Filomat 38:30 (2024), 10747–10763
https://doi.org/10.2298/FIL2430747B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bullen-Mercer type inequalities for the h-convex function with twice differentiable functions

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Abstract. Bullen-type inequalities for h-convex functions using conformable fractional operators are established in this study on the cone of twice-differentiable functions. This is a novel fractional version of the existing Bullen-type inequalities with simple procedures using the B-function. Furthermore, new results on Bullen-type inequalities are presented for several specific cases of convexity, generalizing existing inequalities known in the literature.

1. Introduction

Explorations in numerical integration and the definition of error bounds are critical in mathematical literature. Researchers have thoroughly investigated error boundaries for functions with variable differentiability, from once to many times. The Bullen-type inequality is a significant mathematical tool for integral estimate. The well-known Hermite-Hadamard inequality is defined as follows [10], for the convex function:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

In [4], Bullen improved the right side of (1) by the following inequality, which is known as Bullen's inequality:

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

The estimation of Bullen-type inequalities for functions whose first derivative absolute values are convex is as follows. [13, Remark 4.2].

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]}{16}.$$

2020 *Mathematics Subject Classification*. Primary 26D10; Secondary 26A51, 26A33, 26D15.
Keywords. h-convex function, B-function, twice-differentiable functions, Bullen-Mercer inequality
Received: 30 April 2024; Revised: 19 July 2024; Accepted: 21 July 2024
Communicated by Miodrag Spalević
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The following is an estimation of Bullen-type inequality for functions whose second derivative absolute values are convex: [19, Proposition.4] and [9, Corollary 1.].

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{96}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2}$$

Bullen's inequalities provide an estimate of the average value of a function that is convex on both sides while simultaneously ensuring that the function is integrable. This inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [13]-[9]).

The analysis of fractional calculations is a generalization of classical analysis, and it advanced rapidly thanks to the exciting concept of convexity. Its extensive applications in functional analysis and optimization theory have made it a very popular research area. The author of [20] introduces a novel class of functions called h-convex functions.

Definition 1.1. *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, if f is non-negative and for all $x, y \in I$, $\lambda \in (0,1)$ we have*

$$f(\lambda x+(1-\lambda)y) \leq h(\lambda)f(x)+h(1-\lambda)f(y). \tag{3}$$

If the inequality (3) is reversed, then f is said to be h-concave.

By setting

- $h(\lambda)=\lambda$, Definition 1.1 reduces to convex function [16].
- $h(\lambda)=1$, Definition 1.1 reduces to P-functions [7, 17].
- $h(\lambda)=\lambda^s$, Definition 1.1 reduces to s-convex functions [3].
- $h(\lambda)=\frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$, Definition 1.1 reduces to polynomial n-fractional convex functions [14].

In [15], the famous Jensen-Mercer inequality was presented as follows: If f is a convex function on $[a,b]$, then

$$f\left(a+b-\sum_{j=1}^{n}\lambda_j z_j\right) \leq f(a)+f(b)-\sum_{j=1}^{n}\lambda_j f(z_j),$$

for each $z_j \in [a,b]$ and $\lambda_j \in [0,1]$ $(j=\overline{1;n})$ with $\sum\limits_{j=1}^{n}\lambda_j=1$.

In [1], the authors presented the following interesting result (Lemma.4.1).

Lemma 1.2. *Let f be an h-convex function. Then for every $z \in [a,b]$, there exists $\lambda \in [0,1]$ such that*

$$f(a+b-z) \leq [h(\lambda)+h(1-\lambda)]\,[f(a)+f(b)]-f(z). \tag{4}$$

Recently, [2] the authors present a new class of function called B-function defined as:

Definition 1.3. *Let $a<b$ and $g : (a,b) \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. The function g is a B-function, or that g belongs to the class $B(a,b)$, if for all $x \in (a,b)$ that $g(x-a)$, $g(b-x)$ are defined and*

$$g(x-a)+g(b-x) \leq 2\,g\left(\frac{a+b}{2}\right). \tag{5}$$

If the inequality (5) is reversed, g is called A-function, or that g belongs to the class $A(a,b)$.
If we have equality in (5), g is called AB-function, or that g belongs to the class $AB(a,b)$.

Corollary 1.4. *Let $h : (0,1) \to \mathbb{R}$ be a non-negative function. The function h is a B-function, if and only if for all $\lambda \in (0,1)$, we have*

$$h(\lambda) + h(1-\lambda) \leq 2\,h\left(\frac{1}{2}\right). \tag{6}$$

- *The functions $h(\lambda) = \lambda$ and $h(\lambda) = 1$ are AB-function, B-function and A-function.*
- *The function $h(\lambda) = \lambda^s$, $s \in (0,1]$ is B-function.*
- *The function $h(\lambda) = \dfrac{1}{n}\sum_{k=1}^{n} \lambda^{\frac{1}{k}}$, $n, k \in \mathbb{N}$ is B-function.*

The conformable fractional integral operators with orders $\alpha > 0$ and $\rho \in (0,1]$ are represented as follows.

$${}^{\rho}\mathfrak{I}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{(x-a)^{\rho} - (t-a)^{\rho}}{\rho}\right)^{\alpha-1} (t-a)^{\rho-1} f(t)dt, \quad x > a,$$

$${}^{\rho}\mathfrak{I}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\frac{(b-x)^{\rho} - (b-t)^{\rho}}{\rho}\right)^{\alpha-1} (b-t)^{\rho-1} f(t)dt, \quad x < b.$$

For $\rho = 1$, the preceding operators are reduced to Riemann-Liouville fractional operators of order $\alpha > 0$ as follows:

$$\mathfrak{I}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$\mathfrak{I}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

For $p, q > 0$, the beta function $\beta(.,.)$ is defined as follows:

$$\beta(q,p) = \int_0^1 (1-y)^{q-1}\; y^{p-1}\, dy.$$

In 2024, the authors established a new Bullen's inequality for twice-differentiable function involving the conformable fractional integral operators with orders $\alpha > 0$ and $\rho \in (0,1]$ as follows [12, Theorem 1].

Theorem 1.5. *If $|f''|$ is a convex mapping on $[a,b]$, then*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{a^+} f\left(\frac{a+b}{2}\right) + {}^{\rho}\mathfrak{I}^{\alpha}_{b^-} f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{(b-a)^2}{8}\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \int_0^1 \left|\int_t^1 \left((1-(1-z)^{\rho})^{\alpha} - \frac{1}{2}\right) dz\right| dt. \end{aligned} \tag{7}$$

Based on previous research, our current study employs an exploratory approach to discovering Bullen-type inequality for h-convex functions by characterizing twice differentiable functions using Riemann-Liouville integral operators.

2. Bullen-Mercer type inequalities

First, we introduce the main Lemma, which is an essential resource for our results.

Lemma 2.1. *Let $\alpha > 0$, $\rho \in (0, 1]$ and $x, y \in [a, b]$ where $x < y$. If $f : [x, y] \to \mathbb{R}$ is a twice differentiable mapping such that $f'' \in L_1([x, y])$, then the following identity holds.*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right] \\
&-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right] \\
&=\frac{(y-x)^2}{8}\int_0^1\left[\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right] \\
&\times\left[f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)+f''\left(\left(\frac{1+t}{2}\right)(a+b-y)+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right]dt.
\end{aligned} \tag{8}$$

Proof. Let $x, y \in [a, b]$ with $x < y$ and taking $\tau = \frac{1-t}{2}(a+b-y)+\frac{1+t}{2}(a+b-x)$, we get

$$\begin{aligned}
&\int_0^1(1-(1-t)^{\rho})^{\alpha-1}(1-t)^{\rho-1}f\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)dt \\
&=\left(\frac{2}{y-x}\right)^{\rho\alpha}\int_{a+b-\frac{x+y}{2}}^{a+b-x}\left[\left((a+b-x)-\left(a+b-\frac{x+y}{2}\right)\right)^{\rho}-((a+b-x)-\tau)^{\rho}\right]^{\alpha-1} \\
&\quad\times((a+b-x)-\tau)^{\rho-1}f(\tau)d\tau \\
&=\left(\frac{2}{y-x}\right)^{\rho\alpha}\rho^{\alpha-1}\Gamma(\alpha)\;{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right),
\end{aligned}$$

and

$$\int_0^1(1-(1-t)^{\rho})^{\alpha}\,dt=\int_0^1(1-s^{\rho})^{\alpha}\,ds=\frac{1}{\rho}\int_0^1(1-s)^{\alpha}s^{\frac{1}{\rho}-1}\,ds=\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right).$$

By utilizing the integration by parts method, we can obtain the following expression:

$$\begin{aligned}
J_1 &= \int_0^1\left[\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right]f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)dt \\
&=\left(\frac{2}{y-x}\right)\left[\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right]f'\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)\Big|_0^1 \\
&+\left(\frac{2}{y-x}\right)\int_0^1\left((1-(1-t)^{\rho})^{\alpha}-\frac{1}{2}\right)f'\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)dt
\end{aligned}$$

$$= -\left(\frac{2}{y-x}\right)\left[\int_0^1 \left((1-(1-z)^\rho)^\alpha - \frac{1}{2}\right)dz\right] f'\left(a+b-\frac{x+y}{2}\right) + \left(\frac{2}{y-x}\right)^2$$

$$\times\left\{\left((1-(1-t)^\rho)^\alpha - \frac{1}{2}\right) f\left(\left(\frac{1-t}{2}\right)(a+b-y) + \left(\frac{1+t}{2}\right)(a+b-x)\right)\Big|_0^1\right.$$

$$\left. - \alpha\rho \int_0^1 (1-(1-t)^\rho)^{\alpha-1}(1-t)^{\rho-1} f\left(\left(\frac{1-t}{2}\right)(a+b-y) + \left(\frac{1+t}{2}\right)(a+b-x)\right)dt\right\}$$

$$= -\left(\frac{2}{y-x}\right)\left[\frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right) - \frac{1}{2}\right] f'\left(a+b-\frac{x+y}{2}\right) + \left(\frac{2}{y-x}\right)^2$$

$$\times\left\{\frac{1}{2} f(a+b-x) + \frac{1}{2} f\left(a+b-\frac{x+y}{2}\right) - \left(\frac{2}{y-x}\right)^{\rho\alpha} \rho^\alpha \Gamma(\alpha+1)\ {}^\rho\mathfrak{I}^\alpha_{(a+b-x)^-} f\left(a+b-\frac{x+y}{2}\right)\right\}.$$

Similarly, putting $\tau = \frac{1+t}{2}(a+b-y) + \frac{1-t}{2}(a+b-x)$, we get

$$\int_0^1 (1-(1-t)^\rho)^{\alpha-1}(1-t)^{\rho-1} f\left(\frac{1+t}{2}(a+b-y) + \frac{1-t}{2}(a+b-x)\right)dt$$

$$= \left(\frac{2}{y-x}\right)^{\rho\alpha} \int_{a+b-y}^{a+b-\frac{x+y}{2}} \left[\left(\left(a+b-\frac{x+y}{2}\right) - (a+b-y)\right)^\rho - (\tau-(a+b-y))^\rho\right]^{\alpha-1}$$

$$\times (\tau-(a+b-y))^{\rho-1} f(\tau)d\tau$$

$$= \left(\frac{2}{y-x}\right)^{\rho\alpha} \rho^{\alpha-1}\Gamma(\alpha)\ {}^\rho\mathfrak{I}^\alpha_{(a+b-y)^+} f\left(a+b-\frac{x+y}{2}\right).$$

Applying integration by parts yields

$$J_2 = \int_0^1 \left[\int_t^1 \left((1-(1-z)^\rho)^\alpha - \frac{1}{2}\right)dz\right] f''\left(\left(\frac{1+t}{2}\right)(a+b-y) + \left(\frac{1-t}{2}\right)(a+b-x)\right)dt$$

$$= -\left(\frac{2}{y-x}\right)\left[\int_t^1 \left((1-(1-z)^\rho)^\alpha - \frac{1}{2}\right)dz\right] f'\left(\left(\frac{1+t}{2}\right)(a+b-y) + \left(\frac{1-t}{2}\right)(a+b-x)\right)\Big|_0^1$$

$$-\left(\frac{2}{y-x}\right)\int_0^1 \left((1-(1-t)^\rho)^\alpha - \frac{1}{2}\right) f'\left(\left(\frac{1+t}{2}\right)(a+b-y) + \left(\frac{1-t}{2}\right)(a+b-x)\right)dt$$

$$= \left(\frac{2}{y-x}\right)\left[\int_0^1 \left((1-(1-z)^\rho)^\alpha - \frac{1}{2}\right)dz\right] f'\left(a+b-\frac{x+y}{2}\right) + \left(\frac{2}{y-x}\right)^2$$

$$\times\left\{\left((1-(1-t)^\rho)^\alpha - \frac{1}{2}\right) f\left(\left(\frac{1+t}{2}\right)(a+b-y) + \left(\frac{1-t}{2}\right)(a+b-x)\right)\Big|_0^1\right.$$

$$\left. -\alpha\rho \int_0^1 (1-(1-t)^\rho)^{\alpha-1}(1-t)^{\rho-1} f\left(\left(\frac{1+t}{2}\right)(a+b-y) + \left(\frac{1-t}{2}\right)(a+b-x)\right)dt\right\}$$

$$= \left(\frac{2}{y-x}\right)\left[\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)-\frac{1}{2}\right]f'\left(a+b-\frac{x+y}{2}\right)+\left(\frac{2}{y-x}\right)^{2}$$

$$\times\left\{\frac{1}{2}f(a+b-y)+\frac{1}{2}f\left(a+b-\frac{x+y}{2}\right)-\left(\frac{2}{y-x}\right)^{\rho\alpha}\rho^{\alpha}\Gamma(\alpha+1)\ {}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)\right\}.$$

Consequently, the following equality is valid:

$$\frac{(y-x)^2}{8}(J_1+J_2)= \frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]$$

$$-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right].$$

This concludes the proof. □

The first result for Bullen-Mercer inequality with conformable fractional integral operators is given below.

Theorem 2.2. *Assume that the assumptions of Lemma 8 hold and h is B-function. If $|f''|$ is a h-convex mapping on $[a,b]$, then the following Bullen-Mercer type inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$

$$\leq\frac{(y-x)^2}{4}h\left(\frac{1}{2}\right)\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt \tag{9}$$

$$\leq\frac{(y-x)^2}{4}h\left(\frac{1}{2}\right)\left\{4h\left(\frac{1}{2}\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt.$$

Proof. Let h be a B-function. Using the h-convexity of the function $|f''|$ and the absolute value of identity (8) to deduce

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$

$$\leq\frac{(y-x)^2}{8}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|$$

$$\times\left[\left|f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)\right|+\left|f''\left(\left(\frac{1+t}{2}\right)(a+b-y)+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right|\right]dt$$

$$\leq\frac{(y-x)^2}{8}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|$$

$$\times\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]dt.$$

By applying inequality (6) for $\lambda = \frac{1-t}{2}$, we result

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$
$$\leq \frac{(y-x)^2}{8}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|\left(2h\left(\frac{1}{2}\right)\right)\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]dt.$$

This accomplishes the first inequality in (9). Applying (4) and (6) yields to the second inequality in (9). □

By taking $x=a$ and $y=b$, we establish the following Bullen inequality.

Corollary 2.3. *Assume that the assumptions of Lemma 8 hold. If $|f''|$ is a h-convex mapping on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{a^{+}}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{b^{-}}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{(b-a)^2}{4}h\left(\frac{1}{2}\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt. \tag{10}$$

Remark 2.4. *Corollary 2.3 is a generalization of Theorem 1 in [12]. One can easily see it assuming $h(\lambda)=\lambda$.*

Next, consider some particular cases of Theorem 2.2 with h-convexity involving conformable fractional integral operators.

1. Given $h(t)=t^s$ with $s\in(0,1]$ in Theorem 2.2, we obtain the following Corollary.

   Corollary 2.5. *Assume α, ρ and f are defined according to Theorem 2.2. If $|f''|$ is a s-convex function on $[x,y]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$
$$\leq \frac{(y-x)^2}{4}\left(\frac{1}{2}\right)^{s}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt$$
$$\leq \frac{(y-x)^2}{4}\left(\frac{1}{2}\right)^{s}\left\{4\left(\frac{1}{2}\right)^{s}[\left|f''(a)\right|+\left|f''(b)\right|]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt. \tag{11}$$

   Remark 2.6. • *Taking $\rho=1$ in inequality (11), we get Bullen-Mercer inequality via Riemann-Liouville* *operators for s-convex function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\Im^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+\Im^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|\\
&\leq\frac{(y-x)^2}{4}\left(\frac{1}{2}\right)^{s}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|dt\\
&\leq\frac{(y-x)^2}{4}\left(\frac{1}{2}\right)^{s}\left\{4\left(\frac{1}{2}\right)^{s}[\left|f''(a)\right|+\left|f''(b)\right|]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|dt.
\end{aligned}\tag{12}$$

The above inequality (12) is a generalization of Corollary 1 in [12], simply by setting $s = 1$, $x = a$ and $y = b$.

- *Using $\alpha = 1$ in inequality (12) yields Bullen-Mercer inequality via Riemann integral for s-convex function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|\\
&\leq\frac{(y-x)^2}{48}\left(\frac{1}{2}\right)^{s}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\\
&\leq\frac{(y-x)^2}{48}\left(\frac{1}{2}\right)^{s}\left\{4\left(\frac{1}{2}\right)^{s}[\left|f''(a)\right|+\left|f''(b)\right|]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}.
\end{aligned}\tag{13}$$

The above inequality (13) is a generalization of [19, Propesition 4]. It suffices to set $s = 1$, $x = a$, and $y = b$.

2. Setting $h(\lambda) = 1$ in Theorem 2.2 yields the following new result for the class P-function. Take into account $s \to 0^+$ in the inequalities (11), (12), and (13).

Corollary 2.7. *Assume α, ρ and f are defined according to Theorem 2.2. If $|f''|$ is a P-function on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.\\
&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\Im^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\Im^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|\\
&\leq\frac{(y-x)^2}{4}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt\\
&\leq\frac{(y-x)^2}{4}\left\{4\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt.
\end{aligned}\tag{14}$$

Remark 2.8. *Using $\rho = 1$ in (14), we can produce the following Bullen-Mercer inequality using Riemann-*

Liouville operators, where $|f''|$ is a P-function.

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right. \\
&\left.-\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \\
&\leq\frac{(y-x)^2}{4}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|dt \\
&\leq\frac{(y-x)^2}{4}\left\{4\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|dt.
\end{aligned}\tag{15}$$

Remark 2.9. *By setting $\alpha = 1$ in (15), we use the Riemann integral to derive the following Bullen-Mercer inequality for the class P-function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right| \\
&\leq\frac{(y-x)^2}{48}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right] \\
&\leq\frac{(y-x)^2}{48}\left\{4\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}.
\end{aligned}\tag{16}$$

Remark 2.10. *Bullen inequalities for the class P-function can also be obtained by setting $x = a$ and $y = b$ in the preceding inequalities (14), (15), and (16).*

3. Putting $h(\lambda)=\frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$ in Theorem 2.2, we obtain the following new results for n-fractional polynomial convex functions.

Corollary 2.11. *Assume α, ρ and f are defined according to Theorem 2.2. If $|f''|$ is a n-fractional polynomial convex mapping on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right. \\
&\left.-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \\
&\leq\frac{(y-x)^2}{4n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt \\
&\leq\frac{(y-x)^2}{4n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left\{\frac{4}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|dt.
\end{aligned}\tag{17}$$

Remark 2.12. • *Taking $\rho = 1$ in inequality (17), we get Bullen-Mercer inequality via Riemann-Liouville operators for n-fractional polynomial convex function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\Im^{\alpha}_{(a+b-y)^+}f\left(a+b-\frac{x+y}{2}\right)+\Im^{\alpha}_{(a+b-x)^-}f\left(a+b-\frac{x+y}{2}\right)\right]\right|\\
&\le \frac{(y-x)^2}{4\,n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|dt\\
&\le \frac{(y-x)^2}{4\,n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left\{\frac{4}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}[\left|f''(a)\right|+\left|f''(b)\right|]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|dt.
\end{aligned} \tag{18}$$

The above inequality (18) is the second generalization of Corollary 1 in [12]. It suffices to set $n = 1$, $x = a$, and $y = b$.

• *Putting $\alpha = 1$ in inequality (18), we deduce Bullen-Mercer inequality via Riemann integral for n-fractional polynomial convex function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|\\
&\le \frac{(y-x)^2}{48\,n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left[\left|f''(a+b-y)\right|+\left|f''(a+b-x)\right|\right]\\
&\le \frac{(y-x)^2}{48\,n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left\{\frac{4}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}[\left|f''(a)\right|+\left|f''(b)\right|]-\left[\left|f''(x)\right|+\left|f''(y)\right|\right]\right\}.
\end{aligned} \tag{19}$$

Inequality (19) is a new generalization of [19, Propesition 4]. It suffices to put $n = 1$, $x = a$, and $y = b$.

Theorem 2.13. *Let h be a B-function on $(0, 1)$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that α, ρ, f are defined as in Lemma 2.1. If $\left|f''\right|^q$ is a h-convex mapping on $[a, b]$, the following Bullen-Mercer type inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.\\
&\left.-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\Im^{\alpha}_{(a+b-y)^+}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\Im^{\alpha}_{(a+b-x)^-}f\left(a+b-\frac{x+y}{2}\right)\right]\right|
\end{aligned} \tag{20}$$

$$\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}$$

$$\times\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}}$$

$$\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}$$

$$\times\left\{4\,h\left(\frac{1}{2}\right)[\left|f''(a)\right|^q+\left|f''(b)\right|^q]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}.$$

Proof. Using the absolute value of identity (8), we obtain

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\,\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{(a+b-y)^+}f\left(a+b-\frac{x+y}{2}\right)+{}^\rho\mathfrak{I}^\alpha_{(a+b-x)^-}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$

$$\leq \frac{(y-x)^2}{8}\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|\left|f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)\right|dt$$

$$+\frac{(y-x)^2}{8}\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|\left|f''\left(\left(\frac{1+t}{2}\right)(a+b-y)+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right|dt.$$

Applying Hölder inequality and $A^{\frac{1}{q}}+B^{\frac{1}{q}}\leq 2^{1-\frac{1}{q}}(A+B)^{\frac{1}{q}}$ gives

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\,\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{(a+b-y)^+}f\left(a+b-\frac{x+y}{2}\right)+{}^\rho\mathfrak{I}^\alpha_{(a+b-x)^-}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$

$$\leq \frac{(y-x)^2}{8}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left(\frac{1+t}{2}\right)(a+b-x)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(y-x)^2}{8}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}2^{1-\frac{1}{q}}\left[\int_0^1\left|f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\right.\right.\right.$$

$$\left.\left.\left.\left(\frac{1+t}{2}\right)(a+b-x)\right)\right|^q dt+\int_0^1\left|f''\left(\left(\frac{1-t}{2}\right)(a+b-y)+\left(\frac{1+t}{2}\right)(a+b-x)\right)\right|^q dt\right]^{\frac{1}{q}}.$$

Since $|f''|^q$ is a h-convex function, we conclude

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$
$$\leq\frac{(y-x)^2}{8}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}2^{\frac{1}{p}}\left[\int_0^1\left(h\left(\frac{1-t}{2}\right)\left|f''(a+b-y)\right|^q\right.\right.$$
$$\left.\left.+h\left(\frac{1+t}{2}\right)\left|f''(a+b-x)\right|^q\right)dt+\int_0^1\left(h\left(\frac{1+t}{2}\right)\left|f''(a+b-y)\right|^q+h\left(\frac{1-t}{2}\right)\left|f''(a+b-x)\right|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq\frac{(y-x)^2}{8}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}2^{\frac{1}{p}}$$
$$\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}}.$$

Applying (6) for $\lambda=\frac{1-t}{2}$, we result

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$
$$\leq\frac{(y-x)^2}{8}\left(2\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}2^{\frac{1}{p}}\left(2\,h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}$$
$$\times\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}}.$$

This results in the first inequality in (20). Applying (4) and (6) yields to the second inequality in (20). □

By taking $x=a$ and $y=b$, we establish the following Bullen inequality.

Corollary 2.14. *Assume that the assumptions of Lemma 8 hold. If $|f''|^q$ is a h-convex mapping on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{a^{+}}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{b^{-}}f\left(\frac{a+b}{2}\right)\right]\right| \tag{21}$$
$$\leq\frac{(b-a)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

Remark 2.15. *Corollary 2.14 is a generalization of Theorem 2 in [12]. One can easily see assuming $h(\lambda) = \lambda$.*

Next, examine some particular cases of Theorem 2.13 with h-convexity and conformable fractional integral operators.

1. By applying Theorem 2.13 to $h(t) = t^s$ with $s \in (0, 1]$, we get the following result.

Corollary 2.16. *Assume α, ρ and f are defined according to Theorem 2.13. If $|f''|^q$ is a s-convex function on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right. \\
&\left.-\frac{2^{\rho\alpha-1}\,\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \\
&\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{s}{q}} \\
&\times\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}} \\
&\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{s}{q}} \\
&\times\left\{4\left(\frac{1}{2}\right)^{s}[\left|f''(a)\right|^q+\left|f''(b)\right|^q]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}.
\end{aligned} \tag{22}$$

Remark 2.17. •

- *Taking $\rho = 1$ in inequality (22), we establish Bullen-Mercer inequality via Riemann-Liouville operators for s-convex function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right. \\
&\left.-\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \\
&\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{s}{q}}\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}} \qquad (23) \\
&\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left(z^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{s}{q}} \\
&\times\left\{4\left(\frac{1}{2}\right)^{s}[\left|f''(a)\right|^q+\left|f''(b)\right|^q]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}.
\end{aligned}$$

The above inequality (23) is a generalization of Corollary 2 in [12], simply by setting $s = 1$, $x = a$ and $y = b$.

- *Putting $\alpha = 1$ in inequality (23), we get*

$$\int_0^1 \left|t-t^2\right|^p dt = \int_0^1 (t-t^2)^p dt = \int_0^1 t^p(1-t)^p dt = \beta(p+1,p+1),$$

where $\beta(.\,,.)$ is the beta function. Therefore, we establish the next Bullen-Mercer inequality via Riemann integral for s-convex function .

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(t)dt\right| \\
&\le \frac{(y-x)^2}{8}\,(\beta(p+1,p+1))^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{s}{q}}\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}} \qquad (24)\\
&\le \frac{(y-x)^2}{8}\,(\beta(p+1,p+1))^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{s}{q}}\left\{4\left(\frac{1}{2}\right)^s[\left|f''(a)\right|^q+\left|f''(b)\right|^q]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}.
\end{aligned}$$

The above inequality (24) is a generalization of of Corollary 3 in [12]. It suffices to set $s = 1$, $x = a$, and $y = b$.

2. Setting $h(\lambda) = 1$ in Theorem 2.13 gives the following new result about the class P-function. Consider $s \to 0^+$ in the inequalities (22), (23) and (24).

Corollary 2.18. *Assume α, ρ and f are defined according to Theorem 2.13. If $|f''|^q$ is a P-function on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.\\
&\left.-\frac{2^{\rho\alpha-1}\,\rho^\alpha\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^+}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^-}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \qquad (25)\\
&\le \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}}\\
&\le \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^\rho)^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left\{4[\left|f''(a)\right|^q+\left|f''(b)\right|^q]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}.
\end{aligned}$$

Remark 2.19. *Taking $\rho = 1$, we derive the following Bullen-Mercer inequality via Riemann-Liouville operators, where f is a P-function.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{(a+b-y)^+}f\left(a+b-\frac{x+y}{2}\right)+\mathfrak{I}^{\alpha}_{(a+b-x)^-}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \qquad (26)\\
&\le \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left(z^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}}\\
&\le \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left(z^\alpha-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left\{4[\left|f''(a)\right|^q+\left|f''(b)\right|^q]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}.
\end{aligned}$$

Remark 2.20. *By putting $\rho = 1$ and $\alpha = 1$, we apply the Riemann integral to derive the following Bullen-Mercer inequality for the class P-function.*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right| \\ &\leq \frac{(y-x)^2}{8}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}} \\ &\leq \frac{(y-x)^2}{8}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left\{4\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}. \end{aligned} \tag{27}$$

3. Setting $h(\lambda) = \frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$ yields the following new result for the class n-fractional polynomial convex function.

Corollary 2.21. *Assume α, ρ and f are defined according to Theorem 2.2. If $|f''|^q$ is a n-fractional polynomial convex mapping on $[a, b]$, then*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right. \\ &\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-x)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right| \\ &\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}} \\ &\times\left[\left|f''(a+b-y)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}} \\ &\leq \frac{(y-x)^2}{4}\left(\int_0^1\left|\int_t^1\left((1-(1-z)^{\rho})^{\alpha}-\frac{1}{2}\right)dz\right|^p dt\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}} \\ &\times\left\{\left(\frac{4}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]-\left[\left|f''(x)\right|^q+\left|f''(y)\right|^q\right]\right\}^{\frac{1}{q}}. \end{aligned} \tag{28}$$

Remark 2.22. •

- *Taking $\rho = 1$ in inequality (28), we get Bullen-Mercer inequality via Riemann-Liouville operators for* *n-fractional polynomial convex function.*

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\Im^{\alpha}_{(a+b-y)^{+}}f\left(a+b-\frac{x+y}{2}\right)+\Im^{\alpha}_{(a+b-x)^{-}}f\left(a+b-\frac{x+y}{2}\right)\right]\right|$$
$$\leq\frac{(y-x)^{2}}{4}\left(\int_{0}^{1}\left|\int_{t}^{1}\left(z^{\alpha}-\frac{1}{2}\right)dz\right|^{p}dt\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}}\left[\left|f''(a+b-y)\right|^{q}+\left|f''(a+b-x)\right|^{q}\right]^{\frac{1}{q}} \tag{29}$$
$$\leq\frac{(y-x)^{2}}{4}\left(\int_{0}^{1}\left|\int_{t}^{1}\left(z^{\alpha}-\frac{1}{2}\right)dz\right|^{p}dt\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}}$$
$$\times\left\{\left(\frac{4}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)[\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}]-\left[\left|f''(x)\right|^{q}+\left|f''(y)\right|^{q}\right]\right\}^{\frac{1}{q}}.$$

- *Putting $\rho=1$ and $\alpha=1$ in inequality (28), we get Bullen-Mercer inequality via Riemann integral for s-convex function.*

$$\left|\frac{1}{2}\left[\frac{f(a+b-y)+f(a+b-x)}{2}+f\left(a+b-\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(t)dt\right|$$
$$\leq\frac{(y-x)^{2}}{8}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}}\left[\left|f''(a+b-y)\right|^{q}+\left|f''(a+b-x)\right|^{q}\right]^{\frac{1}{q}}$$
$$\leq\frac{(y-x)^{2}}{8}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{q}}\left\{\left(\frac{4}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)[\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}]-\left[\left|f''(x)\right|^{q}+\left|f''(y)\right|^{q}\right]\right\}^{\frac{1}{q}}. \tag{30}$$

References

[1] M. Abbasi, A. Morassaei, F. Mirzapour, *Jensen-Mercer Type Inequalities for Operator h-Convex Functions*, Bull. Iran. Math. Soc. **48** (2022), 2441–2462. https://doi.org/10.1007/s41980-021-00652-1 ,

[2] B. Benaissa, N. Azzouz, H. Budak, *Hermite-Hadamard type inequalities for new conditions on h-convex functions via ψ-Hilfer integral operators*, Anal. Math. Phys. **14**, 35 (2024). https://doi.org/10.1007/s13324-024-00893-3

[3] W.W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*, Publ. Inst. Math. **23** (1978) 13–20.

[4] P.S. Bullen, *Error Estimates for Some Elementary Quadrature Rules*, No. 602/633, University of Belgrade: Belgrade, Serbia, (1978), 97–103.

[5] M. Çakmak, *The differentiable h-convex functions involving the Bullen inequality*, Acta Univ. Apulensis Math. Inform. **65** (2021), 29–36.

[6] M. Çakmak, *On some Bullen-type inequalities via conformable fractional integrals*, J. Sci. Perspect. **3(4)** (2019), 285–298. https://doi.org/10.26900/jsp.3.030

[7] S.S. Dragomir, J. Pecaric, L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21(3)** (1995), 335–341.

[8] T. Du, C. Luo and Z. Cao,, *On the Bullen-type inequalities via generalized fractional integrals and their applications*, Fractals. **29(07)** (2021), 2150188.

[9] A. Fahad, S. I. Butt, B. Bayraktar, M. Anwar and Y. Wang, *Some new Bullen-type inequalities obtained via fractional integral operators*, Axioms. **12(7)** (2023), 691.

[10] J. Hadamard, *Etude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann*, J Math Pures Appl. **58** (1893), 171–215 (in French).

[11] F. Hezenci, H. Budak and H. Kara, *A study on conformable fractional version of Bullen-type inequalities*, Turkish J. Math. **47(4)** (2023), 1306–1317.

[12] F. Hezenci, H. Budak, *Bullen-type inequalities for twice-differentiable functions by using conformable fractional integrals*, J. Inequal. Appl. **2024(45)** (2024). https://doi.org/10.1186/s13660-024-03130-4

[13] H. Hsiow Ru, T. Kuei Lin, H. Kai Chen, *New inequalities for fractional integrals and their applications*, Turkish J. Math. **40(3)**, Article 1, (2016). https://doi.org/10.3906/mat-1411-61

[14] I. Isçan, *Construction of a new class of functions with their some properties and certain inequalities: n-fractional polynomial convex functions*, Miskolc Math. Notes. **24(3)** (2023),1389–1404. https://doi.org/10.18514/MMN.2023.4142

[15] A. McD. Mercer, *A variant of Jensen's inequality*, JIPAM. J. Inequal. Pure. App. Math. **4 (4)** (2003) Article 73. http://eudml.org/doc/123826

[16] D. S. Mitrinovic, J. E. Pecariç, A. M. Fink, *Classical and new inequalities in analysis*, Springer Dordrecht, 1993. https://doi.org/10.1007/978-94-017-1043-5

[17] C.E.M. Pearce, A.M. Rubinov, *P-functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl. **240** (1999), 92–104.

[18] M. Z. Sarikaya, *On the some generalization of inequalities associated with Bullen, Simpson, midpoint and trapezoid type*, Acta Univ. Apulensis Math. Inform. **73** (2023).

[19] M. Z. Sarikaya, N. Aktan, *On the generalization of some integral inequalities and their applications*, Math. Comput. Model. **54(9-10)** (2011), 2175–2182.

[20] S. Varosanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086.