf{A}_B$, and this implies that $\mathcal{L}_B$ is a self-adjoint dilation of $\widetilde{\mathfrak{L}}_B$. For this purpose set the equality

$$\mathcal{P}(\mathcal{L}_B - \lambda I)^{-1}\mathcal{P}_1 y = (\widetilde{\mathfrak{L}}_B - \lambda I)^{-1} y,\ y \in \mathfrak{H},\ \Im\lambda < 0. \tag{3.5}$$

Construct the vector-function g as $(\mathcal{L}_B - \lambda I)^{-1}\mathcal{P}_1 y = g = \langle \vartheta_-, z, \vartheta_+ \rangle$. Therefore $(\mathcal{L}_B - \lambda I)g = \mathcal{P}_1 y$, and hence, $L(z) - \lambda z = y$, $\vartheta_-(\xi) = \vartheta_-(0)e^{-i\lambda\xi}$, $\vartheta_+(\zeta) = \vartheta_+(0)e^{-i\lambda\zeta}$. Since $g \in \mathfrak{D}(\mathcal{L}_B)$, hence, $\vartheta_- \in W_2^1(\mathbb{R}_-; E)$, and so $\vartheta_-(0) = 0$ and, consequently, z satisfies the boundary condition $F_2 z + BF_1 z = 0$. Therefore, $z \in \mathfrak{D}(\widetilde{\mathfrak{L}}_B)$, and since a point λ with $\mathrm{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = (\widetilde{\mathfrak{L}}_B - \lambda I)y$. Thus, for $y \in \mathfrak{H}$ and $\mathrm{Im}\lambda < 0$, we have

$$(\mathcal{L}_B - \lambda I)^{-1}\mathcal{P}_1 y = \langle 0, (\widetilde{\mathfrak{L}}_B - \lambda I)^{-1} y, K^{-1}(F_2 y + B^* F_1 y)e^{-i\lambda\zeta} \rangle.$$

Using the mapping $\mathcal{P}$ and (3.5) we obtain that

$$(\widetilde{\mathfrak{L}}_B - \lambda I)^{-1} = \mathcal{P}(\mathcal{L}_B - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathbb{U}(s)e^{-i\lambda s} ds \mathcal{P}_1$$

$$= -i\int_0^\infty \mathcal{V}(s)e^{-i\lambda s} ds = (\mathbf{A}_B - \lambda I)^{-1},\ Im\lambda < 0$$

and therefore $\widetilde{\mathfrak{L}}_B = \mathbf{A}_B$. This completes the proof. □

4. Scattering theory of the dilation and functional model of the dissipative operator

Lax–Phillip's theory makes us possible to construct the model operator (see [21]). In fact, the unitary group $\{\mathbb{U}(s)\}$ and the 'incoming' and 'outgoing' subspaces $\mathcal{D}_- = \langle \mathfrak{L}^2(\mathbb{R}_-; E \oplus E), 0, 0\rangle$ and $\mathcal{D}_+ = \langle 0, 0, \mathfrak{L}^2(\mathbb{R}_+; E \oplus E)\rangle$ have the following properties:

(**1**) $\mathbb{U}(s)\mathcal{D}_- \subset \mathcal{D}_-$, $s \leq 0$ and $\mathbb{U}(s)\mathcal{D}_+ \subset \mathcal{D}_+$, $s \geq 0$;

(**2**) $\overline{\cap_{s\leq 0}\mathbb{U}(s)\mathcal{D}_-} = \overline{\cap_{s\geq 0}\mathbb{U}(s)\mathcal{D}_+} = \{0\}$;

(**3**) $\overline{\cup_{s\geq 0}\mathbb{U}(s)\mathcal{D}_-} = \overline{\cup_{s\leq 0}\mathbb{U}(s)\mathcal{D}_+} = \mathbb{H}$;

(**4**) $\mathcal{D}_- \perp \mathcal{D}_+$.

The property (**4**) is obvious. To prove the property (**1**) for $\mathcal{D}_+$ (for $\mathcal{D}_-$, the proof is analogous) we shall set $R_\lambda = (\mathcal{L}_B - \lambda I)^{-1}$. For all λ with $\mathrm{Im}\lambda < 0$ and for all $h = \langle 0, 0, \phi_+\rangle \in \mathcal{D}_+$, we have

$$R_\lambda h = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\phi_+(s)\,ds\rangle$$

and therefore $R_\lambda h \in \mathcal{D}_+$. Hence, if $g \perp \mathcal{D}_+$, then

$$0 = (R_\lambda h, g)_{\mathbb{H}} = -i\int_0^\infty e^{-i\lambda s}(\mathbb{U}(s)h, g)_{\mathbb{H}}\,ds,\ \Im\lambda < 0.$$

This implies that $(\mathbb{U}(s)h, g)_{\mathbb{H}} = 0$ for all $s \geq 0$ and therefore $\mathbb{U}(s)\mathcal{D}_+ \subset \mathcal{D}_+$ for $s \geq 0$. This proves the property (**1**).

To show that the property (**2**) holds let us define the mappings $\mathcal{P}^+ : \mathbb{H} \to \mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ and $\mathcal{P}_1^+ : \mathfrak{L}^2(\mathbb{R}_+; E \oplus E) \to \mathcal{D}_+$ as $\mathcal{P}^+ : \langle \phi_- y, \phi_+\rangle \to \phi_+$ and $\mathcal{P}_1^+ : \phi \to \langle 0, 0, \phi\rangle$, respectively. Note that the semigroup of isometries $\mathbb{U}^+(s) = \mathcal{P}^+\mathbb{U}(s)\mathcal{P}_1^+, s \geq 0$ is the one-side shift in $\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$. Namely, the generator of the semigroup of the shift $\mathcal{Z}(s)$ in $\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator $\mathbf{S}$ of semigroup of isometries $\mathbb{U}^+(s)$, $s \geq 0$ is the operator

$$\mathbf{S}\varphi = \mathcal{P}^+\mathcal{L}_B\mathcal{P}_1^+\varphi = \mathcal{P}^+\mathcal{L}_B\langle 0, 0, \varphi\rangle = \mathcal{P}^+\langle 0, 0, i\frac{d\varphi}{d\xi}\rangle = i\frac{d\varphi}{d\xi},$$

where $\varphi \in W_2^1(\mathbb{R}_+; E \oplus E)$ and $\varphi(0) = 0$. However, a semigroup is uniquely determined by its generator. Therefore we have that $\mathbb{U}^+(s) = \mathcal{Z}(s)$, and hence,

$$\bigcap_{s\geq 0}\mathbb{U}(s)\mathcal{D}_+ = \langle 0, 0, \bigcap_{s\geq 0}\mathcal{Z}(s)\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)\rangle = \{0\}.$$

Thus the property (**2**) is proved.

According to the Lax–Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. Now we apply this theory to construct these representations. In these process, we also prove the property (**3**) of the incoming and outgoing subspaces.

We shall remind that the linear operator $\mathbf{A}$ (with domain $\mathfrak{D}(A)$) acting in the Hilbert space $\mathbf{H}$ is called *completely non-self-adjoint* (or *simple)* if there is no invariant subspace $M \subseteq \mathfrak{D}(\mathbf{A})$ ($M \neq \{0\}$) of the operator $\mathbf{A}$ on which the restriction $\mathbf{A}$ to M is self-adjoint.

Lemma 4.1. *The operator $\widetilde{\mathfrak{L}}_B$ is completely non-self-adjoint (simple).*

Proof. Consider that $\mathfrak{H}_0 \subseteq \mathfrak{H}$ be a subspace and let the operator $\widetilde{\mathfrak{L}}_B$ indicates the self-adjoint operator $\widetilde{\mathfrak{L}}'_B$. For $h \in \mathfrak{H}_0 \cap \mathfrak{D}(\widetilde{\mathfrak{L}}_B)$ we have $h \in \mathfrak{D}(\widetilde{\mathfrak{L}}^*_B)$. Therefore

$$0 = (\widetilde{\mathfrak{L}}'_B h, h)_{E\oplus E} - (h, \widetilde{\mathfrak{L}}'_B h)_{E\oplus E} = (F_1 h, F_2 h)_E - (F_2 h, F_1 h)_E = (F_1 h, -BF_1 h)_{E\oplus E}$$

$$-(-BF_1 h, F_1 h)_{E\oplus E} = ((B - B^*)F_1 h, F_1 h)_{E\oplus E} = 2i(\Im BF_1 h, F_1 h)_{E\oplus E}.$$

This implies that $F_1 h = 0$. For eigenvectors $y_\lambda \in \mathfrak{H}_0$ of the operator $\widetilde{\mathfrak{L}}_B$, we have $F_1 y_\lambda = 0$ and the boundary condition $F_2 y + BF_1 y = 0$ implies that $F_2 y_\lambda = 0$, i.e., $y_\lambda(a) = 0$, $(Py'_\lambda)(a) = 0$. Then by the uniqueness theorem of the Cauchy problem for the equation $L(y) = \lambda y$ $(x \in \mathcal{I})$, we have $y_\lambda \equiv 0$. Since all solutions of $L(y) = \lambda y$ $(x \in \mathcal{I})$ belong to $\mathfrak{H}$, from this it can be concluded that the resolvent $R_\lambda(\widetilde{\mathfrak{L}}_B)$ of the operator $\widetilde{\mathfrak{L}}_B$ is a Hilbert–Schmidt operator, and hence the spectrum of $\widetilde{\mathfrak{L}}_B$ is purely discrete. Therefore expansion theorem in eigenvectors of the self-adjoint operator shows that $\mathfrak{H}_0 = \{0\}$, i.e., the operator $\widetilde{\mathfrak{L}}_B$ is simple. The lemma is proved. □

Now let us set

$$\mathbb{H}_- = \overline{\cup_{s\geq 0}\mathbb{U}(s)\mathcal{D}_-},\ \mathbb{H}_+ = \overline{\cup_{s\leq 0}\mathbb{U}(s)\mathcal{D}_+}.$$

This setting allows us to prove the property (**3**).

Lemma 4.2. *The equality* $\mathbb{H}_- + \mathbb{H}_+ = \mathbb{H}$ *holds.*

Proof. Let $\mathbb{H}' = \mathbb{H} \ominus (\mathbb{H}_- + \mathbb{H}_+)$. Then one can consider that $\mathbb{H}'$ is of the form $\mathbb{H}' = \langle 0, \mathfrak{H}', 0\rangle$, where $\mathfrak{H}'$ is a subspace of $\mathfrak{H}$. Our assertion is that $\mathbb{H}' = \{0\}$. Otherwise, if the subspace $\mathbb{H}'$ (and hence, also $\mathfrak{H}'$) were nontrivial, then the unitary group $\{\mathbb{U}(s)\}$ restricted to this subspace, would be a unitary part of the group $\{\mathbb{U}(s)\}$. Consequently, the restriction $\widetilde{\mathfrak{L}}'_B$ of the operator $\widetilde{\mathfrak{L}}_B$ to $\mathfrak{H}'$ would be the self-adjoint operator in $\mathfrak{H}'$. However the operator $\widetilde{\mathfrak{L}}_B$ is simple and therefore $\mathfrak{H}' = \{0\}$. This completes the proof. □

Consider the matrix-valued solutions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ of the equation (2.1) satisfying the conditions

$$\Phi(a, \lambda) = O,\ (P\Phi')(a, \lambda) = -I,\ \Psi(a, \lambda) = I,\ (P\Psi')(a, \lambda) = O. \tag{4.1}$$

Let $M(\lambda)$ be the matrix-valued function (Weyl function) (see [12]) with the rule

$$M(\lambda) F_1\Phi = F_2\Phi,\ M(\lambda) F_1\Psi = F_2\Psi. \tag{4.2}$$

It is easy to show that the matrix-valued function $M(\lambda)$ is meromorphic in $\mathbb{C}$ with all its poles on real axis $\mathbb{R}$, and that it has the following properties (see [12]):

(a) $\Im M(\lambda) \leq 0$ for $\Im\lambda > 0$, and $\Im M(\lambda) \geq 0$ for $\Im\lambda < 0$;

(b) $M^*(\lambda) = M(\bar{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of $M(\lambda)$.

Let $u_j(x, \lambda)$ and $v_j(x, \lambda)$ $(j = 1, 2, ..., 2m)$ be solutions of the equation (2.1) satisfying the conditions

$$F_1 u_j = (M(\lambda) + B)^{-1} Ke_j,\ F_1 v_j = (M(\lambda) + B^*)^{-1} Ke_j\ (j = 1, 2, ..., 2m), \tag{4.4}$$

where $e_1, e_2, ..., e_{2m}$ are the orthonormal basis for $E \oplus E$.

Consider the vectors $\Theta^-_{\lambda j}$ $(j = 1, 2, ..., 2m)$ defined by

$$\Theta^-_{\lambda j}(x, \xi, \zeta) = \langle e^{-i\lambda\xi} e_j, u_j(x, \lambda), K^{-1}(M + B^*)(M + B)^{-1} Ke^{-i\lambda\zeta} e_j\rangle.$$

It is clear that vectors $\Theta^-_{\lambda j}$ $(j = 1, 2, ..., 2m)$ for all $\lambda \in \mathbb{R}$ do not belong to $\mathbb{H}$. Beside this, these vectors $\Theta^-_{\lambda j}$ $(j = 1, 2, ..., 2m)$ satisfies the equation $\mathcal{L}V = \lambda V$ and the boundary conditions (3.2). The transformation $\mathbb{F}_- : h \to \tilde{h}_-(\lambda)$ for the vectors $h = \langle\phi_-, y, \phi_+\rangle$ is determined using the vectors $\Theta^-_{\lambda j}$ $(j = 1, 2, ..., 2m)$ by the formula

$$(\mathbb{F}_- h)(\lambda) := \tilde{h}_-(\lambda) := \sum_{j=1}^{2m} \tilde{h}^-_j(\lambda) e_j,$$

where ϕ_-, ϕ_+ and y are smooth, compactly supported vector-valued functions, and

$$\tilde{h}^-_j(\lambda) = \frac{1}{\sqrt{2\pi}}(h, \Theta^-_{\lambda j})_{\mathbb{H}}\ (j = 1, 2, ..., 2m).$$

Lemma 4.3. *The transformation* $\mathbb{F}_-$ isometrically *maps* $\mathbb{H}_-$ *onto* $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$. *For all vectors* $h, g \in \mathbb{H}_-$, *the Parseval equality*

$$(h,g)_{\mathbb{H}} = (\tilde{h}_-, \tilde{g}_-)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2m} \tilde{h}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)} d\lambda,$$

and the inversion formula

$$h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2m} \Theta_{\lambda j}^- \tilde{h}_j^-(\lambda)\, d\lambda$$

hold, where $\tilde{h}_-(\lambda) = (\mathbb{F}_- h)(\lambda)$, $\tilde{g}_-(\lambda) = (\mathbb{F}_- g)(\lambda)$.

Proof. It should be shown that the transformation $\mathbb{F}_-$ maps $\mathcal{D}_-$to $H_-^2(E \oplus E)$. Let $H_\pm^2(E \oplus E)$ denote the Hardy classes in $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$ consisting of the vector-valued functions analytically extendable to the upper and lower half-planes, respectively. Then for arbitrary two vectors $h, g \in \mathcal{D}_-$, $h = \langle h_-, 0, 0 \rangle$, $g = \langle g_-, 0, 0 \rangle$, $h_-, g_- \in \mathfrak{L}^2(\mathbb{R}_-; E \oplus E)$, we have

$$\tilde{h}_j^-(\lambda) = \frac{1}{\sqrt{2\pi}} (h, \Theta_{\lambda j}^-)_{\mathbb{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \left(h_-(\xi), e^{-i\lambda\xi} e_j \right)_{E\oplus E} d\xi \in H_-^2,$$

$$\tilde{h}_-(\lambda) = \sum_{j=1}^{2m} \tilde{h}_j^-(\lambda) e_j \in H_-^2(E \oplus E),$$

and the Parseval equality:

$$(h,g)_{\mathbb{H}} = (\tilde{h}_-, \tilde{g}_-)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2m} \tilde{h}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)} d\lambda.$$

Now, we want to extend this equality to the whole $\mathbb{H}_-$. For this aim, we consider in $\mathbb{H}_-$ the dense set $\mathbb{H}_-'$ of vectors, obtained on smooth, compactly supported vector-valued functions belonging to $\mathcal{D}_-$ by the following way: $h \in \mathbb{H}_-'$, $h = \mathbb{U}(s)h_0$, $h_0 = \langle \phi_-, 0, 0 \rangle$, $\phi_- \in C_0^\infty(\mathbb{R}_-; E \oplus E))$. For these vectors, noting $\mathcal{L}_B = \mathcal{L}_B^*$ and using the fact that $\mathbb{U}(-s)h \in \langle C_0^\infty(\mathbb{R}_-; E \oplus E), 0, 0 \rangle$ and $(\mathbb{U}(-s)h, \Theta_{\lambda j}^-)_{\mathbb{H}} = e^{-i\lambda s}(h, \Theta_{\lambda j}^-)_{\mathbb{H}}$ $(j = 1, 2, ..., 2m)$ for $s > s_h, s_g$, we have

$$(h,g)_{\mathbb{H}} = (\mathbb{U}(-s)h, \mathbb{U}(-s)g)_{\mathbb{H}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{2m} (\mathbb{U}(-s)h, \Theta_{\lambda j}^-)_{\mathbb{H}} \overline{(\mathbb{U}(-s)g, \Theta_{\lambda j}^-)_{\mathbb{H}}} d\lambda = \int_{-\infty}^{\infty} \sum_{j=1}^{2m} \tilde{h}_j^-(\lambda) \overline{\tilde{g}_j^-(\lambda)} d\lambda.$$

Passing to the closure, it is obtained that the Parseval equality holds for the whole space $\mathbb{H}_-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of the integrals on a finite intervals. Consequently, we get that

$$\mathbb{F}_- \mathbb{H}_- = \overline{\cup_{s\geq 0} \mathbb{F}_- \mathbb{U}(s) \mathcal{D}_-} = \overline{\cup_{s\geq 0} e^{i\lambda s} H_-^2(E \oplus E)} = \mathfrak{L}^2(\mathbb{R}; E \oplus E).$$

Hence $\mathbb{F}_-$ maps $\mathbb{H}_-$ onto whole $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$. So, the lemma is proved. □

Consider the vectors

$$\Theta_{\lambda j}^+(x, \xi, \zeta) = \langle S_B(\lambda) e^{-i\lambda\xi} e_j, v_j(\lambda), e^{-i\lambda\zeta} e_j \rangle \ (j = 1, 2, ..., 2m),$$

where

$$S_B(\lambda) = K^{-1}(M(\lambda) + B)(M(\lambda) + B^*)^{-1} K. \tag{4.5}$$

Using the vectors $\Theta^+_{\lambda j}$ $(j = 1, 2, ..., 2m)$, we will see that the transformation $\mathbb{F}_+ : h \to \tilde{h}_+(\lambda)$ for the vectors $h = \langle \phi_-, y, \phi_+ \rangle$ is determined by the formula

$$(\mathbb{F}_+ h)(\lambda) := \tilde{h}_+(\lambda) := \sum_{j=1}^{2m} \tilde{h}^+_j(\lambda) e_j,$$

where ϕ_-, ϕ_+, and y are smooth, compactly supported functions, and

$$\tilde{h}^+_j(\lambda) = \frac{1}{\sqrt{2\pi}} (h, \Theta^+_{\lambda j})_{\mathbb{H}} \ (j = 1, 2, ..., 2m).$$

The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. *The transformation $\mathbb{F}_+$ isometrically maps $\mathbb{H}_+$ onto $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$. For all vectors $h, g \in \mathbb{H}_+$, the Parseval equality*

$$(h, g)_{\mathbb{H}} = (\tilde{h}_+, \tilde{g}_+)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2m} \tilde{h}^-_j(\lambda) \overline{\tilde{g}^-_j(\lambda)} d\lambda,$$

and the inversion formula

$$h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2m} \Theta^+_{\lambda j} \tilde{h}^+_j(\lambda) d\lambda,$$

are valid, where $\tilde{h}^+_j(\lambda) = (\mathbb{F}_+ h)(\lambda)$, $\tilde{g}_+(\lambda) = (\mathbb{F}_+ g)(\lambda)$.

It is not so hard to see that the matrix-valued function $S_B(\lambda)$ is meromorphic in $\mathbb{C}$, and all poles are in the lower half-plane. From (4.5) it is obvious that $\|S_B(\lambda)\|_E \leq 1$ for $\Im\lambda > 0$ and $S_B(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$.

Since $S_B(\lambda)$ is the unitary matrix for $\lambda \in \mathbb{R}$, using the vectors $\Theta^+_{\lambda j}$ and $\Theta^-_{\lambda j}$ $(j = 1, 2, ..., 2m)$ we get that

$$\Theta^+_{\lambda j} = \sum_{k=1}^{2m} S_{jk}(\lambda) \Theta^-_{\lambda k} \ (j = 1, 2, ..., 2m),$$

where $S_{jk}(\lambda)$ $(j, k = 1, 2, ..., 2m)$ are entries of the matrix $S_B(\lambda)$. Lemmas 4.3 and 4.4 show that $\mathbb{H}_- = \mathbb{H}_+$. Together with Lemma 4.2, this implies that $\mathbb{H}_- = \mathbb{H}_+ = \mathbb{H}$. Hence, the property (**3**) for $\{\mathbb{U}(s)\}$ above has been established for the incoming and outgoing subspaces.

Hence the transformation $\mathbb{F}_-$ maps $\mathbb{H}$ isometrically onto $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$; the subspace $\mathcal{D}_-$ is mapped onto $H^2_-(E \oplus E)$, and the operators $\mathbb{U}(s)$ mapped to operators of multiplication by $e^{i\lambda s}$. This means that $\mathbb{F}_-$ is an incoming spectral representation of the group $\{\mathbb{U}(s)\}$. Similarly, $\mathbb{F}_+$ is an outgoing spectral representation of $\{\mathbb{U}(s)\}$. From the explicit formulas for $\Theta^-_{\lambda j}$ and $\Theta^+_{\lambda j}$ $(j = 1, 2, ..., 2m)$, it follows that the passage from the $\mathbb{F}_-$-representation of a vector $h \in \mathbb{H}$ to its $\mathbb{F}_+$-representation is accomplished as follows: $\tilde{h}_+(\lambda) = S_B^{-1}(\lambda)\tilde{h}_-(\lambda)$. Thus by [21], we have now proved

Theorem 4.5. *The matrix $S_B^{-1}(\lambda)$ is the scattering matrix of the group $\{\mathbb{U}(s)\}$ (of the operator $\mathcal{L}_B$).*

Let $S(\lambda)$ be an arbitrary non-constant inner matrix-valued function on the upper half-plane (the analytic matrix-valued function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $\|S(\lambda)\| \leq 1$ for $\lambda \in \mathbb{C}_+$

and $S(\lambda)$ is a unitary matrix for almost all $\lambda \in \mathbb{R}$). Define $\mathbf{K} = H_+^2 \ominus SH_+^2$. Then $\mathbf{K} \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of the operators $\mathbf{V}(s)$ $(s \geq 0)$ acting in $\mathbf{K}$ according to the formula $\mathbf{V}(s)\phi = \mathbf{P}\left[e^{i\lambda s}\phi\right]$, $\phi := \phi(\lambda) \in \mathbf{K}$, where $\mathbf{P}$ is the orthogonal projection from H_+^2 onto $\mathbf{K}$. The generator of the semigroup $\{\mathbf{V}(s)\}$ is denoted by $\mathbf{B} : \mathbf{B}\phi = \lim_{s\to+0}(is)^{-1}(\mathbf{V}(s)\phi-\phi)$. $\mathbf{B}$ is a maximal dissipative operator acting in $\mathbf{K}$ and its domain $\mathfrak{D}(\mathbf{B})$ consists of all vectors $\phi \in \mathbf{K}$ for which the above limit exists. The operator $\mathbf{B}$ is called a *model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax and Phillips [21], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [24]). We claim that $S(\lambda)$ is the *characteristic function* of the operator $\mathbf{B}$.

Hence it is obtained with the unitary transformation $\mathbb{F}_-$ that

$$\mathbb{H} \to \mathfrak{L}^2(\mathbb{R}; E \oplus E),\ h \to \tilde{h}_-(\lambda) = (\mathbb{F}_- h)(\lambda),\ \mathcal{D}_- \to H_-^2(E \oplus E),$$

$$\mathcal{D}_+ \to S_B H_+^2(E \oplus E),\ \mathbb{H} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+) \to H_+^2(E \oplus E) \ominus S_B H_+^2(E \oplus E),$$

$$\mathbb{U}(s)h \to (\mathbb{F}_- \mathbb{U}(s) \mathbb{F}_-^{-1} \tilde{h}_-)(\lambda) = e^{i\lambda s}\tilde{h}_-(\lambda).$$

These formulas show that the operator $\widetilde{\mathfrak{L}}_B$ $(\mathfrak{L}_T)$ is a unitary equivalent to the model dissipative operator with the characteristic function $S_B(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide with [24, 26- 28], we have proved

Theorem 4.6. *The characteristic function of the maximal dissipative operator $\widetilde{\mathfrak{L}}_B$ $(\mathfrak{L}_T)$ coincides with the matrix-valued function $S_B(\lambda)$ determined by formula (4.5). The matrix-valued function $S_B(\lambda)$ is meromorphic in the complex plane $\mathbb{C}$ and is an inner function in the upper half-plane.*

5. Completeness of the system of root vectors of the dissipative operator

Characteristic function may help us to find the answer of the questions of the spectral analysis of the maximal dissipative operator $\mathfrak{L}_T$ $(\widetilde{\mathfrak{L}}_B)$. It is known that the absence of the singular factor $s(\lambda)$ in the factorization $\det S_B(\lambda) = s(\lambda)\mathcal{B}(\lambda)$ ($\mathcal{B}(\lambda)$ is the Blaschke product) proves the completeness of the system of eigenvectors and associated vectors of the operator $\mathfrak{L}_T$ $(\widetilde{\mathfrak{L}}_B)$ in the space $\mathfrak{H}$ (see [24, 26]).

We first use the following

Lemma 5.1. *The characteristic function $\tilde{S}_T(\lambda)$ of the operator $\mathfrak{L}_T$ has the form*

$$\tilde{S}_T(\lambda) := S_B(\lambda) = Y_1\left(I - T_1T_1^*\right)^{-\frac{1}{2}}(\Upsilon(\xi) - T_1)\left(I - T_1^*\Upsilon(\xi)\right)^{-1}\left(I - T_1^*T_1\right)^{\frac{1}{2}}Y_2,$$

where $T_1 = -T$ is the Cayley transformation of the dissipative operator B, and $\Upsilon(\xi)$ is the Cayley transformation of the matrix-valued function $M(\lambda)$, $\xi = (\lambda - i)(\lambda + i)^{-1}$, and

$$Y_1 = (\Im B)^{-\frac{1}{2}}(I - T_1)^{-1}\left(I - T_1T_1^*\right)^{\frac{1}{2}},\ Y_2 = \left(I - T_1^*T_1\right)^{-\frac{1}{2}}\left(I - T_1^*\right)(\Im B)^{\frac{1}{2}},$$

$$|\det Y_1| = |\det Y_2| = 1.$$

Proof. Using Theorem 4.6, we get that

$$S_B(\lambda) = (\Im B)^{-\frac{1}{2}}(M(\lambda) + B)(M(\lambda) + B^*)^{-1}(\Im B)^{\frac{1}{2}}.$$

Hence

$$\Im B = \frac{1}{2i}(B - B^*) = \frac{1}{2}[(I - T_1)^{-1}(I + T_1) + \left(I + T_1^*\right)\left(I - T_1^*\right)^{-1}]$$

$$= \frac{1}{2}[(I - T_1)^{-1} + (I - T_1)^{-1} T_1 + \left(I - T_1^*\right)^{-1} + T_1^* \left(I - T_1^*\right)^{-1}]$$
$$= \frac{1}{2}[(I - T_1)^{-1} + (I - T_1)^{-1} - I + \left(I - T_1^*\right)^{-1} + \left(I - T_1^*\right)^{-1} - I]$$
$$= (I - T_1)^{-1} + \left(I - T_1^*\right)^{-1} - I$$
$$= (I - T_1)^{-1} \left[I - T_1^* + I - T_1 - (I - T_1)\left(I - T_1^*\right)\right] \left(I - T_1^*\right)^{-1}$$
$$= (I - T_1)^{-1} \left(I - T_1 T_1^*\right) \left(I - T_1^*\right)^{-1}. \tag{5.1}$$

Similarly we get that

$$\Im B = \left(I - T_1^*\right)^{-1} \left(I - T_1^* T_1\right) (I - T_1)^{-1}. \tag{5.2}$$

Consider the Cayley transformation $\Upsilon_1(\lambda)$ of the accumulative operator $M(\lambda)$ for $\Im\lambda > 0$. Therefore we get that $M(\lambda) = -i(I - \Upsilon_1(\lambda))^{-1}(I + \Upsilon_1(\lambda))$. Hence

$$M(\lambda) + B = -i[(I - \Upsilon_1(\lambda))^{-1}(I + \Upsilon_1(\lambda)) - (I - T_1)^{-1}(I + T_1)]$$
$$= -i[-(I - \Upsilon_1(\lambda))^{-1}(I - \Upsilon_1(\lambda) - 2I) + (I - T_1)^{-1}(I - T_1 - 2I)]$$
$$= -i[-I + 2(I - \Upsilon_1(\lambda))^{-1} + I - 2(I - T_1)^{-1}]$$
$$= -2i[(I - \Upsilon_1(\lambda))^{-1} - (I - T_1)^{-1}]$$
$$= -2i(I - T_1)^{-1}(\Upsilon_1(\lambda) - T_1)(I - \Upsilon_1(\lambda))^{-1}. \tag{5.3}$$

Similarly,

$$M(\lambda) + B^* = -2i\left(I - T_1^*\right)^{-1}\left(I - T_1^*\Upsilon_1(\lambda)\right)(I - \Upsilon_1(\lambda))^{-1},$$

and

$$(M(\lambda) + B^*)^{-1} = -\frac{1}{2i}(I - \Upsilon_1(\lambda))\left(I - T_1^*\Upsilon_1(\lambda)\right)^{-1}\left(I - T_1^*\right). \tag{5.4}$$

Therefore from (5.1)-(5.4) we obtain that

$$\tilde{S}_T(\lambda) = S_B(\lambda) = Y_1\left(I - T_1 T_1^*\right)^{-\frac{1}{2}}(\Upsilon(\xi) - T_1)\left(I - T_1^*\Upsilon(\xi)\right)\left(I - T_1^* T_1\right)^{\frac{1}{2}} Y_2.$$

Here

$$\Upsilon(\xi) := \Upsilon_1(-i(\xi + 1)(\xi - 1)^{-1}),$$
$$Y_1 := (\Im B)^{-\frac{1}{2}}(I - T_1)^{-1}\left(I - T_1^* T_1\right)^{\frac{1}{2}},$$
$$Y_2 := \left(I - T_1^* T_1\right)^{-\frac{1}{2}}\left(I - T_1^*\right)(\Im B)^{\frac{1}{2}}.$$

Obviously $|\det Y_1| = |\det Y_2| = 1$. Hence, the lemma is proved. □

It is known [16, 24, 26] that the inner matrix-valued function $\tilde{S}_T(\lambda)$ is a Blaschke–Potapov product if and only if $\det \tilde{S}_T(\lambda)$ is a Blaschke product. Then it follows from Lemma 5.1 that the characteristic function $\tilde{S}_T(\lambda)$ is a Blaschke–Potapov product if and only if the matrix-valued function

$$Y_T(\xi) = \left(I - T_1 T_1^*\right)^{-\frac{1}{2}}(\Upsilon(\xi) - T_1)\left(I - T_1^*\Upsilon(\xi)\right)^{-1}\left(I - T_1^* T_1\right)^{\frac{1}{2}}$$

is a Blaschke–Potapov product in a unit disk.

In order to state the completeness theorem, we will first define a suitable form for the Γ-capacity (see [16, 32]).

Let $\mathbf{E}$ be an N-dimensional ($N < +\infty$) Euclidean space. In $\mathbf{E}$, we fix an orthonormal basis $\varphi_1, \varphi_2, ..., \varphi_N$ and denote by $\mathbf{E}_k$ ($k = 1, 2, ..., N$) the linear span vectors $\varphi_1, \varphi_2, ..., \varphi_k$. If $\mathbf{M} \subset \mathbf{E}_k$, then the set of $v \in \mathbf{E}_{k-1}$ with the property $Cap\{\mu : \mu \in \mathbb{C},\ (v + \mu\varphi_k) \in \mathbf{M}\} > 0$ will be denoted by $\Gamma_{k-1}\mathbf{M}$. ($CapG$ is the inner logarithmic capacity of the set $G \subset \mathbb{C}$). The Γ-*capacity of the set* $\mathbf{M} \subset \mathbf{E}$ is a number

$$\Gamma - Cap\mathbf{M} := \sup Cap\{\mu : \mu \in \mathbb{C},\ \mu\varphi_1 \subset \Gamma_1\Gamma_2...\Gamma_{N-1}\mathbf{M}\},$$

where the sup is taken with respect to all orthonormal basics in $\mathbf{E}$. It is known [16, 32] that every set $M \subset \mathbf{E}$ of zero Γ-capacity has zero $2N$-dimensional Lebesgue measure (in the decomplexified space $\mathbf{E}$), however, the converse is false.

Let $\mathcal{L}[E \oplus E]$ be the set consisting of all linear operators acting in $E \oplus E$. To convert $\mathcal{L}[E \oplus E]$ into the $4m^2$-dimensional Euclidean space, we introduce the inner product $\langle T, S\rangle = trS^*T$ for $T, S \in \mathcal{L}[E \oplus E]$ (trS^*T is the trace of the operator S^*T). Hence, we may introduce the Γ-capacity of a set of $\mathcal{L}[E \oplus E]$.

We will utilize the following important result of [16].

Lemma 5.2. *Let $Y(\xi)$ ($|\xi| < 1$) be a holomorphic function with the values to be contractive operators in $\mathcal{L}[E \oplus E]$* (i.e., $\|Y(\xi)\| \leq 1$). *Then for Γ-quasi-every strictly contractive operators T in $\mathcal{L}[E \oplus E]$ (i.e., for all strictly contractive $T \in \mathcal{L}[E \oplus E]$ with the possible exception of a set of Γ-capacity zero), the inner part of the contractive function*

$$Y_T(\xi) = (I - TT^*)^{-\frac{1}{2}} (Y(\xi) - T)(I - T^*Y(\xi))^{-1} (I - T^*T)^{\frac{1}{2}}$$

is a Blaschke–Potapov product.

Denote by $\mathbf{S}$ being the linear operator in the Hilbert space $\mathbf{H}$ with the domain $\mathfrak{D}(\mathbf{S})$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathbf{S}$ if there exist a nonzero element $z_0 \in \mathfrak{D}(\mathbf{S})$ such that $\mathbf{S}z_0 = \lambda_0 z_0$. Such element z_0 is called the *eigenvector* of the operator $\mathbf{S}$ corresponding to the eigenvalue λ_0. The elements $z_1, z_2, ..., z_k$ are called the *associated vectors* of the eigenvector z_0 if they belong to $\mathfrak{D}(\mathbf{S})$ and $\mathbf{S}z_j = \lambda_0 z_j + z_{j-1}$, $j = 1, 2, ..., k$. The element $z \in \mathfrak{D}(\mathbf{S})$, $z \neq 0$ is called a *root vector* of the operator A corresponding to the eigenvalue λ_0, if all powers of $\mathbf{S}$ are defined on this element and $(\mathbf{S} - \lambda_0 I)^m z = 0$ for some integer m. The set of all root vectors of $\mathbf{S}$ corresponding to the same eigenvalue λ_0 with the vector $z = 0$ forms a linear set $\mathbf{K}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathbf{K}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathbf{K}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathbf{S}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathbf{S}$ is equivalent to the completeness of the system of all root vectors of this operator.

Summarizing all the obtained results for the maximal dissipative operators $\mathfrak{L}_T$ ($\widetilde{\mathfrak{L}_B}$), we have proved the following

Theorem 5.3. *For Γ-quasi-every strictly contractive $T \in \mathcal{L}[E \oplus E]$, the characteristic function $\tilde{S}_T(\lambda)$ of the maximal dissipative operator $\mathfrak{L}_T$ is a Blaschke–Potapov product, and the spectrum of $\mathfrak{L}_T$ is purely discrete and belongs to the open upper half-plane. For Γ-quasi-every strictly contractive $T \in \mathcal{L}[E \oplus E]$, the operator $\mathfrak{L}_T$ has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit point at infinity, and the system of all eigenvectors and associated vectors (or root vectors) of this operator is complete in the space $\mathfrak{L}^2_A(\mathcal{I}; E)$.*

Remarks: 1. Since a linear operator $\mathbf{S}$ acting in the Hilbert space $\mathbf{H}$ is maximal accumulative if and only if $-\mathbf{S}$ is maximal dissipative, all results concerning maximal dissipative operators can be immediately transferred to maximal accumulative operators.

2. The results are valid for regular matrix Sturm–Liouville operators (with regular end points a and b). In this case the maps F_1 and F_2 have the form

$$F_1 y = \begin{pmatrix} -y(a) \\ y(b) \end{pmatrix},\ F_2 y = \begin{pmatrix} (Py')(a) \\ (Py')(b) \end{pmatrix},\ y \in \mathfrak{D}_{\max}.$$

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