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A type of mean-value interpolation of holomorphic functions on an annulus

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Abstract. We study mean-value interpolation of Hermite type by a polynomial of degree m in z^{-1} and n in z. We show that the interpolation problem corresponding to the integrals over the segments of the form $\{te^{i\theta} : \rho \le t \le 1\}$ always has a unique solution. We point out that the sequence of interpolation functions of a holomorphic function in a neighborhood of a closed annulus converges uniformly to the function when the angles that define the line segments are equally spaced.

1. Introduction

Let $\mathcal{F}$ be a finite-dimensional vector space of functions defined on a subset Ω of $\mathbb{R}^n$ or $\mathbb{C}^n$ with $d := \dim \mathcal{F}$. The classical problem in the theory of interpolation is to find d functionals $\{\mu_1, \mu_2, \dots, \mu_d\}$ in the dual space of $\mathcal{F}$ for which the interpolation problem

$$\mu_k(P) = c_{\mu_k}, \quad 1 \le k \le d,$$

has a unique solution $P \in \mathcal{F}$ for any given preassigned data $\{c_{\mu_k}\}$. The set $\{\mu_1, \mu_2, \dots, \mu_d\}$ can be regarded as the interpolation conditions. It reduces to the Lagrange interpolation problem when the μ_i's are the Dirac delta functionals, i.e., $\mu_k = \delta_{\mathbf{a}_k}$, $\mathbf{a}_k \in \Omega$, $1 \le k \le d$. More precisely, $\mu_k(f) = f(\mathbf{a}_k)$ for $1 \le k \le d$. In the case where the μ_i's are induced from differential operators of the form

$$f \mapsto \Big(\sum_{\alpha} c_\alpha D^\alpha\Big)(f)(\mathbf{a}_k), \quad \mathbf{a}_k \in \Omega,$$

it becomes the Hermite–Birkhoff interpolation problem. Many explicit solutions of the two particular interpolation problems mentioned above are available in literature. For example, some types of unisolvent sets, which solve the Lagrange interpolation problem, can be found in [3, 4, 10] and regular Hermite interpolation schemes were constructed in [6, 7, 15, 19]. Another type of interpolation problem is the mean-value interpolation problem, where the functionals $f \mapsto \mu_k(f)$ are given by the integrals. In [13, 14]

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the authors investigated the interpolation problem the case where the $\mu_k(f)$ are the integrals of derivatives of f over the standard simplices $\Delta_k = \{(u_1, \dots, u_k) : u_j \geq 0, \sum_{j=1}^{k} u_j \leq 1\}$. Some authors considered the mean-value interpolation problem based on Radon projection defined as follows. Let $\mathbb{D}$ be the open unit disk in the plane. For $\theta \in [0, 2\pi)$ and $t \in [0, 1)$, we denote by $I(\theta, t)$ the line segment of $\mathbb{D}$ that passes through the point $(t\cos\theta, t\sin\theta)$ and is perpendicular to the vector $(\cos\theta, \sin\theta)$. The Radon projection $\mathcal{R}_\theta(f; t)$ of a real-valued function or complex-valued function f defined on $\overline{\mathbb{D}}$ is the line integral of f over $I(\theta, t)$,

$$\mathcal{R}_\theta(f; t) = \int_{I(\theta,t)} f ds = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) ds.$$

In [1, 2], the authors solved the problem of determining a bivaritate polynomial from its finite Radon projections. Some classical and important results focusing on Radon projections in two and several variables can be found in [8, 9, 16, 17, 25]. Georgieva and Hofreither investigated the Lagrange interpolation problem using Radon projection, where $\mathcal{F}$ is the space of bivariate harmonic polynomials of total degree at most n (see [11, 12]). The authors obtained interesting results. They gave sets of chords which determine an element of $\mathcal{F}$ uniquely. In [21, 22], we generalized some results of Georgieva and Hofreither to Hermite type interpolation. In a recent work [23], we studied polynomial interpolation of holomorphic functions based on Radon projections. We constructed two types of regular Hermite interpolation schemes. The first Hermite scheme contains the chords which are equidistant from the origin along with the derivatives with respect to angles. The other one is formed by parallel chords. We also studied the convergence property of interpolation polynomial of holomorphic functions in neighborhoods of $\overline{\mathbb{D}}$. We proved in [23, Theorem 3.7] that the interpolation polynomials converge geometrically on $\overline{\mathbb{D}}$ to the functions as $n \to \infty$ when $n + 1$ angles of chords are equispaced and the distance is sufficiently near 1.

In this paper, we give a new type of mean-value interpolation on an annulus and investigate the convergence of the interpolation functions. For $0 \leq \rho_1 < \rho_2$, let $\overline{A}(\rho_1, \rho_2)$ be the closed annulus defined by two circles $\{z \in \mathbb{C} : |z| = \rho_1\}$ and $\{z \in \mathbb{C} : |z| = \rho_2\}$,

$$\overline{A}(\rho_1, \rho_2) = \{z \in \mathbb{C} : \rho_1 \leq |z| \leq \rho_2\}.$$

In the special case $\rho_1 = 0$ and $\rho_2 = 1$, $\overline{A}(0, 1)$ is identical with the closed unit disk $\overline{\mathbb{D}}$. Let $\rho \in [0, 1)$ be a fixed radius. For $0 \leq \theta < 2\pi$ and a continuous complex-valued function f defined on $\overline{A}(\rho, 1)$, we denote by $\mu_\theta\big(\overline{A}(\rho, 1); f\big)$ the line integral of f over the segment $\{te^{i\theta} : \rho \leq t \leq 1\}$. More precisely,

$$\mu_\theta\big(\overline{A}(\rho, 1); f\big) = \int_\rho^1 f(te^{i\theta}) dt. \tag{1}$$

For two natural numbers m, n, let $\mathcal{R}_{m,n}$ be the space of rational functions defined by

$$\mathcal{R}_{m,n} = \operatorname{span}_{\mathbb{C}}\big\{z^{-m}, z^{-m+1}, \dots, z^n\big\}.$$

Remark that $\mathcal{R}_{0,n}$ becomes the space $\mathcal{P}_n$ of all polynomials (of complex coefficients) of degree at most n in $\mathbb{C}$. Our purpose is to study the interpolation problem for the space $\mathcal{R}_{m,n}$ based on the functionals defined in (1). The following theorem is our first main result

Theorem 1.1. *Let $0 < \rho < 1$. Let m, d be positive integers and n be a natural number. Let $\nu_1, \dots, \nu_d$ be positive integers such that $\nu_1 + \cdots + \nu_d = m + n + 1$. Let $\theta_1, \dots, \theta_d \in [0, 2\pi)$ be pairwise distinct angles. Then, for arbitrary complex numbers $\gamma_{k,j}$, there exists a unique $R \in \mathcal{R}_{m,n}$ such that*

$$\frac{d^j}{d\theta^j} \mu_\theta\big(\overline{A}(\rho, 1); R\big)\Big|_{\theta=\theta_k} = \gamma_{k,j}, \quad k = 1, \dots, d, \quad j = 0, \dots, \nu_k - 1.$$

Next, we wish to investigate the convergence property of the interpolation functions. Let $\theta_1, \ldots, \theta_{m+n+1} \in [0, 2\pi)$ be pairwise distinct angles. Let f be holomorphic in a neighborhood of $\overline{A}(\rho, 1)$. Theorem 1.1 points out that there exists a unique $R \in \mathcal{R}_{m,n}$ such that

$$\mu_{\theta_k}\Big(\overline{A}(\rho,1); R\Big) = \mu_{\theta_k}\Big(\overline{A}(\rho,1); f\Big), \quad k = 1, \ldots, m+n+1. \tag{2}$$

The rational function R in (2) is denoted by $\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f]$, where

$$\Theta = \Big\{\theta_1, \ldots, \theta_{m+n+1}\Big\}.$$

We show that the sequence of interpolation functions converge uniformly on the closed annulus when angles are equispaced. It is our second main result.

Theorem 1.2. *Let $0 < \rho < 1$ and f be holomorphic in a neighborhood of $\overline{A}(\rho, 1)$. Let m, n be positive integers. Let Φ_{m+n+1} be the set of equally spaced angles*

$$\Phi_{m+n+1} = \Big\{\varphi_j^{(m+n+1)} = \frac{2j\pi}{m+n+1} : 0 \le j \le m+n\Big\}.$$

Then there exist three positive constants A, B, C and $\delta \in (0, 1)$ depending only on f such that

$$\|\mathbf{I}[\mathcal{R}_{m,n}, \Phi_{m+n+1}; f] - f\|_{\overline{A}(\rho,1)} \le (m+n+1)(A\delta^m + B\delta^n) + C\delta^{m+n+1}.$$

Consequently, if $n_k\delta^{m_k}$ and $m_k\delta^{n_k}$ tend to 0 as $k \to \infty$, then $\mathbf{I}[\mathcal{R}_{m_k,n_k}, \Phi_{m_k+n_k+1}; f]$ converges to f uniformly on $\overline{A}(\rho, 1)$ as $k \to \infty$.

Our article is organized as follows. In the next section we compute the value $\mu_\theta\Big(\overline{A}(\rho, 1); z^k\Big)$ for $k \in \mathbb{Z}$ and use the result to prove Theorem 1.1. Section 3 presents the proof of Theorem 1.2. We first give a formula for the rational function $\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f]$. Using the formula, we can estimate the error between the interpolation function and the finite Laurent expansion of f. The estimate leads to an upper bound for the uniform norm of $\mathbf{I}[\mathcal{R}_{m,n}, \Phi_{m+n+1}; f] - f$, and Theorem 1.2 follows. In Section 3 we also study the convergence of interpolation polynomials based on the functional $f \mapsto \mu_\theta\Big(\overline{\mathbb{D}}; f\Big)$. We show in Theorem 3.4 that the interpolation polynomials converge geometrically on $\overline{\mathbb{D}}$ to the functions as $n \to \infty$ when $n + 1$ angles are equispaced.

Finally, we note that the Lagrange interpolation problem by an element in $\mathcal{R}_{n,n-1}$ was studied in [5, 24]. The interpolation set consists of $2n$ points, equally spaced around two circles of an annulus. Also in [24] Mason conjectured that the Lebesgue function of the Lagrange interpolation on an annulus $\overline{A}(\rho^{-1}, \rho)$, $\rho \ge 1$, attains its maximal value on the inner circle and the Lebesgue constant grows like $2\log n/\pi$. Pan proved that the conjecture is true (see [18]).

2. Mean-value interpolation of Hermite type

In this section, we calculate the value $\mu_\theta\Big(\overline{A}(\rho, 1); z^k\Big)$ and prove Theorem 1.1. We also give some useful properties of the parameter β_k.

Lemma 2.1. *If $q_k(z) = z^k$ for $k \in \mathbb{Z}$, then*

$$\mu_\theta\Big(\overline{A}(\rho,1); q_k\Big) = \frac{1-\rho^{k+1}}{k+1}e^{ik\theta}, \quad k \ge 0, \quad 0 \le \rho < 1,$$

$$\mu_\theta\Big(\overline{A}(\rho,1); q_{-1}\Big) = -(\ln\rho)e^{-i\theta}, \quad 0 < \rho < 1,$$

and

$$\mu_\theta\Big(\overline{A}(\rho,1); q_{-k}\Big) = \frac{1-\rho^{-k+1}}{-k+1}e^{-ik\theta}, \quad k \ge 2, \quad 0 < \rho < 1.$$

Proof. The proof is immediate. For $k \geq 0$ and $0 \leq \rho < 1$, we have

$$\mu_\theta\Big(\overline{A}(\rho,1);q_k\Big) = \int_\rho^1 (te^{i\theta})^k dt = \frac{1-\rho^{k+1}}{k+1}e^{ik\theta}.$$

Similarly, for $0 < \rho < 1$, we can write

$$\mu_\theta\Big(\overline{A}(\rho,1);q_{-1}\Big) = \int_\rho^1 (te^{i\theta})^{-1} dt = -(\ln\rho)e^{-i\theta}$$

and

$$\mu_\theta\Big(\overline{A}(\rho,1);q_{-k}\Big) = \int_\rho^1 (te^{i\theta})^{-k} dt = \frac{1-\rho^{-k+1}}{-k+1}e^{-ik\theta}, \quad k \geq 2.$$

The proof is complete. □

For convenience, we set

$$\beta_k(\rho) = \int_\rho^1 t^k dt, \quad k \in \mathbb{Z}.$$

We have

$$\beta_{-1}(\rho) = -\ln(\rho), \quad \beta_k(\rho) = \frac{1-\rho^{k+1}}{k+1}, \quad k \neq -1. \tag{3}$$

Note that, for $k < 0$, $\beta_k(\rho)$ is well-defined only if $0 < \rho < 1$. However, $\beta_k(\rho)$ is well-defined for $k \geq 0$ and $\rho \in [0,1)$. We will write $\beta_k(\rho)$ simply β_k when no confusion can arise. In this setting, Lemma 2.1 gives

$$\mu_\theta\Big(\overline{A}(\rho,1);q_k\Big) = \beta_k e^{ik\theta}, \quad k \in \mathbb{Z}. \tag{4}$$

Proof. [Proof of Theorem 1.1] Since the number of interpolation conditions is equal to $m+n+1 = \dim \mathcal{R}_{m,n}$. It is sufficient to show that if $R \in \mathcal{R}_{m,n}$ satisfies the following conditions

$$\frac{d^j}{d\theta^j}\mu_\theta\Big(\overline{A}(\rho,1);R\Big)\Big|_{\theta=\theta_k} = 0, \quad k = 1,\dots,d, \quad j = 0,\dots,\nu_k - 1. \tag{5}$$

then $R \equiv 0$. Let us set $R(z) = \sum_{k=-m}^n c_k z^k$. By Lemma 2.1 (see also (4)), we can write

$$\mu_\theta\Big(\overline{A}(\rho,1);R\Big) = \sum_{k=-m}^n c_k\beta_k e^{ik\theta}.$$

The polynomial $Q(z) = \sum_{k=0}^{m+n} c_{k-m}\beta_{k-m}z^k$ belongs to the space $\mathcal{P}_{m+n}$ and admits the relation

$$e^{-im\theta}Q(e^{i\theta}) = \mu_\theta\Big(\overline{A}(\rho,1);R\Big).$$

Hence, relation (5) is equivalent to

$$\frac{d^j}{d\theta^j}\Big(e^{-im\theta}Q(e^{i\theta})\Big)\Big|_{\theta=\theta_k} = 0, \quad k = 1,\dots,d, \quad j = 0,\dots,\nu_k - 1. \tag{6}$$

Using [20, Lemma 2.6] in (6) we obtain

$$\frac{d^j}{d\theta^j}Q(e^{i\theta})\Big|_{\theta=\theta_k} = 0, \quad k = 1,\dots,d, \quad j = 0,\dots,\nu_k - 1.$$

Lemma 3.3 in [23] now implies

$$Q^{(j)}(e^{i\theta_k}) = 0, \quad k = 1,\dots,d, \quad j = 0,\dots,\nu_k - 1. \tag{7}$$

By the uniqueness of univariate Hermite interpolation, we conclude from (7) that $Q = 0$. It follows that $c_{k-m}\beta_{k-m} = 0$ for $k = 0, 1, \dots, m+n$. Since $\beta_k > 0$ for any $k \in \mathbb{Z}$, we get $c_k = 0$ for $-m \le k \le n$, and hence $R = 0$, which completes the proof. □

Examining the proof of Theorem 1.1, we see that it still holds true when $\rho = 0$ and $m = 0$. In this case, $\mathcal{R}_{0,n}$ is replaced by $\mathcal{P}_n$. We state the result without proof.

Corollary 2.2. *Let n be a natural number. Let $\nu_1,\dots,\nu_d$ be positive integers such that $\nu_1 + \cdots + \nu_d = n + 1$. Let $\theta_1,\dots,\theta_d \in [0, 2\pi)$ be pairwise distinct angles. Then, for arbitrary complex numbers $\gamma_{k,j}$, there exists a unique $P \in \mathcal{P}_n$ such that*

$$\frac{d^j}{d\theta^j}\mu_\theta\big(\overline{\mathbb{D}}; P\big)\Big|_{\theta=\theta_k} = \gamma_{k,j}, \quad k = 1,\dots,d, \quad j = 0,\dots,\nu_k - 1.$$

Example 2.3. *We compute the interpolation function $R \in \mathcal{R}_{1,1}$ of the function $f(z) = \frac{e^{\frac{1}{z}}}{z^2}$ on the annulus $\overline{A}(\frac{1}{2}, 1)$ corresponding to the set of angles $\{0, \frac{\pi}{2}, \pi\}$. The interpolation function is determined by the following three equations*

$$\mu_0\big(\overline{A}(\tfrac{1}{2},1); R\big) = \mu_0\big(\overline{A}(\tfrac{1}{2},1); f\big), \quad \mu_{\frac{\pi}{2}}\big(\overline{A}(\tfrac{1}{2},1); R\big) = \mu_{\frac{\pi}{2}}\big(\overline{A}(\tfrac{1}{2},1); f\big)$$

$$\mu_\pi\big(\overline{A}(\tfrac{1}{2},1); R\big) = \mu_\pi\big(\overline{A}(\tfrac{1}{2},1); f\big).$$

We first evaluate the integrals of f over three line segments. We have

$$\mu_0\big(\overline{A}(\tfrac{1}{2},1); f\big) = \int_{\frac{1}{2}}^{1} \frac{e^{\frac{1}{t}}}{t^2}dt = e^2 - e, \quad \mu_\pi\big(\overline{A}(\tfrac{1}{2},1); f\big) = \int_{\frac{1}{2}}^{1} \frac{e^{-\frac{1}{t}}}{t^2}dt = \frac{1}{e} - \frac{1}{e^2},$$

$$\mu_{\frac{\pi}{2}}\big(\overline{A}(\tfrac{1}{2},1); f\big) = \int_{\frac{1}{2}}^{1} \frac{e^{\frac{1}{it}}}{(it)^2}dt = -\int_{\frac{1}{2}}^{1} \frac{\cos\frac{1}{t}}{t^2} + i\int_{\frac{1}{2}}^{1} \frac{\sin\frac{1}{t}}{t^2} = \sin 1 - \sin 2 + i(\cos 1 - \cos 2),$$

Let us set $R(z) = c_{-1}z^{-1} + c_0 + c_1 z$. We see that

$$\mu_0\big(\overline{A}(\tfrac{1}{2},1); R\big) = \int_{\frac{1}{2}}^{1} (\frac{c_{-1}}{t} + c_0 + c_1 t)dt = c_{-1}\ln 2 + \frac{c_0}{2} + \frac{3c_1}{8},$$

$$\mu_\pi\big(\overline{A}(\tfrac{1}{2},1); R\big) = \int_{\frac{1}{2}}^{1} (\frac{c_{-1}}{-t} + c_0 - c_1 t)dt = -c_{-1}\ln 2 + \frac{c_0}{2} - \frac{3c_1}{8},$$

$$\mu_{\frac{\pi}{2}}\big(\overline{A}(\tfrac{1}{2},1); R\big) = \int_{\frac{1}{2}}^{1} (\frac{c_{-1}}{it} + c_0 + c_1(it))dt = -ic_{-1}\ln 2 + \frac{c_0}{2} + \frac{3ic_1}{8}.$$

Hence, we obtain a system of equations

$$\begin{cases} c_{-1}\ln 2 + \frac{c_0}{2} + \frac{3c_1}{8} = e^2 - e \\ -c_{-1}\ln 2 + \frac{c_0}{2} - \frac{3c_1}{8} = \frac{1}{e} - \frac{1}{e^2} \\ -ic_{-1}\ln 2 + \frac{c_0}{2} + \frac{3ic_1}{8} = \sin 1 - \sin 2 + i(\cos 1 - \cos 2) \end{cases}$$

The exact solution of the system of equations is

$$c_{-1} = \frac{e^4 - e^3 - e + 1 - 2e^2(\cos 1 - \cos 2) + i\Big(-e^4 + e^3 - e + 1 + 2e^2(\sin 1 - \sin 2)\Big)}{4e^2\ln 2},$$

$$c_0 = \frac{e^4 - e^3 + e - 1}{e^2},$$

$$c_1 = \frac{2}{3}\cdot\frac{e^4 - e^3 - e + 1 + 2e^2(\cos 1 - \cos 2) + i\Big(e^4 - e^3 + e - 1 - 2e^2(\sin 1 - \sin 2)\Big)}{3e^2}.$$

The above solution gives precise formula for $R(z)$.

We need some elementary properties of the sequence $\{\beta_k\}$ that are used in the next section.

Lemma 2.4. *Let* $0 < \rho < 1$.
i) For $j, k \in \mathbb{Z}$ *and* $n \in \mathbb{Z}^+$, *we have*

$$\beta_j > \beta_k > 0, \quad j < k$$

and

$$\beta_{k-n} < \rho^{-n}\beta_k.$$

ii) If $k \geq 1$ *and* $N \geq k+2$, *then* $\beta_{k-N} \leq N\rho^{k+1-N}\beta_k$.
iii) If $k \geq 0$ *and* $N \geq k+2$, *then* $\beta_{-k+N} \leq N\rho^{k-1}\beta_{-k}$.

Proof. i) For $j < k$, we have $t^j > t^k$ for all $t \in (\rho, 1)$. Hence, $\int_\rho^1 t^j dt > \int_\rho^1 t^k dt$. Consequently, $\beta_j > \beta_k$. To prove the second inequality, we write

$$\beta_{k-n} = \int_\rho^1 t^{k-n}dt < \int_\rho^1 \rho^{-n}t^k dt = \rho^{-n}\beta_k.$$

ii) Since $\rho \in (0,1)$, we see that

$$\frac{\beta_{k-N}}{\beta_k} = \frac{1-\rho^{k+1-N}}{k+1-N}\cdot\frac{k+1}{1-\rho^{k+1}} = \frac{k+1}{N-k-1}\cdot\frac{1-\rho^{N-k-1}}{1-\rho^{k+1}}\cdot\frac{1}{\rho^{N-k-1}}$$

$$= \frac{1}{1+\rho+\cdots+\rho^k}\cdot\frac{1+\rho+\cdots+\rho^{N-k-2}}{N-k-1}\cdot\frac{k+1}{\rho^{N-k-1}} < \frac{k+1}{\rho^{N-k-1}} < \frac{N}{\rho^{N-k-1}}.$$

iii) We first consider the case $k \geq 2$. We can write

$$\frac{\beta_{-k+N}}{\beta_{-k}} = \frac{1-\rho^{N+1-k}}{N+1-k}\cdot\frac{-k+1}{1-\rho^{-k+1}} = \frac{k-1}{N+1-k}\cdot\frac{1-\rho^{N+1-k}}{1-\rho^{k-1}}\cdot\rho^{k-1}$$

$$= \frac{1}{1+\rho+\cdots+\rho^{k-2}}\cdot\frac{1+\rho+\cdots+\rho^{N-k}}{N+1-k}\cdot(k-1)\rho^{k-1} \leq (k-1)\rho^{k-1} < N\rho^{k-1}.$$

If $k = 1$, then

$$\frac{\beta_{-1+N}}{\beta_{-1}} = \frac{1-\rho^N}{N(-\ln\rho)} = \frac{1-\rho}{-\ln\rho}\cdot\frac{1+\rho+\cdots+\rho^{N-1}}{N} \le 1 < N,$$

because $\rho \in (0,1)$ and $1-\rho \le -\ln\rho$.
Finally, for $k = 0$, we write

$$\frac{\beta_N}{\beta_0} = \frac{1-\rho^{N+1}}{(N+1)(1-\rho)} = \frac{1+\rho+\cdots+\rho^N}{N+1} < 1 < \frac{N}{\rho}.$$

The inequalities are proved completely. □

3. Convergence of interpolation functions

3.1. Convergence of interpolation functions on an annulus

In this subsection, we assume that $0 < \rho < 1$. Let f be holomorphic in a neighborhood of $\overline{A}(\rho, 1)$. We can find ρ_1 and ρ_2 with $0 < \rho_1 < \rho < 1 < \rho_2$ such that f is holomorphic on a neighborhood of $\overline{A}(\rho_1, \rho_2)$. By the Laurent theorem we can write

$$f(z) = \sum_{\ell=-\infty}^{\infty} a_\ell z^\ell, \quad \rho_1 \le |z| \le \rho_2, \tag{8}$$

where

$$a_\ell = \frac{1}{2\pi i}\int\limits_{|t|=\rho_2} \frac{f(t)dt}{t^{\ell+1}}, \quad \ell \ge 0 \quad \text{and} \quad a_{-l} = \frac{1}{2\pi i}\int\limits_{|t|=\rho_1} t^{l-1} f(t)dt, \quad l \ge 1.$$

Let us set

$$M = \max\{\|f\|_{|t|=\rho_1}, \|f\|_{|t|=\rho_2}\},$$

where $\|f\|_X = \sup\{|f(t)| : t \in X\}$. Applying standard arguments we obtain the estimate for the coefficients of the Laurent expansion

$$|a_\ell| \le \frac{M}{\rho_2^\ell}, \quad \ell \ge 0, \quad |a_{-l}| \le M\rho_1^l, \quad l \ge 1. \tag{9}$$

The above settings and results for f will be used throughout the subsection. We also use the following simple relations

$$\sum_{k=0}^{d} \frac{1}{\rho_2^k} < \sum_{k=0}^{\infty} \frac{1}{\rho_2^k} = \frac{\rho_2}{\rho_2 - 1} \quad \text{and} \quad \sum_{k=0}^{d} \Big(\frac{\rho_1}{\rho}\Big)^k < \sum_{k=0}^{\infty} \Big(\frac{\rho_1}{\rho}\Big)^k = \frac{\rho}{\rho - \rho_1}.$$

Let us define the generalized Vandermonde determinant corresponding to the set of functions $\mathcal{F}_{m,n} := \{f_{-m}, f_{-m+1}, \dots, f_n\}$ and the set of points $\exp(i\Theta) := \{e^{i\theta_1}, \dots, e^{i\theta_{m+n+1}}\}$. We set

$$\mathrm{VDM}(\mathcal{F}_{m,n}; \exp(i\Theta)) = \begin{vmatrix} f_{-m}(e^{i\theta_1}) & f_{-m+1}(e^{i\theta_1}) & \cdots & f_n(e^{i\theta_1}) \\ f_{-m}(e^{i\theta_2}) & f_{-m+1}(e^{i\theta_2}) & \cdots & f_n(e^{i\theta_2}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{-m}(e^{i\theta_{m+n+1}}) & f_{-m+1}(e^{i\theta_{m+n+1}}) & \cdots & f_n(e^{i\theta_{m+n+1}}) \end{vmatrix}$$

A useful formula $\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f]$ is given in the following lemma.

Lemma 3.1. *If* $f(z) = \sum_{\ell=-\infty}^{\infty} a_\ell z^\ell$*, then* $\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f](z) = \sum_{k=-m}^{n} c_k z^k$*, where*

$$c_k = \frac{1}{\beta_k} \sum_{\ell=-\infty}^{\infty} a_\ell \beta_\ell \frac{\mathrm{VDM}(\mathcal{M}_{m,n}[q_k \leftarrow q_\ell]; \exp(i\Theta))}{\mathrm{VDM}(\mathcal{M}_{m,n}; \exp(i\Theta))}, \quad -m \le k \le n. \tag{10}$$

Here $\mathcal{M}_{m,n} := \{q_{-m}, q_{-m+1}, \ldots, q_n\}$ *with* $q_j(z) = z^j$ *and* $\mathcal{M}_{m,n}[q_k \leftarrow q_\ell]$ *means that we substitute* q_ℓ *for* q_k *in* $\mathcal{M}_{m,n}$*.*

Proof. Since the series at the right hand side of (8) converges uniformly on the closed annulus $\overline{A}(\rho, 1)$ to f, Lemma 2.1 (see also (4)) yields

$$\mu_\theta(f) = \sum_{\ell=-\infty}^{\infty} a_\ell \beta_\ell e^{i\ell\theta}, \quad \theta \in [0, 2\pi). \tag{11}$$

Observe that the series at the right hand side of (11) converges absolutely. Indeed, from Lemma 2.4(i) we see that

$$\beta_\ell \le \beta_0 = 1 - \rho, \quad \ell \ge 0$$

and

$$\beta_{-l} \le \rho^{-l} \beta_0 = \rho^{-l}(1 - \rho), \quad l \ge 1.$$

It follows that

$$\sum_{\ell=-\infty}^{\infty} |a_\ell \beta_\ell e^{i\ell\theta}| = \sum_{\ell=0}^{\infty} |a_\ell \beta_\ell| + \sum_{l=1}^{\infty} |a_{-l} \beta_{-l}| \le \sum_{\ell=0}^{\infty} \frac{M(1-\rho)}{\rho_2^\ell} + \sum_{l=1}^{\infty} M(1-\rho)\Big(\frac{\rho_1}{\rho}\Big)^l < \infty,$$

where we use (9) in the second relation.

Since $\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f](z) = \sum_{k=-m}^{n} c_k z^k$, we can use Lemma 2.1 again to obtain

$$\mu_\theta\Big(\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f]\Big) = \sum_{k=-m}^{n} c_k \beta_k e^{ik\theta}, \quad \theta \in [0, 2\pi). \tag{12}$$

Since $\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f]$ satisfies the interpolation conditions

$$\mu_{\theta_j}(\mathbf{I}[\mathcal{R}_{m,n}, \Theta; f]) = \mu_{\theta_j}(f), \quad j = 1, \ldots, m+n+1,$$

we conclude from (11) and (12) that

$$\sum_{k=-m}^{n} c_k \beta_k (e^{i\theta_j})^k = \sum_{\ell=-\infty}^{\infty} a_\ell \beta_\ell (e^{i\theta_j})^\ell, \quad j = 1, \ldots, m+n+1. \tag{13}$$

Hence, we get a system of $m+n+1$ linear equations, where $c_{-m}\beta_{-m}, c_{-m+1}\beta_{-m+1}, \ldots, c_n\beta_n$ are unknown. We can use the Cramer rule to compute its solution. The determinant of the coefficient matrix is equal to

$$\mathrm{VDM}(\mathcal{M}_{m,n}; \exp(i\Theta)).$$

Hence, from (13), we have

$$c_k \beta_k = \sum_{\ell=-\infty}^{\infty} a_\ell \beta_\ell \frac{\mathrm{VDM}(\mathcal{M}_{m,n}[q_k \leftarrow q_\ell]; \exp(i\Theta))}{\mathrm{VDM}(\mathcal{M}_{m,n}; \exp(i\Theta))}. \tag{14}$$

The proof is complete. □

Proof. [Proof of Theorem 1.2] The proof is inspired from the proof of Theorem 3.7 in [23]. For simplicity of notation, we let N stand for $m+n+1$. Since $\mathbf{I}[\mathcal{R}_{m,n}, \Phi_N; f] \in \mathcal{R}_{m,n}$, we can write

$$\mathbf{I}[\mathcal{R}_{m,n}, \Phi_N; f](z) = \sum_{k=-m}^{n} c_k^{(m,n)} z^k.$$

By Lemma 3.1, we have

$$c_k^{(m,n)} = \frac{1}{\beta_k} \sum_{\ell=-\infty}^{\infty} a_\ell \beta_\ell \frac{\mathrm{VDM}(\mathcal{M}_{m,n}[q_k \leftarrow q_\ell]; \exp(i\Phi_N))}{\mathrm{VDM}(\mathcal{M}_{m,n}; \exp(i\Phi_N))}, \quad -m \le k \le n. \tag{15}$$

Since $\Phi_N = \left\{\varphi_j^{(N)} = \frac{2j\pi}{N} : 0 \le j \le m+n\right\}$, we see that if $\ell \in j + N\mathbb{Z} = \{j + Nk : k \in \mathbb{Z}\}$, then

$$\left(q_\ell\left(e^{i\varphi_0^{(N)}}\right), \ldots, q_\ell\left(e^{i\varphi_{m+n}^{(N)}}\right)\right) = \left(q_j\left(e^{i\varphi_0^{(N)}}\right), \ldots, q_j\left(e^{i\varphi_{m+n}^{(N)}}\right)\right).$$

It follows that

$$\mathrm{VDM}(\mathcal{M}_{m,n}[q_k \leftarrow q_\ell]; \exp(i\Phi_N)) = \begin{cases} \mathrm{VDM}(\mathcal{M}_{m,n}; \exp(i\Phi_N)) & \text{if } \ell \in k + N\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Substituting this into (15) we obtain

$$c_k^{(m,n)} = \frac{1}{\beta_k} \sum_{l=-\infty}^{\infty} a_{k+lN} \beta_{k+lN}, \quad -m \le k \le n. \tag{16}$$

We denote by $e_{m,n}(z)$ the error between the interpolation function and the finite Laurent expansion of f,

$$e_{m,n}(z) := \mathbf{I}[\mathcal{R}_{m,n}, \Phi_N; f](z) - \sum_{k=-m}^{n} a_k z^k = \sum_{k=-m}^{n} \left(c_k^{(m,n)} - a_k\right) z^k. \tag{17}$$

In view of (17), we get

$$\begin{aligned} \|e_{m,n}\|_{\overline{A(\rho,1)}} &\le \sum_{k=-m}^{n} |c_k^{(m,n)} - a_k| \cdot \|z^k\|_{\overline{A(\rho,1)}} \\ &\le \sum_{k=1}^{n} |c_k^{(m,n)} - a_k| + \sum_{k=0}^{m} |c_{-k}^{(m,n)} - a_{-k}| \cdot \frac{1}{\rho^k} \end{aligned} \tag{18}$$

We need to estimate the quantity $|c_k^{(m,n)} - a_k|$ for $-m \le k \le n$. From inequality (9) and relation (16), we get, for $-m \le k \le n$,

$$\begin{aligned} |c_k^{(m,n)} - a_k| &= \Big| \sum_{l=-\infty, l\neq 0}^{\infty} \frac{\beta_{k+lN}}{\beta_k} a_{k+lN} \Big| \\ &\le \sum_{l=1}^{\infty} \left| \frac{\beta_{k+lN}}{\beta_k} a_{k+lN} \right| + \sum_{l=1}^{\infty} \left| \frac{\beta_{k-lN}}{\beta_k} a_{k-lN} \right| \\ &\le \sum_{l=1}^{\infty} \frac{\beta_{k+lN}}{\beta_k} \cdot \frac{M}{\rho_2^{k+lN}} + \sum_{l=1}^{\infty} \frac{\beta_{k-lN}}{\beta_k} \cdot M\rho_1^{lN-k}. \end{aligned} \tag{19}$$

We will use Lemma 2.4 to get upper bounds for

$$\frac{\beta_{k+lN}}{\beta_k} \quad \text{and} \quad \frac{\beta_{k-lN}}{\beta_k}, \quad -m \le k \le n, \quad l \ge 1.$$

We first consider the case $k \in \{1, \dots, n\}$. Since $lN \geq m + n + 1 \geq k + 2$, the inequality in Lemma 2.4(ii) yields $\frac{\beta_{k-lN}}{\beta_k} < N\rho^{-lN+k+1}$. In addition, $\frac{\beta_{k+lN}}{\beta_k} < 1$ by Lemma 2.4(i). It follows that

$$\begin{aligned}
|c_k^{(m,n)} - a_k| &\leq \sum_{l=1}^{\infty} \frac{M}{\rho_2^{k+lN}} + \sum_{l=1}^{\infty} N\rho^{-lN+k+1} \cdot M\rho_1^{lN-k} \\
&= \frac{M}{\rho_2^{N+k}} \sum_{l=0}^{\infty} \frac{1}{\rho_2^{lN}} + \frac{MN\rho_1^{N-k}}{\rho^{N-k-1}} \sum_{l=0}^{\infty} \Big(\frac{\rho_1}{\rho}\Big)^{lN} \\
&\leq \frac{M}{\rho_2^{N+k}} \sum_{l=0}^{\infty} \frac{1}{\rho_2^{l}} + MN\rho \cdot \frac{\rho_1^{N-k}}{\rho^{N-k}} \sum_{l=0}^{\infty} \Big(\frac{\rho_1}{\rho}\Big)^{l} \\
&= \frac{M}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{N+k-1}} + \frac{MN\rho^2}{\rho - \rho_1} \cdot \frac{\rho_1^{N-k}}{\rho^{N-k}}. \qquad (20)
\end{aligned}$$

Next, we treat the case $-m \leq k \leq 0$. Since $lN \geq m + n + 1 \geq -k + 2$, Lemma 2.4(iii) gives $\frac{\beta_{k+lN}}{\beta_k} < N\rho^{-k-1}$. Using Lemma 2.4 (i), we get $\frac{\beta_{k-lN}}{\beta_k} < \rho^{-lN}$. Substituting these estimates into (19) we obtain

$$\begin{aligned}
|c_k^{(m,n)} - a_k| &\leq \sum_{l=1}^{\infty} N\rho^{-k-1} \cdot \frac{M}{\rho_2^{k+lN}} + \sum_{l=1}^{\infty} \rho^{-lN} \cdot M\rho_1^{lN-k} \\
&= \frac{MN\rho^{-k-1}}{\rho_2^{N+k}} \sum_{l=0}^{\infty} \frac{1}{\rho_2^{lN}} + \frac{M\rho_1^{N-k}}{\rho^{N}} \sum_{l=0}^{\infty} \Big(\frac{\rho_1}{\rho}\Big)^{lN} \\
&\leq \frac{MN\rho^{-k-1}}{\rho_2^{N+k}} \sum_{l=0}^{\infty} \frac{1}{\rho_2^{l}} + \frac{M\rho_1^{N-k}}{\rho^{N}} \sum_{l=0}^{\infty} \Big(\frac{\rho_1}{\rho}\Big)^{l} \\
&= \frac{MN\rho^{-k-1}}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{N+k-1}} + \frac{M\rho}{\rho - \rho_1} \cdot \frac{\rho_1^{N-k}}{\rho^{N}}, \quad -m \leq k \leq 0. \qquad (21)
\end{aligned}$$

We will use (20) to estimate the first term in (18). We have

$$\begin{aligned}
\sum_{k=1}^{n} |c_k^{(m,n)} - a_k| &\leq \sum_{k=1}^{n} \Big(\frac{M}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{N+k-1}} + \frac{MN\rho^2}{\rho - \rho_1} \cdot \frac{\rho_1^{N-k}}{\rho^{N-k}}\Big) \\
&= \frac{M}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{N}} \sum_{l=0}^{n-1} \frac{1}{\rho_2^{l}} + \frac{MN\rho^2}{\rho - \rho_1} \Big(\frac{\rho_1}{\rho}\Big)^{m+1} \sum_{l=0}^{n-1} \Big(\frac{\rho_1}{\rho}\Big)^{l} \\
&\leq \frac{M\rho_2}{(\rho_2 - 1)^2} \cdot \frac{1}{\rho_2^{N}} + \frac{MN\rho^3}{(\rho - \rho_1)^2} \Big(\frac{\rho_1}{\rho}\Big)^{m+1}.
\end{aligned}$$

Similarly, using relation (21) we obtain

$$\begin{aligned}
\sum_{k=0}^{m} |c_{-k}^{(m,n)} - a_{-k}| \cdot \frac{1}{\rho^k} &\leq \sum_{k=0}^{m} \Big(\frac{MN\rho^{k-1}}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{N-k-1}} + \frac{M\rho}{\rho - \rho_1} \cdot \frac{\rho_1^{N+k}}{\rho^{N}}\Big) \cdot \frac{1}{\rho^k} \\
&\leq \sum_{k=0}^{m} \Big(\frac{MN\rho^{-1}}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{N-k-1}} + \frac{M\rho}{\rho - \rho_1} \cdot \frac{\rho_1^{N+k}}{\rho^{N+k}}\Big) \\
&\leq \frac{MN\rho^{-1}\rho_2}{(\rho_2 - 1)^2} \cdot \frac{1}{\rho_2^{n}} + \frac{M\rho^2}{(\rho - \rho_1)^2} \cdot \Big(\frac{\rho_1}{\rho}\Big)^{N}.
\end{aligned}$$

We conclude from (18) and the above estimates that

$$\|e_{m,n}\|_{\overline{A}(\rho,1)} \le \frac{M\rho_2}{(\rho_2-1)^2}\cdot\frac{1}{\rho_2^N} + \frac{MN\rho^3}{(\rho-\rho_1)^2}\Big(\frac{\rho_1}{\rho}\Big)^{m+1} + \frac{MN\rho^{-1}\rho_2}{(\rho_2-1)^2}\frac{1}{\rho_2^n} + \frac{M\rho^2}{(\rho-\rho_1)^2}\Big(\frac{\rho_1}{\rho}\Big)^N. \tag{22}$$

On the other hand, using inequality (9), we get

$$\|\sum_{\ell=n+1}^{\infty} a_\ell z^\ell\|_{\overline{A}(\rho,1)} \le \sum_{\ell=n+1}^{\infty}\frac{M}{\rho_2^\ell} = \frac{M}{\rho_2-1}\cdot\frac{1}{\rho_2^n} \tag{23}$$

and

$$\|\sum_{\ell=m+1}^{\infty} a_{-\ell} z^{-\ell}\|_{\overline{A}(\rho,1)} \le \sum_{\ell=m+1}^{\infty} M\rho_1^\ell\frac{1}{\rho^\ell} = \frac{M\rho_1}{\rho-\rho_1}\Big(\frac{\rho_1}{\rho}\Big)^{m+1}. \tag{24}$$

Combining (22), (23) and (24) we obtain

$$\begin{aligned}\|\mathbf{I}[\mathcal{R}_{m,n},\Phi_N;f]-f\|_{\overline{A}(\rho,1)} &\le \|\mathbf{I}[\mathcal{R}_{m,n},\Theta_N;f](z)-\sum_{k=-m}^{n} a_k z^k\|_{\overline{A}(\rho,1)}\\ &+ \|\sum_{\ell=n+1}^{\infty} a_\ell z^\ell\|_{\overline{A}(\rho,1)} + \|\sum_{\ell=m+1}^{\infty} a_{-\ell} z^{-\ell}\|_{\overline{A}(\rho,1)}\\ &\le AN\delta^m + BN\delta^n + C\delta^N,\end{aligned}$$

where $\delta = \max\{1/\rho_2, \rho_1/\rho\} \in (0,1)$ and the constants A, B, C depend only on M, ρ, ρ_1, ρ_2. The last estimate directly implies the desired assertion, and the proof is complete. □

The following corollary is a simple consequence of Theorem 1.2.

Corollary 3.2. *Let $0 < \rho < 1$ and f be holomorphic in a neighborhood of $\overline{A}(\rho,1)$. Let m, n be positive integers such that $m - n \in \{0, \pm 1\}$. Let Φ_{m+n+1} be the set of equally spaced angles*

$$\Phi_{m+n+1} = \Big\{\varphi_j^{(m+n+1)} = \frac{2j\pi}{m+n+1} : 0 \le j \le m+n\Big\}.$$

Then there exists $\delta \in (0,1)$ depending only on f such that

$$\limsup_{n\to\infty}\Big(\|\mathbf{I}[\mathcal{R}_{m,n},\Phi_{m+n+1};f]-f\|_{\overline{A}(\rho,1)}\Big)^{\frac{1}{n}} \le \delta.$$

3.2. Convergence of interpolation polynomials on the unit disk

In this subsection, we consider the case $\rho = 0$. In this case $\overline{A}(0,1) = \overline{\mathbb{D}}$.

Let n be a positive integers. Let $\theta_1, \ldots, \theta_{n+1} \in [0, 2\pi)$ be pairwise distinct angles. Let f be holomorphic in a neighborhood of $\overline{\mathbb{D}}$. By Corollary 2.2, there is a unique $P \in \mathcal{P}_n$ such that

$$\mu_{\theta_k}\big(\overline{\mathbb{D}};P\big) = \mu_{\theta_k}\big(\overline{\mathbb{D}};f\big), \quad k = 1,\ldots,n+1. \tag{25}$$

The polynomial P in (25) is denoted by $\mathbf{I}[\mathcal{P}_n,\Theta;f]$, where

$$\Theta = \Big\{\theta_1,\ldots,\theta_{n+1}\Big\}.$$

We can repeat the arguments in the proof of Lemma 3.1 to get a formula for $\mathbf{I}[\mathcal{P}_n,\Theta;f]$.

Lemma 3.3. *If* $f(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$*, then* $\mathbf{I}[\mathcal{P}_n, \Theta; f](z) = \sum_{k=0}^{n} c_k z^k$*, where*

$$c_k = \frac{1}{\beta_k} \sum_{\ell=0}^{\infty} a_\ell \beta_\ell \frac{\mathrm{VDM}(\mathcal{M}_{0,n}[q_k \leftarrow q_\ell]; \exp(i\Theta))}{\mathrm{VDM}(\mathcal{M}_{0,n}; \exp(i\Theta))}, \quad 0 \le k \le n,$$

where $\exp(i\Theta) = \{e^{i\theta_1}, \dots, e^{i\theta_{n+1}}\}$*,* $\mathcal{M}_{0,n} = \{q_0, q_1, \dots, q_n\}$ *with* $q_j(z) = z^j$ *and the* β_ℓ*'s are given in (3).*

Theorem 3.4. *Let* f *be holomorphic in a neighborhood of* $\overline{\mathbb{D}}$*. Let* n *be a positive integer. Let* Φ_{n+1} *be the set of equally spaced angles*

$$\Phi_{n+1} = \left\{\varphi_j^{(n+1)} = \frac{2j\pi}{n+1} : 0 \le j \le n\right\}.$$

Then there exists a positive constant $\delta \in (0, 1)$ *such that*

$$\limsup_{n\to\infty} \left(\|\mathbf{I}[\mathcal{P}_n, \Phi_{n+1}; f] - f\|_{\overline{\mathbb{D}}}\right)^{\frac{1}{n}} \le \delta. \tag{26}$$

Proof. The proof is similar to the proof of Theorem 1.2, For convenience to the readers, we give the precise arguments. Since f be holomorphic in a neighborhood of $\overline{\mathbb{D}}$, there exists $\rho_2 > 1$ and f is holomorphic on a neighborhood of $\overline{D}(0, \rho_2)$. We can write

$$f(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell, \quad |z| \le \rho_2.$$

By the Cauchy inequality, we have

$$|a_\ell| \le \frac{M}{\rho_2^\ell}, \quad \ell \ge 0, \quad M = \|f\|_{|t|=\rho_2}. \tag{27}$$

Since $\mathbf{I}[\mathcal{P}_n, \Phi_{n+1}; f] \in \mathcal{P}_n$, we can write

$$\mathbf{I}[\mathcal{P}_n, \Phi_{n+1}; f](z) = \sum_{k=0}^{n} c_k^{(n)} z^k.$$

Using Lemma 3.3, we get a formula for coefficients

$$c_k^{(n)} = \frac{1}{\beta_k} \sum_{\ell=0}^{\infty} a_\ell \beta_\ell \frac{\mathrm{VDM}(\mathcal{M}_{0,n}[q_k \leftarrow q_\ell]; \exp(i\Phi_{n+1}))}{\mathrm{VDM}(\mathcal{M}_{0,n}; \exp(i\Phi_{n+1}))}, \quad 0 \le k \le n. \tag{28}$$

Since $\Phi_{n+1} = \left\{\varphi_j^{(n+1)} = \frac{2j\pi}{n+1} : 0 \le j \le n\right\}$, we can check at once that if $\ell \in j + (n+1)\mathbb{Z}$ then

$$\left(q_\ell\left(e^{i\varphi_0^{(n+1)}}\right), \dots, q_\ell\left(e^{i\varphi_n^{(n+1)}}\right)\right) = \left(q_j\left(e^{i\varphi_0^{(n+1)}}\right), \dots, q_j\left(e^{i\varphi_n^{(n+1)}}\right)\right).$$

Consequently,

$$\mathrm{VDM}(\mathcal{M}_{0,n}[q_k \leftarrow q_\ell]; \exp(i\Phi_{n+1})) = \begin{cases} \mathrm{VDM}(\mathcal{M}_{0,n}; \exp(i\Phi_{n+1})) & \text{if } \ell \in k + (n+1)\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Combining the above relation with (28) we obtain

$$c_k^{(n)} = \frac{1}{\beta_k} \sum_{l=0}^{\infty} a_{k+l(n+1)} \beta_{k+l(n+1)}, \quad 0 \le k \le n. \tag{29}$$

From relations (27) and (29), we get, for $0 \le k \le n$,

$$\begin{aligned} |c_k^{(n)} - a_k| &= \Big|\sum_{l=1}^{\infty} \frac{\beta_{k+l(n+1)}}{\beta_k} a_{k+l(n+1)}\Big| \le \sum_{l=1}^{\infty} \frac{\beta_{k+l(n+1)}}{\beta_k} \cdot \frac{M}{\rho_2^{k+l(n+1)}} \\ &\le \sum_{l=1}^{\infty} \frac{M}{\rho_2^{k+l(n+1)}} = \frac{M}{\rho_2^{k+n+1}} \sum_{l=0}^{\infty} \frac{1}{\rho_2^{l(n+1)}} \\ &\le \frac{M}{\rho_2^{k+n+1}} \sum_{l=0}^{\infty} \frac{1}{\rho_2^{l}} = \frac{M}{\rho_2 - 1} \frac{1}{\rho_2^{k+n}}, \end{aligned}$$

where we use Lemma 2.4(i) in the second relation, that is $\beta_{k+l(n+1)}/\beta_k \le 1$. Therefore, we get an estimate for the error between the interpolation polynomial and the finite Taylor expansion

$$\begin{aligned} \|\mathbf{I}[\mathcal{P}_n, \Phi_{n+1}; f](z) - \sum_{k=0}^{n} a_k z^k\|_{\overline{\mathbb{D}}} &= \|\sum_{k=0}^{n} \Big(c_k^{(n)} - a_k\Big) z^k\|_{\overline{\mathbb{D}}} \\ &\le \sum_{k=0}^{n} |c_k^{(n)} - a_k| \cdot \|z^k\|_{\overline{\mathbb{D}}} \\ &\le \sum_{k=0}^{n} \frac{M}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{k+n}} \\ &\le \frac{M\rho_2}{(\rho_2 - 1)^2} \cdot \frac{1}{\rho_2^{n}} \end{aligned} \tag{30}$$

In addition, relation (27) gives

$$\|\sum_{\ell=n+1}^{\infty} a_\ell z^\ell\|_{\overline{\mathbb{D}}} \le \sum_{\ell=n+1}^{\infty} \frac{M}{\rho_2^{\ell}} = \frac{M}{\rho_2 - 1} \cdot \frac{1}{\rho_2^{n}} \tag{31}$$

Combining (30) and (31), we obtain

$$\begin{aligned} \|\mathbf{I}[\mathcal{P}_n, \Phi_{n+1}; f] - f\|_{\overline{\mathbb{D}}} &\le \|\mathbf{I}[\mathcal{P}_n, \Phi_{n+1}; f](z) - \sum_{k=0}^{n} a_k z^k\|_{\overline{\mathbb{D}}} + \|\sum_{\ell=n+1}^{\infty} a_\ell z^\ell\|_{\overline{\mathbb{D}}} \\ &\le \Big(\frac{M\rho_2}{(\rho_2 - 1)^2} + \frac{M}{\rho_2 - 1}\Big) \frac{1}{\rho_2^{n}}. \end{aligned}$$

The desired assertion follows directly from the last estimate. The proof is complete. □

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