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Strict s-numbers of weighted Hardy type operators on trees

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Abstract. In this paper we calculate the strict s-numbers of Hardy type operators $T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o)$ for $1 < p < \infty$, defined by

$$T_o f(x) := v(x) \int_o^x f(t)u(t)dt, \quad for \;\; o \in \Upsilon_o,$$

where u and v are measurable functions on Υ_o satisfying the conditions $u \in L^{p'}(\kappa), v \in L^p(\Upsilon_o)$, $f \in L^p(\Upsilon_o)$ and $x \in \Upsilon_o$, for every subtree κ of a tree Υ_o such that the closure of κ is compact subset of Υ_o. We obtain the equality among strict s-numbers.

1. Introduction

Let X be a Banach space and $T : X \to X$ be an operator. There is a question to ask whether the operator T is compact or not. If it is compact then one is interested in learning the degree of its compactness. The s-numbers can be used as a tool to answer these questions [5, 9]. The s-numbers can also be used to determine the degree of non-compactness of operators [2]. Among all strict s-numbers, the calculation of Bernstein numbers (defined in Section 2) is applied to investigate the finite strict singularity of operators, a weaker property than compactness (e.g. see [1, 16, 18, 19] or [26]). In 1974, A. Pietsch presented the axiomatic theory of s-numbers [22] and later more general version of this definition came up [23]. By generalizing the source and the target spaces in definition of s-numbers, we get strict s-numbers: Approximation numbers, Kolmogorov numbers, Gelfand Numbers, Bernstein numbers, Mityagin numbers and Isomorphism numbers. When X is an infinite dimensional Hilbert space and T a compact operator, then all n^{th} strict s-numbers of T coincide and these are equal to the n^{th} eigen value of the operator $(T^*T)^{\frac{1}{2}}$ (when arranged in decreasing order) [22]. This is not true if X is not a Hilbert space [23]. However, in [11], the coincidence of strict s-numbers for the simplest case of Hardy operators and for the embedding involving L^p and Sobolev spaces, has been proved. For the weighted Hardy operators $H : L^p(I) \to L^p(I), 1 < p < \infty$, ($I$ is an interval of reals), defined by $H(f) = v(x) \int_a^x u(t)f(t)dt$, it was shown in [10] that the strict s-numbers for

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H coincide. This operator from L^p to L^q, for different values of p and q, has been studied in [8, 20, 24] and [25]. The boundedness of this operator on trees is proved in [14]. The asymptotic estimates and bounds for the approximation numbers of weighted Hardy operators on trees have been obtained by Evans *et. al.* in [13].
In this paper we calculate the exact values of all strict s-numbers of weighted Hardy operators by generalizing the results of [10] to a tree.

2. Elementary Material

Tree

A tree Υ is a connected graph without cycles or loops, where the edges are non-degenerate closed line segments whose end points are vertices. Each vertex of Υ is of finite degree, which means that only finite number of edges can generate from a vertex. For every $x_1, x_2 \in \Upsilon$, there is a unique polygonal path in Υ which joins x_1 and x_2, denoted by $(x_1 : x_2)$. The length of this polygonal path defines the distance between x_1 and x_2 and hence Υ is endowed with the metric topology. For a subtree κ of Υ, $E(\kappa)$ and $V(\kappa)$ are used to denote, respectively, the sets of edges and vertices of κ. By $\delta(\kappa)$ we denote the set of the boundary points of κ in Υ. A subtree κ of Υ is said to be compact if it meets only a finite number of edges of Υ. Let κ be the measurable subset of tree Υ and $|\kappa|$ denotes its Lebesgue measure. Then, norm on Lebesgue space $L^p(\kappa)$ is defined by

$$\left\|f\right\|_{p,\kappa} = \left(\int_\kappa |f(t)|^p dt\right)^{\frac{1}{p}}.$$

We will denote, for short, $\left\|f\right\|_{p,\Upsilon} = \left\|f\right\|_p$. A connected subset of Υ is a subtree if we add its boundary points to the set of vertices of Υ and hence form the new edges from the existing ones. Hereafter, we adopt this convention when we refer to subtrees. The characteristic function of a set K will be denoted by χ_K. We need the following important results from [12, Lemma 2.1, p. 495]. Let $\tau(\Upsilon)$ be the metric topology on Υ. Then
(i) The set $A \subset \Upsilon$ is compact if and only if it is closed and meets only a finite number of edges;
(ii) $\tau(\Upsilon)$ is locally compact;
(iii) Υ is the union of countable number of edges; thus, if Υ is endowed with the natural one-dimensional Lebesgue measure, it is a σ-finite measure space.
For the proof of the above see [12].
For $o \in \Upsilon$, the notation $t \succeq_o z$ (*or* $z \preceq_o t$) means that z lies on the path $(o : t)$ joining o and t. We write $z \prec_o t$ for $z \preceq_o t$ and $z \neq t$. This defines the partial ordering on tree Υ and the ordered graph so formed is referred to as a tree rooted at o and it will be denoted by Υ_o. If o is not a vertex, then we split the edge containing o in two edges emanating from o, making o a vertex. In this way Υ_o is the unique finite union of subtrees $\Upsilon_{o,i}$ which intersect only at o. Let η_o be the degree of the root o. Then we can write

$$\Upsilon_o = \cup_{k=1}^{\eta_o} \Upsilon_{o,k}.$$

Note that if $z \notin (a : b)$ then $z \preceq_a y$ if and only if $z \preceq_b y$.

Let $f \in L^p(\Upsilon_o)$ and $\mathcal{S}$ be a measurable subset of Υ_o. Then

$$\int_{\mathcal{S}} f = \sum_{e \in \mathcal{S}} \int_e f,$$

where e denotes an edge of $\mathcal{S}$. By $|x|$ we denote the length of the path $(o : x)$. The distance between two points $x, y \in \Upsilon_o$ is the length of the path $(x : y)$, where $x \preceq_o y$, denoted by $|(x : y)|$. For a detailed study of trees, we refer the reader to [12, 14, 21].

Definition 2.1. *A point $\theta \in \delta(\kappa)$, where κ is a subtree of Υ containing o, is said to be maximal if every $z \succ_o \theta$ lies in $\Upsilon \setminus \kappa$. By τ_o we denote the set of all subtrees κ of Υ containing o whose boundary points are all maximal.*

Definition 2.2. *Let Υ_o be a tree. Then for $x \in \Upsilon_o$ and $f \in L^p(\Upsilon_o)$, the Hardy operator $T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o)$ for $1 \leq p \leq \infty$, is defined by*

$$T_o f(x) := v(x) \int_o^x f(t)u(t)dt,$$

where u and v are measurable functions on Υ_o satisfying the conditions $u \in L^{p'}(\kappa)$ and $v \in L^p(\Upsilon_o)$, where $p'^{-1}+p^{-1} = 1$, for subtrees κ of Υ_o whose closures are compact subsets of Υ_o.

The operator T_o is bounded in view of [13, Theorem 2.4] for the proof of which one is referred to [14]. For compactness of T_o, one is referred to [6, 13].
From [13] we have the following.

Definition 2.3. *Let κ be a subtree of tree Υ_o and $T_{o,\kappa} : L^p(\kappa) \to L^p(\kappa)$ be the operator. Then*

$$\mathcal{A}(\kappa) := \begin{cases} \sup\limits_{f \in L^p(\kappa), f \neq 0} \inf\limits_{\alpha \in \mathbb{C}} \dfrac{\left\|T_{o,\kappa} f - \alpha v\right\|_{p,\kappa}}{\left\|f\right\|_{p,\kappa}}, & \text{if } \mu(\kappa) > 0, \\ 0, & \text{if } \mu(\kappa) = 0, \end{cases}$$

where

$$T_{o,\kappa} f(x) := v(x)\chi_\kappa(x) \int_o^x f(t)u(t)\chi_\kappa(t)dt,$$

and

$$\mu(\kappa) := \begin{cases} \int_\kappa |v(t)|^p dt & \text{if } 1 \leq p < \infty, \\ ess \sup\limits_\kappa |v(t)| & \text{if } p = \infty. \end{cases}$$

Let $\Gamma \subset \Upsilon_o$ be a subtree with $\gamma \in \Gamma$ being the nearest point to o. Then we have ([13], P. 394), $T_{o,\Gamma} = T_{\gamma,\Gamma}$. Let η_γ be the degree of $\gamma \in \Gamma$. Then $\Gamma = \cup_{k=1}^{\eta_\gamma} \Gamma_{\gamma,k}$, where $\Gamma_{\gamma,k}$ are subtrees of Γ intersecting at γ only. We will call $\Gamma_{\gamma,k}$ to be a norming subtree of Γ if $\left\|T_{\gamma,\Gamma}\right\| := \left\|T_{\gamma,\Gamma}|L^p(\Gamma) \to L^p(\Gamma)\right\| = \left\|T_{\gamma,\Gamma_{\gamma,k}}\right\|$ for some $1 \leq k \leq \eta_\gamma$, and denote it by Γ^*_γ. In this way, $\|T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o)\| := \|T_o\|_{\Upsilon_o} = \|T_o\|_{\Upsilon^*_o}$. Note that if γ is the root of a subtree $\Gamma \subseteq \Upsilon_o$, then $\left\|T_{\gamma,\Gamma}\right\| = \left\|T_\gamma\right\|_\Gamma$. A point $x \in \Upsilon_o$ with degree η_x is said to be simple if there is a subtree Υ_{x,i_0} such that $\|T_x\|_{\Upsilon_{x,i_0}} > \|T_x\|_{\Upsilon_{x,i}}, 1 \leq i \leq \eta_x, i \neq i_0$.

Definition 2.4. ***(The s-Numbers)**[10]*
Let $B(X, Y)$ denote the Banach space of all bounded linear operators acting between Banach spaces X and Y. For an operator $T \in B(X, Y)$, we associate a sequence $s_n(T)$ of scalars satisfying the following properties:

- *(S1) Monotonicity: $\|T\| = s_1(T) \geq s_2(T) \geq s_3(T) \geq ... \geq 0$,*
- *(S2) $s_n(T + S) \leq s_n(T) + \|S\|$ for every $S \in B(X, Y)$,*
- *(S3) Ideal Property: $s_n(B \circ T \circ A) \leq \|B\|s_n(T)\|A\|$ for every $A \in B(Z_1, X)$ and $B \in B(Y, Z_2)$, where Z_1, Z_2 are Banach spaces,*
- *(S4) Norming Property: $s_n(Id : \ell^2_n \to \ell^2_n) = 1$,*
- *(S5) Rank Property: $s_n(T) = 0$ whenever rank $T < n$.*

Then, $s_n(T)$ is called the n-th s-number of T. The number $s_n(T)$ is called the n-th strict s-number of T when the following condition

- *(S6) $s_n(Id : E \to E) = 1$ for every Banach space E of $\dim E \geq n$,*

is considered in place of (S4).

The s-numbers have varied definitions in literature. Initially, A. Pietsch gave the definition of s-numbers (see [22]) which makes use of condition (S6). Later, the definition was refined so that a larger class of s-numbers (such as Chang, Hilbert, Weyl numbers etc.) can be included. For more details of s-numbers, we refer to [3, 7, 23] or [17].

For $T \in B(X, Y)$ and $n \in \mathbb{N}$, we define the n-th Approximation, Gelfand, Kolmogorov, Bernstein, Mityagin and Isomorphism numbers by

$$\begin{aligned}
\mathfrak{a}_n(T) &= \inf_{\substack{F \in B(X,Y)\\ rank\,F<n}} \|T - F\|,\\
\mathfrak{c}_n(T) &= \inf_{\substack{M \subseteq X\\ codim\,M<n}} \sup_{x \in B_M} \|Tx\|_Y,\\
\mathfrak{d}_n(T) &= \inf_{\substack{N \subseteq Y\\ dim\,N<n}} \sup_{x \in B_X} \|Tx\|_{Y/N},\\
\mathfrak{b}_n(T) &= \sup_{\substack{M \subseteq X\\ dim\,M \geq n}} \inf_{x \in S_M} \|Tx\|_Y,\\
\mathfrak{m}_n(T) &= \sup_{\substack{N \subseteq Y\\ codim\,N \geq n}} \sup\{\alpha \geq 0 : \alpha B_{Y/N} \subseteq (\pi_N \circ T)B_X\},
\end{aligned}$$

where $\pi_N : Y \to Y/N$ is a canonical surjection of closed subspace N of Y (see [4, 26] or [15]),

$$\mathfrak{i}_n(T) = \sup_{dim\,(E) \geq n} \|P\|^{-1}\|Q\|^{-1},$$

respectively, where E is Banach space and $P \in B(Y, E)$, $Q \in B(E, X)$ such that $P \circ T \circ Q$ defines identity map on E. The above s-numbers are connected through some inequalities which are bounded below by Isomorphism numbers and bounded from above by Approximation numbers. To be concrete, for $T \in B(X, Y)$ and $n \in \mathbb{N}$, the following relation is obtained (see [10])

$$\begin{aligned}
\mathfrak{i}_n(T) \leq min\{\mathfrak{b}_n(T), \mathfrak{m}_n(T)\} &\leq min\{\mathfrak{c}_n(T), \mathfrak{d}_n(T)\}\\
&\leq max\{\mathfrak{c}_n(T), \mathfrak{d}_n(T)\} \leq \mathfrak{a}_n(T). \qquad (1)
\end{aligned}$$

3. Auxiliary Results

From now onwards we will assume $1 < p < \infty$.

Lemma 3.1. *Let $T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o)$, $1 < p < \infty$, be compact and Γ, Γ' be two subtrees of Υ_o such that $\Gamma' \subset \Gamma$, and $|\Gamma \setminus \Gamma'| > 0$ with $|\Gamma'| > 0$. Suppose that $u, v \neq 0$ almost everywhere on Υ_o and $\int_{\Upsilon_o} |v^p| d(x) < \infty$. Then*

$$\left\|T_{o,\Gamma}\right\|_p \geq \left\|T_{o,\Gamma'}\right\|_p > 0 \qquad (2)$$

and

$$\mathcal{A}(\Gamma) \geq \mathcal{A}(\Gamma') > 0. \qquad (3)$$

Proof. Let $\Gamma' \subset \Gamma$ be the subtrees of Υ_o with $o_{\Gamma'}$ and o_Γ are the nearest point to o such that $|\Gamma^*_{o_\Gamma} \setminus \Gamma'^*_{o_{\Gamma'}}| > 0$. If possible, suppose $\left\|T_{o,\Gamma'}\right\|_p = 0$. Then there exists an $f \neq 0$ such that $\left\|T_{o,\Gamma'} f\right\|_p = 0$. This gives $\int_{\Gamma'} |T_{o,\Gamma'} f(x)|^p dx = 0$, providing $T_{o,\Gamma'} f(x) = 0$ almost everywhere on Γ'. Let $b := o_{\Gamma'}$ be the nearest point of Γ' to o. Then we can write

$$v(x)\chi_{\Gamma'}(x) \int_b^x u(t) f(t) \chi_{\Gamma'}(t) dt = 0, \text{ for almost every } x \in \Gamma'$$

which implies that either $v = 0$ on Γ' or $u = 0$ almost everywhere on Γ', leading to contradiction, since $|\Gamma'| > 0$. Therefore, $\left\|T_{o,\Gamma'}\right\|_p > 0$ for $|\Gamma'| > 0$. Next, on considering $\Gamma = \Gamma' \cup (\Gamma \setminus \Gamma')$, we have

$$\begin{aligned}\left\|T_{o,\Gamma}f\right\|_p^p &= \int_{\Gamma} |T_{o,\Gamma}f(x)|^p dx \\ &= \int_{\Gamma'\cup(\Gamma\setminus\Gamma')} |T_{o,\Gamma}f(x)|^p dx \\ &= \int_{\Gamma'} |T_{o,\Gamma}f(x)|^p dx + \int_{\Gamma\setminus\Gamma'} |T_{o,\Gamma}f(x)|^p dx.\end{aligned}$$

We note that $\Gamma \setminus \Gamma'$ is either a subtree of Γ with $|\Gamma \setminus \Gamma'| > 0$ or it is a finite union of subtrees Γ_i of Γ such that $|\Gamma_i \cap \Gamma_j| = 0$ for $i \neq j$ and at least one of them is positive, say $|\Gamma_{i_k}| > 0$. Therefore, in both the cases, $\int_{\Gamma\setminus\Gamma'} |T_{o,\Gamma}f(x)|^p dx > 0$, and therefore

$$\begin{aligned}\left\|T_{o,\Gamma}f\right\|_p^p &> \int_{\Gamma'} |T_{o,\Gamma}f(x)|^p dx \\ &= \left\|T_{o,\Gamma}f\right\|_{p,\Gamma'}^p \\ &= \left\|T_{o,\Gamma'}f\right\|_p^p\end{aligned}$$

yielding

$$\left\|T_{o,\Gamma}\right\| > \left\|T_{o,\Gamma'}\right\| \text{ for } \Gamma' \subset \Gamma.$$

The equality holds when $\Gamma'$and Γ have a common root o' such that $\Gamma_{o',1} \subset \Gamma' \subset \Gamma$, where $\Gamma_{o',1}$ is the connected component of Γ rooted at o', the nearest point to o, such that $\left\|T_{o,\Gamma}\right\| = \left\|T_{o,\Gamma_{o',1}}\right\|$. Now, to prove (3.2), let $x \in \Gamma' \subset \Gamma$. Then, by [13, Lemma 3.5, Theorem 3.8], there exist k, l such that $\Gamma'_{x,k} \neq \Gamma'_{x,l}$ and

$$\max\left\{\left\|T_{o,\Gamma'_{x,k}}\right\|, \left\|T_{o,\Gamma'_{x,l}}\right\|\right\} \le \min_{x\in\Gamma'} \left\|T_{x,\Gamma'}\right\| = \mathcal{A}(\Gamma').$$

Since $|\Gamma'| > 0$, we may assume that $|\Gamma'_{x,k}|, |\Gamma'_{x,l}| > 0$ and therefore $\mathcal{A}(\Gamma') > 0$. Let $\zeta \neq \zeta'$ be two non-simple points in Γ and Γ' respectively, such that $\mathcal{A}(\Gamma) = \left\|T_{\zeta,\Gamma}\right\|$ and $\mathcal{A}(\Gamma') = \left\|T_{\zeta',\Gamma'}\right\|$, Then $\mathcal{A}(\Gamma) = \left\|T_{\zeta,\Gamma}\right\| \geq \left\|T_{\zeta,\Gamma'}\right\| \geq \left\|T_{\zeta',\Gamma'}\right\| = \mathcal{A}(\Gamma')$. The case of $\zeta = \zeta'$ is obvious. This completes the proof. □

Lemma 3.2. *Let $T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o)$ be compact. Then there exists a path $\Delta_o \subseteq \Upsilon_o$ such that $\|T_o\|_{\Upsilon_o} = \|T_o\|_{\Delta_o}$.*

Proof. Let Υ_o^* be the norming subtree of Υ_o. Then $\left\|T_{o,\Upsilon_o}\right\| = \left\|T_{o,\Upsilon_o^*}\right\|_{\Upsilon_o^*} = \|T_o\|_{\Upsilon_o^*}$ and by compactness of T_o, there is an $f_1 \in L^p(\Upsilon_o)$ such that $\|T_o\|_{\Upsilon_o^*} = \left\|T_o f_1\right\|_{\Upsilon_o^*}$. Let o_1 be the nearest point of subtree $\Upsilon_{o_1} \subset \Upsilon_o^*$ to o such that $\Upsilon_o^* = (o : o_1) \cup \Upsilon_{o_1}$. Then

$$\begin{aligned}\left\|T_o f_1\right\|_{\Upsilon_o^*}^p &= \int_o^{o_1} |T_o f_1(x)|^p dx + \int_{\Upsilon_{o_1}} |T_o f_1(x)|^p dx \\ &= \int_o^{o_1} |T_o f_1(x)|^p dx + \left\|T_o f_1\right\|_{\Upsilon_{o_1}}^p \\ &\le \int_o^{o_1} |T_o f_1(x)|^p dx + \|T_o\|_{\Upsilon_{o_1}}^p\end{aligned}$$

and by compactness of T_o, there is an $f_2 \in L^p(\Upsilon_o)$ and norming subtree $\Upsilon^*_{o_1} \subseteq \Upsilon_{o_1}$ such that

$$\left\|T_o f_1\right\|^p_{\Upsilon^*_o} \leq \int_o^{o_1} |T_o f_1(x)|^p dx + \left\|T_o f_2\right\|^p_{\Upsilon^*_{o_1}}.$$

By the similar arguments as above, there is a point $o_2 \in \Upsilon_{o_2} \subseteq \Upsilon^*_{o_1}$ such that

$$\begin{aligned}\left\|T_o f_1\right\|^p_{\Upsilon^*_o} &= \int_o^{o_1} |T_o f_1(x)|^p dx + \int_{o_1}^{o_2} |T_o f_2(x)|^p dx + \int_{\Upsilon_{o_2}} |T_o f_2(x)|^p dx \\ &= \int_o^{o_1} |T_o f_1(x)|^p dx + \int_{o_1}^{o_2} |T_o f_2(x)|^p dx + \left\|T_o f_2\right\|^p_{\Upsilon_{o_2}} \\ &\leq \int_o^{o_1} |T_o f_1(x)|^p dx + \int_{o_1}^{o_2} |T_o f_2(x)|^p dx + \|T_o\|^p_{\Upsilon_{o_2}}.\end{aligned}$$

Continuing in this manner, we obtain a path $\Delta_o = \underset{i\in\Lambda}{\cup}(o_{i-1} : o_i)$ contained in Υ^*_o emanating from o ($:= o_0$) and an $f \in L^p(\Upsilon_o)$ such that $f = \sum_{i\in\Lambda} f_i \chi_{(o_{i-1}:o_i)}$ for some index set $\Lambda \subseteq \mathbb{N}$, so that we have $\|T_o\|^p = \left\|T_o f_1\right\|^p_{\Upsilon^*_o} \leq \left\|T_o f\right\|^p_{\Delta_o} \leq \|T_o\|^p_{\Delta_o}$. Since $\Delta_o \subseteq \Upsilon^*_o$, therefore by Lemma 3.1, we have $\|T_o\|_{\Upsilon_o} = \|T_o\|_{\Delta_o}$, proving the Lemma 3.2. □

We have the following definitions.

Definition 3.3. *Let Γ_b be a subtree of Υ_o rooted at $b \in \Delta_o$ which is the nearest point of Γ_b to o. We define*

$$\mathcal{P} := \{\Gamma_b \subseteq \Upsilon_o : |\Delta_o \cap \Gamma_b| > 0\}$$

and

$$\mathcal{P}' := \left\{\Gamma_b \in \mathcal{P} : \|T_b\|_{\Gamma_b} \text{ is attained on the path } \Delta_o \cap \Gamma_b\right\}.$$

It is easy to see that Υ_o belongs to $\mathcal{P}$.

Remark 3.4. *For $\Gamma_b \in \mathcal{P}'$, by [13, Theorem 3.8] and by Lemma 3.2, there is a non-simple point $\theta \in \Gamma_b \cap \Delta_o$ such that $\mathcal{A}(\Gamma_b) = \|T_\theta\|_{\Gamma_b} = \|T_\theta\|_{\Delta_o\cap\Gamma_b}$.*

Remark 3.5. *In view of Definition 3.3 and Remark 3.4, there are $b_1, b_2, ..., b_n \in \Delta_o$ with $b_1 = o$ (say) and $b_{l-1} \leq_o b_l$ for $3 \leq l \leq n$, such that $\Gamma_{b_i} \in \mathcal{P}'$ with $|\Gamma_{b_i} \cap \Gamma_{b_j}| = 0$ for $i \neq j$, and a subtree $\widehat{\Gamma}_o$ of Υ_o containing Δ_o such that $\cup_{i=1}^n \Gamma_{b_i} = \widehat{\Gamma}_o$. In this way $\{\Gamma_{b_i} : i = 1, 2, 3, ..., n\}$ forms a partition of $\widehat{\Gamma}_o$. Denote all such partitions of $\widehat{\Gamma}_o$ by $\wp_n(\widehat{\Gamma}_o)$.*

Definition 3.6. *For each $N \in \mathbb{N} \setminus \{1\}$, we define*

$$\epsilon_N = \{\epsilon > 0 : \left\|T_{b_1}\right\|_{\Gamma_{b_1}} = \mathcal{A}(\Gamma_{b_i}) = \epsilon, 2 \leq i \leq N, \text{ where } \Gamma_{b_i} \in \mathcal{P}' \text{ is the largest subtree rooted at } b_i$$
$$\text{such that } \{\Gamma_{b_i}, 1 \leq i \leq N\} \in \wp_N(\widehat{\Upsilon}_o) \text{ for some subtree } \widehat{\Upsilon}_o \text{ such that } \Delta_o \subseteq \widehat{\Upsilon}_o \subseteq \Upsilon_o\}.$$

In the above definition, note that for $N = 1$, $\left\|T_{b_1}\right\|_{\Gamma_{b_1}} = \|T_o\|_{\Upsilon_o} = \epsilon_1$. In future, the closure of $\widehat{\Upsilon}_o$ will be denoted by itself.

Remark 3.7. *Since $\Delta_o \subseteq \mathbb{R}^+$, the existence of ϵ_N is guarenteed by Remarks 3.4, 3.5 and [10, Lemma 3.5].*

We now prove the following lemmas.

Lemma 3.8. *Let $\Gamma \subseteq \Upsilon_o$ be a subtree and $T_\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ be compact. Let $b_1, b_2, ..., b_N \in \Delta_o$ be points such that $\{\Gamma_{b_i} \in \mathcal{P}' : \left\|T_{b_1}\right\|_{\Gamma_{b_1}} = \mathcal{A}(\Gamma_{b_i}) = \epsilon_N,\ 2 \leq i \leq N\} \in \wp_N(\widehat{\Upsilon}_o)$, for some $\epsilon_N > 0$. Then $\mathrm{i}_N(T_{\widehat{\Upsilon}_o}) \geq \epsilon_N$.*

Proof. By compactness of $T_{\widehat{\Upsilon}_o}$ and Remark 3.4, there exist points $\theta_i \in \Gamma_{b_i}$ and functions f_i supported on Γ_{b_i} with $\left\|f_i\right\|_{\Gamma_{b_i}} = 1$ such that $\mathcal{A}(\Gamma_{b_i}) = \left\|T_{\theta_i} f_i\right\|_{\Gamma_{b_i}}$, $2 \leq i \leq N$ and $\left\|T_{b_1}\right\|_{\Gamma_{b_1}} = \left\|T_{b_1} f_1\right\|_{\Gamma_{b_1}}$. Let $\Gamma^1_{\theta_i}$ and $\Gamma^2_{\theta_i}$ be two subtrees of Γ_{b_i} such that $\Gamma_{b_i} = \Gamma^1_{\theta_i} \cup \Gamma^2_{\theta_i}$ and $\Gamma^1_{\theta_i} \cap \Gamma^2_{\theta_i} = \{\theta_i\}$ for all $1 \leq i \leq N$. If $x_1 \in (o : x_2)$ and $\Gamma^2_{x_1} \cap \Gamma^1_{x_2} = \{x\}$, for some $x \in (x_1 : x_2)$, then we write $\Gamma^2_{x_1} << \Gamma^1_{x_2}$. By this convention, we have

$$\Gamma^1_{\theta_1} << \Gamma^2_{\theta_1} << \Gamma^1_{\theta_2} << \Gamma^2_{\theta_2} .... << \Gamma^1_{\theta_N} << \Gamma^2_{\theta_N}.$$

Define $\Omega_1 = \Gamma_{b_1} \cup \Gamma^1_{\theta_2}$, $\Omega_N = \Gamma^2_{\theta_N}$ and $\Omega_j = \Gamma^2_{\theta_j} \cup \Gamma^1_{\theta_{j+1}}$ for $2 \leq j \leq N-1$, and functions $g_j = (\alpha_j f_j + \beta_j f_{j+1})\chi_{\Omega_j}$ for $1 \leq j \leq N-1$, with $g_N = \beta_N f_N$, where α_j and β_j are constants. Then, we have

$$\left\|g_j\right\|_{\Omega_j} = \left\|\alpha_j f_j + \beta_j f_{j+1}\right\|_{\Omega_j} \leq |\alpha_j| \left\|f_j\right\|_{\Gamma^2_{\theta_j}} + |\beta_j| \left\|f_{j+1}\right\|_{\Gamma^1_{\theta_{j+1}}}.$$

Since $\left\|f_j\right\|_{\Gamma_{b_j}} = 1$, so $\left\|f_j\right\|_{\Gamma^2_{\theta_j}} \leq 1$ and $\left\|f_{j+1}\right\|_{\Gamma^1_{\theta_{j+1}}} \leq 1$. Thus, by choosing suitable α_j and β_j, we have $\left\|g_j\right\|_{\Omega_j} = 1$, from which we obtain

$$\begin{aligned}
\frac{\left\|T_{\theta_j} g_j\right\|_{\Gamma^2_{\theta_j}}}{\left\|g_j\right\|_{\Gamma^2_{\theta_j}}} &= \frac{\left\|T_{\theta_j}\left(\left(\alpha_j f_j + \beta_j f_{j+1}\right)\chi_{\Omega_j}\right)\right\|_{\Gamma^2_{\theta_j}}}{\left\|\left(\left(\alpha_j f_j + \beta_j f_{j+1}\right)\chi_{\Omega_j}\right)\right\|_{\Gamma^2_{\theta_j}}} \\
&= \frac{\left\|T_{\theta_j}\left(\alpha_j f_j\right)\right\|_{\Gamma^2_{\theta_j}}}{\left\|\alpha_j f_j\right\|_{\Gamma^2_{\theta_j}}} \\
&\geq \epsilon_N \quad \text{for} \quad 2 \leq j \leq N.
\end{aligned}$$

Similarly, we get

$$\frac{\left\|T_{\theta_j} g_{j-1}\right\|_{\Gamma^1_{\theta_j}}}{\left\|g_{j-1}\right\|_{\Gamma^1_{\theta_j}}} \geq \epsilon_N \quad \text{for} \quad 3 \leq j \leq N.$$

For $\Omega_j \subset \widehat{\Upsilon}_o$,

$$\begin{aligned}
\left\|T_{\widehat{\Upsilon}_o} g_1\right\|_{\Omega_1} &= \left\|T_{\widehat{\Upsilon}_o}\left(\alpha_j f_1 + \beta_1 f_2\right)\right\|_{\Omega_1} \\
&= \left\|T_{\widehat{\Upsilon}_o}\left(\alpha_1 f_1\right)\right\|_{\Gamma_{b_1}} + \left\|T_{\widehat{\Upsilon}_o}\left(\beta_1 f_2\right)\right\|_{\Gamma^1_{\theta_2}} \\
&= |\alpha_1| \left\|T_{\widehat{\Upsilon}_o} f_1\right\|_{\Gamma_{b_1}} + |\beta_1| \left\|T_{\widehat{\Upsilon}_o} f_2\right\|_{\Gamma^1_{\theta_2}} \\
&= \left(|\alpha_1| + |\beta_1|\right)\epsilon_N,
\end{aligned}$$

and for $2 \leq j \leq N-1$,

$$\begin{aligned}
\left\|T_{\widehat{\Upsilon}_o} g_j\right\|_{\Omega_j} &= \left\|T_{\widehat{\Upsilon}_o}(\alpha_j f_j + \beta_j f_{j+1})\right\|_{\Omega_j} \\
&= \left\|T_{\widehat{\Upsilon}_o}(\alpha_j f_j)\right\|_{\Gamma^2_{\theta_j}} + \left\|T_{\widehat{\Upsilon}_o}(\beta_j f_{j+1})\right\|_{\Gamma^1_{\theta_{j+1}}} \\
&= |\alpha_j| \left\|T_{\widehat{\Upsilon}_o} f_j\right\|_{\Gamma^2_{\theta_j}} + |\beta_j| \left\|T_{\widehat{\Upsilon}_o} f_{j+1}\right\|_{\Gamma^1_{\theta_{j+1}}} \\
&= \left(|\alpha_j| + |\beta_j|\right)\epsilon_N.
\end{aligned}$$

For $1 \leq j \leq N-1$, $\left\|g_j\right\|_{\Omega_j} = 1$, $\left|\alpha_j\right| + \left|\beta_j\right| \geq 1$, and for $j = N$, $\Omega_N = \Gamma^2_{\theta_N}$, and for $\alpha_N = 0$, $\beta_N \geq 1$, give us

$$\frac{\left\|T_{\widehat{\Upsilon}_o} g_j\right\|_{\Omega_j}}{\left\|g_j\right\|_{\Omega_j}} \geq \epsilon_N \text{ for } 1 \leq j \leq N.$$

Next, $\mathcal{B}_1 : l_p^N \to L_p(\widehat{\Upsilon}_o)$ and $\mathcal{B}_2 : L_p(\widehat{\Upsilon}_o) \to l_p^N$ are the operators defined by

$$\mathcal{B}_1(\mathbf{x}) = \sum_{j=1}^{N} x_j g_j,$$

for $\mathbf{x} = \{x_1, x_2, x_3, ..., x_N\} \in l_p^N$, and

$$(\mathcal{B}_2 g)(x) = \left\{ \frac{\int_{\Omega_j} g(x)(T_{\widehat{\Upsilon}_o} g_j)_p(x)dx}{\left\|T_{\widehat{\Upsilon}_o} g_j\right\|_{\Omega_j}^p} \right\}_{j=1}^{N},$$

where $(g)_p = |g|^{p-2}g$. Then it is easily seen that $\mathcal{B}_2 \circ T_{\widehat{\Upsilon}_o} \circ \mathcal{B}_1$ is an identity map on l_p^N. We compute

$$||\mathcal{B}_1|| = \sup_{||\mathbf{x}||_{l_p^N}=1} ||\mathcal{B}_1(\mathbf{x})||_{L_p(\widehat{\Upsilon}_o)} = \sup_{||\mathbf{x}||_{l_p^N}=1} \left\| \sum_{j=1}^{N} x_j g_j \right\|_{L_p(\widehat{\Upsilon}_o)} = \sup_{||\mathbf{x}||_{l_p^N}=1} \sum_{j=1}^{N} |x_j| \left\|g_j\right\|_{\Omega_j} = 1.$$

The definition of $\mathcal{B}_2$ implies that the operator norm of $\mathcal{B}_2$ is attained on the functions of the form $g(x) = \sum_{j=1}^{N} r_j T_{\widehat{\Upsilon}_o} g_j(x)$, for constants r_j. We have

$$\left\|g\right\|_{L_p(\widehat{\Upsilon}_o)}^p = \int_{\widehat{\Upsilon}_o} |\sum_{j=1}^{N} r_j T_{\widehat{\Upsilon}_o} g_j(x)|^p dx \geq \epsilon_N^p \left\|\{r_j\}_{j=1}^N\right\|_{l_p^N}^p$$

and

$$\begin{aligned} ||\mathcal{B}_2|| &= \sup_{||g||_{L_p(\widehat{\Upsilon}_o)}=1} \left\| \mathcal{B}_2 \left(\sum_{j=1}^{N} r_j T_{\widehat{\Upsilon}_o} g_j(x) \right) \right\|_{l_p^N} \\ &= \sup_{||g||_{L_p(\widehat{\Upsilon}_o)}=1} \left\| \left\{ \frac{\int_{\Omega_k} \left(\sum_{j=1}^{N} r_j T_{\widehat{\Upsilon}_o} g_j(x) \right) (T_{\widehat{\Upsilon}_o} g_k)_p(x)dx}{\left\|T_{\widehat{\Upsilon}_o} g_k\right\|_{\Omega_k}^p} \right\}_{k=1}^{N} \right\|_{l_p^N} \\ &= \left\|\{r_j\}_{j=1}^N\right\|_{l_p^N} \\ &\leq \frac{1}{\epsilon_N}, \end{aligned}$$

and hence $\mathfrak{i}_N(T_{\widehat{\Upsilon}_o}) = \sup ||\mathcal{B}_1||^{-1} ||\mathcal{B}_2||^{-1} \geq ||\mathcal{B}_1||^{-1} ||\mathcal{B}_2||^{-1} \geq \epsilon_N$ proving the Lemma 3.8.

□

Lemma 3.9. *Let $\Gamma \subseteq \Upsilon_o$ be a subtree and $T_\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ be a compact operator and $\widehat{\Upsilon}_o$ be as in Remark 3.5. Then, for all $n \in \mathbb{N}$, $\mathfrak{i}_n(T_{\widehat{\Upsilon}_o}) \leq \mathfrak{i}_n(T_{\Upsilon_o})$.*

Proof. Let $A_{\Gamma_o} : l_p^n \to L^p(\Gamma_o)$ and $B_{\Gamma_o} : L^p(\Gamma_o) \to l_p^n$ be the operators on a subtree $\Gamma_o \subseteq \Upsilon_o$ containing the root o. When $\Gamma_o = \Upsilon_o$, we simply denote A_{Γ_o} and B_{Γ_o} by A and B, respectively. Define an operator $I : L^p(\Upsilon_o) \to L^p(\widehat{\Upsilon}_o)$ by $I(f) = f\chi_{\widehat{\Upsilon}_o}$. Then we have $\left\|B_{\widehat{\Upsilon}_o} I\right\| \leq \left\|B_{\widehat{\Upsilon}_o}\right\| \|I\| \leq \left\|B_{\widehat{\Upsilon}_o}\right\|$, which implies $\|B\|^{-1} \geq \left\|B_{\widehat{\Upsilon}_o}\right\|^{-1}$. Next, consider subtrees $\widetilde{\Gamma}_{b_k} \in \mathcal{P}$ such that $\widetilde{\Gamma}_{b_k} \cap \widehat{\Upsilon}_o = \Gamma_{b_k} \in \mathcal{P}', 1 \leq k \leq n$. With this, one can easily construct the subtrees $\widetilde{\Omega}_{\sigma_k}$ and functions $\widetilde{g}_k$ as in Lemma 3.8, where $\sigma_k \in \widetilde{\Gamma}_{b_k}$ is same as $\theta_k \in \Gamma_{b_k}$. Now, corresponding to $\mathcal{B}_1$ of Lemma 3.8, we define the operator A by $A(<\zeta_k>) = \sum_{k=1}^n \zeta_k g_k$ for $<\zeta_k> \in l_p^n$. Then it is easy to see that $\|A\| = 1$, which proves the Lemma 3.9. □

Lemma 3.10. *Let $\Gamma \subseteq \Upsilon_o$ be a subtree and $T_\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ be compact operator and $\widehat{\Upsilon}_o$ be as in Remark 3.5. Then, for all $n \in \mathbb{N}$, $\mathfrak{a}_n(T_{\widehat{\Upsilon}_o}) = \mathfrak{a}_n(T_{\Upsilon_o})$.*

Proof. Let $P_\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ be an operator of $rank(P) < n$. When $\Gamma = \Upsilon_o$, we denote T_Γ and P_Γ by T and P respectively. By compactness of T and P, there exists an $f \in \Upsilon_o^*$ such that

$$\|T - P\|_{\Upsilon_o} = \left\|(T - P)f\right\|_{\Upsilon_o^*}.$$

By the same arguments as in Lemma 3.3 and by continiuty of $\|T_x\|$ on Δ_o ([13, Lemma 3.4]), there is a path $\Delta_o' \subseteq \Delta_o \subseteq \Upsilon_o^*$ and a function $\phi_f \in L^p(\Delta_o')$ such that

$$\left\|(T - P)f\right\|_{\Upsilon_o^*} = \left\|(T - P)\phi_f\right\|_{\Delta_o'} \leq \|T - P\|_{\Delta_o'}.$$

But $\Delta_o' \subseteq \Upsilon_o^*$, therefore $\|T - P\|_{\Upsilon_o} = \|T - P\|_{\Delta_o'}$. Since $\Delta_o' \subseteq \widehat{\Upsilon}_o$, we have

$$\begin{aligned}\|T - P\|_{\Upsilon_o} &= \left\|(T - P)\phi_f\right\|_{\Delta_o'} = \left\|\left(T_{\widehat{\Upsilon}_o} - P_{\widehat{\Upsilon}_o}\right)\phi_f\right\|_{\Delta_o'} \\ &\leq \left\|\left(T_{\widehat{\Upsilon}_o} - P_{\widehat{\Upsilon}_o}\right)\right\|_{\widehat{\Upsilon}_o} = \|T - P\|_{\widehat{\Upsilon}_o}.\end{aligned}$$

Noting $\widehat{\Upsilon}_o \subseteq \Upsilon_o$, from above we can write $\|T - P\|_{\Upsilon_o} = \left\|T_{\widehat{\Upsilon}_o} - P_{\widehat{\Upsilon}_o}\right\|_{\widehat{\Upsilon}_o}$. Hence $a_n(T_{\widehat{\Upsilon}_o}) = a_n(T_{\Upsilon_o})$. □

Lemma 3.11. *Let $\Gamma \subseteq \Upsilon_o$ be a subtree and $T_\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ be compact. Let $b_1, b_2, ..., b_N \in \Delta_o$ be points such that $\{\Gamma_{b_i} \in \mathcal{P}' : \left\|T_{b_1}\right\|_{\Gamma_{b_1}} = \mathcal{A}(\Gamma_{b_i}) = \epsilon_N, 2 \leq i \leq N\} \in \wp_N(\widehat{\Upsilon}_o)$ for some $\epsilon_N > 0$. Then $\mathfrak{a}_N(T_{\widehat{\Upsilon}_o}) \leq \epsilon_N$.*

Proof. Let $L : L^p(\widehat{\Upsilon}_o) \to L^p(\widehat{\Upsilon}_o)$ be the operator defined by $L(\psi) = \sum_{i=2}^N L_i(\psi) + 0\chi_{\Gamma_{b_1}}$, where $L_i(\psi)(x) = \chi_{\Gamma_{b_i}}(x)v(x)\int_{\Gamma^1_{\theta_i}} u(t)f(t)\chi_{\Gamma_{b_i}}(t)dt$ ($\Gamma^1_{\theta_i}$ is defined in Lemma 3.8). Then $rank(L) \leq N - 1$. Now by definition of approximation numbers and by compactness of $T_{\widehat{\Upsilon}_o}$ and L, there exists an $f \in L^p(\widehat{\Upsilon}_o)$ with $\left\|f\right\|_{\widehat{\Upsilon}_o} = 1$ such that

$$\begin{aligned}a_N(T_{\widehat{\Upsilon}_o}) &\leq \left\|T_{\widehat{\Upsilon}_o} - L\right\|_{\widehat{\Upsilon}_o} = \left\|\left(T_{\widehat{\Upsilon}_o} - L\right)f\right\|_{\widehat{\Upsilon}_o} \\ &= \left\|T_{\widehat{\Upsilon}_o}f - Lf\right\|_{\widehat{\Upsilon}_o} = \left\|T_{\widehat{\Upsilon}_o}f - \left(\sum_{i=2}^N L_i(f) + 0\chi_{\Gamma_{b_1}}\right)\right\|_{\widehat{\Upsilon}_o} \\ &= \sum_{i=2}^N \left\|T_{\theta_i}f\right\|_{\Gamma_{b_i}} + \left\|Tf\right\|_{\Gamma_{b_1}} \leq \epsilon_N \sum_{i=2}^N \left\|f\right\|_{\Gamma_{b_i}} + \epsilon_N \left\|f\right\|_{\Gamma_{b_1}} \\ &= \epsilon_N \sum_{i=1}^N \left\|f\right\|_{\Gamma_{b_i}} = \epsilon_N,\end{aligned}$$

which proves the lemma. □

4. Main Results

From Lemmas 3.8, 3.11 and Remark 3.7, we obtain the following main result.

Theorem 4.1. *Let $T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o), 1 < p < \infty$, be compact. Then there exists a subtree $\widehat{\Upsilon}_o \subseteq \Upsilon_o$ with a partition $\{\Gamma_{b_i} \in \mathcal{P}' : \left\|T_{b_1}\right\|_{\Gamma_{b_1}} = \mathcal{A}(\Gamma_{b_i}) = \epsilon_N,\ 2 \le i \le N\} \in \wp_N(\widehat{\Upsilon}_o)$ for some $\epsilon_N > 0$, such that $\|T_o\|_{\widehat{\Upsilon}_o} = \|T_o\|_{\Upsilon_o}$ and $\mathfrak{a}_N(T_{\widehat{\Upsilon}_o}) = \epsilon_N = \mathfrak{i}_N(T_{\widehat{\Upsilon}_o})$.*

Using Lemmas 3.9, 3.10 and Theorem 4.1, by inequality (1), we have

Theorem 4.2. *Let $T_o : L^p(\Upsilon_o) \to L^p(\Upsilon_o), 1 < p < \infty$, be compact. Then all strict s-numbers of T_o coincide.*

5. Conclusions

We obtained the exact values of all strict s-numbers of weighted Hardy operators on trees and observed that N^{th} terms of all coincide and equal to ϵ_N.

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