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An optimal approximate solution of the I kind Fredholm singular integral equations

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Abstract. In this work, we study the problem of profile and lattice profile in aerodynamics. The first kind of Fredholm singular integral equation solves this problem. The Cauchy-type singular integrals are solutions to the first kind of Fredholm singular integral equation. However, the antiderivative function of these singular integrals can only be found in some cases. To overcome this, we utilize the Sobolev method to create an optimal quadrature formula for Cauchy-type singular integrals. By doing so, we can approximately solve the Fredholm integral equation of type I with higher accuracy. We compare the exact and approximate solutions of the first kind Fredholm singular integral equation, utilizing both the Sobolev method and another approach.

1. Introduction.

Problems are frequently solved using differential and integral equations in fields such as mechanics, hydrodynamics, aerodynamics, electrodynamics, quantum mechanics, and the theory of elasticity, as well as various other areas of technology. These equations are studied in the course of mathematical physics in science. It is well known that the main boundary value problems for the Laplace equation, namely the Dirichlet and Neumann problems, are solved using the potentials of simple and double layers. These boundary value problems lead to singular integral equations with Cauchy kernel, Hilbert kernel and singularity of a higher order. Analytical solutions of singular integral equations are represented by singular integrals [4, 17, 25]. The antiderivative function of the singular integral in the analytical solution can be found in some cases. Obtaining approximate methods for calculating such singular integrals with high accuracy is one of the important tasks of computational mathematics.

Various methods for approximating singular integral equations have been developed by numerous scientists. Notable researchers in this area include S.M. Belotserkovskii and I.K. Lifanov [4], I.K. Lifanov [22], I.V. Boykov [6], S.A. Dovgy [13], G.V. Milovanović, M.M. Spalević [24, 37], T.Hasegawa [18, 19], C.Dagnino [9, 10] and K.Diethelm [11, 12]. In the book [13], the boundary value problem is introduced

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to the singular integral equation and is applied practically through approximate calculations. The book addresses various task related to this topic. Below is one of the problem from the book.

1.1. Problem for a profile and a lattices of profiles

The flow of an inviscid, incompressible fluid around a profile moving at a speed equal to $\overline{U}_0 = u_{0x}\bar{i} + u_{0y}\bar{j}$ was studied assuming a plane-parallel flow. The profile is considered immobile.

Let's denote $\Phi = \Phi(x, y)$ – the potential of the disturbed flow velocities, and $\overline{V} = grad\Phi$ - the speed of this at a point outside the profile or on its surface. Let us represent the velocity potential in the form of a double layer potential along profile L with density $g(M)$

$$\Phi(x_0, y_0) = \frac{1}{2\pi} \int\limits_L \frac{(x - x_0)y'_s - (y - y_0)x'_s}{(x - x_0)^2 + (y - y_0)^2} g(M) \mathrm{d}s_M \tag{1}$$

for $M_0(x_0, y_0)$ points not lying on the L ($M_0 \notin L$), and in the form

$$\Phi^{\pm}(x_0, y_0) = \frac{1}{2\pi} \int\limits_L \frac{(x - x_0)y'_s - (y - y_0)x'_s}{(x - x_0)^2 + (y - y_0)^2} g(M) \mathrm{d}s_M \pm \frac{1}{2} g(M_0) \tag{2}$$

for M_0 points lying on the L ($M_0 \in L$).

In formulas (1), (2) and further in problems about profiles, we assume that L is specified parametrically.

Since the speed of a point in a potential flow is the gradient of the potential of the velocity field, i.e. $\overline{V} = \nabla\Phi$, then using the formula for the gradient from the potential of the double layer in the flat case [22, 29, 38] and denoting $g'_s(M)$ by $\gamma(M)$, we obtain that the speed of the $\overline{V}$ perturbed flow will now be written in the form

$$\overline{V} = \frac{1}{2\pi} \int\limits_L \frac{(y_0 - y)\bar{i} - (x_0 - x)\bar{j}}{r^2_{MM_0}} \gamma(M) \mathrm{d}s_m \tag{3}$$

for M_0 points not lying on the profile L, or in the form

$$\overline{V}^{\pm} = \frac{1}{2\pi} \int\limits_L \frac{(y_0 - y)\bar{i} - (x_0 - x)\bar{j}}{r^2_{MM_0}} \gamma(M) \mathrm{d}s_m \pm \frac{1}{2} \left(x'_{0s}\bar{i} + y'_{0s}\bar{j} \right) \gamma(M_0), \tag{4}$$

for M_0 points lying on the profile L. Since profile L is stationary, then

$$\overline{V}_{relatively} = \overline{V} + \overline{U}_0$$

and therefore, the problem of finding the field of perturbed velocities will be solved if $\gamma(M)$, $M \in L$, satisfies equality

$$V_{\overline{n}_{M_0}} = -U_0 \overline{n}_{M_0}, \ \ M_0 \in L, \tag{5}$$

or

$$\frac{1}{2\pi} \int\limits_L \frac{y'_{0s}(y_0 - y) + x'_{0s}(x_0 - x)}{r^2_{MM_0}} \gamma(M) \mathrm{d}s_m = U_0 \overline{n}_{M_0}), \ \ M_0 \in L. \tag{6}$$

Let first L be a smooth open curve, defined parametrically – $x = x(t)$, $y = y(t)$, $t \in [-1, 1]$, i.e. function $r'_M = \sqrt{x'^2(t) + y'^2(t)}$ is continuous on $[-1, 1]$ and does not vanish. In this case, equation (6) can be written as

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(t)}{t_0-t}\mathrm{d}t=f(t_0),\quad t_0\in(-1,1),\tag{7}$$

which is called the equation of a thin, slightly curved profile. (7) integral equation is Fredholm singular integral equation of the first kind. In this article, we will explain the problem we aim to solve in the following subsection.

1.2. Statement of the problem

In the work [22] is considered a singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(x)}{x-t}\mathrm{d}x=\varphi(t),\quad t\in(-1,1).\tag{8}$$

Let us recall the definition of singular integrals in the sense of principal value of Cauchy.

Definition 1.1. *The principal Cauchy value of the special integral* $\int_a^b \frac{\phi(x)}{x-t}dx,\ a<t<b$ *is the limit*

$$\lim_{\varepsilon\to 0}\left[\int_{a}^{t-\varepsilon}\frac{\phi(x)}{x-t}dx+\int_{t+\varepsilon}^{b}\frac{\phi(x)}{x-t}dx\right],$$

if it exists.

It is known that if a function ϕ on the interval $[a,b]$ satisfies the Hölder condition with exponent α $(0<\alpha\leq 1)$ and coefficient A, i.e. if

$$\left|\phi(x_1)-\phi(x_2)\right|\leq A|x_1-x_2|^{\alpha},$$

then there exists the integral $\int_a^b \frac{\phi(x)}{x-t}\mathrm{d}x,\ a<t<b.$

Equation (8) has four distinct solutions expressed as follows:

Case I: The function $\phi(t)$ is bounded within a certain range at the two endpoints of t, which are $t=-1$ and $t=1$, then a solution of (8) is

$$\phi_1(t)=-\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{\varphi(x)}{\sqrt{1-x^2}(x-t)}\mathrm{d}x,\tag{9}$$

provided that

$$\int_{-1}^{1}\frac{\varphi(x)}{\sqrt{1-x^2}}\mathrm{d}x=0.$$

Case II: The function $\phi(t)$ is bounded at the end $t=1$, but unbounded at the end $t=-1$, then a solution of (8) is

$$\phi_2(t)=-\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{\varphi(x)}{x-t}\mathrm{d}x,\tag{10}$$

Case III: The function $\phi(t)$ is bounded at the end $t = -1$, but unbounded at the end $t = 1$, then

$$\phi_3(t) = -\frac{1}{\pi}\sqrt{\frac{1+t}{1-t}}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{\varphi(x)}{x-t}\mathrm{d}x, \tag{11}$$

Case IV: The function $\phi(t)$ is unbounded at both the endpoints $t = \pm 1$,

$$\phi_4(t) = -\frac{1}{\pi\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}\mathrm{d}x + \frac{C}{\sqrt{1-t^2}}, \tag{12}$$

where

$$\int_{-1}^{1}\phi(t)\mathrm{d}x = C.$$

It should be noted that if we find an approximate solution to singular integral equation (8), we can derive an approximate solution to singular integral equation (7). For this purpose, the approximate calculation of singular integrals (9)-(12) is sufficient.

Several methods have been developed for the calculation of the singular integrals (9)-(12). They are: Discrete vortex method[13, 22], Interpolation methods [7, 8, 14, 15, 19, 24, 37], Piecewise interpolation method [9, 10, 12] and other numerical quadrature methods [11, 18]. The Discrete Vortex Method employs a singular point in the middle of successive nodes. Numerous scientific studies have focused on constructing quadrature formulas using Interpolation methods. These formulas use the roots of Chebyshev polynomials of the first kind as nodes. However, in several physical applications, singular integral equations need to be solved even when the function g is not very smooth or unknown. In such cases, Gaussian and transformation methods work efficiently only if the function g is smooth enough. The accuracy of numerical integration is significantly influenced by mesh selection. The Gaussian method's utility is constrained since it relies on approximating integrals at Gaussian points. Conversely, the Newton-Cotes method becomes impractical when the singular point is in close proximity to the quadrature formula nodes and even more so when it coincides with them. Scholars have underscored the paramount importance of mesh selection in guaranteeing the accuracy of numerical integration. During the initial phases, considerable emphasis was placed on mesh selection to position the singular point at the centre of a subinterval, highlighting its significant impact on the overall accuracy of the integration. In the reference [8], using the Lagrange interpolation polynomial a quadrature formula was constructed. It is known that the obtained formula is a classical quadrature formula. In our work, an optimal quadrature formula is constructed based on the methods of functional analysis in the Sobolev space $L_2^{(m)}$ of functions which are square integrable with m-th derivative. For existence of such type of quadrature formulas the condition $N+1 \geq m$ should be met, where $N+1$ is the number of nodes in the quadrature formula. It is known that in the case $N+1 = m$ one gets classical quadrature formulas. Hence it can be concluded that classical quadrature formulas can be derived using the condition $N+1 = m$ from optimal quadrature formulas which are constructed based on the methods of functional analysis in the Sobolev space. In the work [16], a quadrature formula was developed by utilizing a linear spline to estimate the singular integrals of the Cauchy type with weight function on the interval. This new quadrature formula is precise for a linear function, providing excellent convergence for any singular point $t \in (-1, 1)$ for all types of solutions of the SIEs (8). There exist better techniques for approximating numerical integration of Cauchy-type singular integrals than the method mentioned. In this paper, we present the construction of an optimal quadrature formula for the space $L_2^{(m)}(-1, 1)$. It is worth noting that our results align with the findings of the work [16] when $m = 1$.

We consider the following quadrature formula

$$\int_{-1}^{1} \frac{\omega_i(x)\varphi(x)}{x-t}\mathrm{d}x \cong \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{N} C_{\alpha,i}[\beta]\varphi^{(\alpha)}(x_\beta - 1), \ \ i = 1,2,3,4, \tag{13}$$

where

$$\omega_1(x) = \frac{1}{\sqrt{1-x^2}}, \ \ \omega_2(x) = \sqrt{\frac{1+x}{1-x}}, \ \ \omega_3(x) = \sqrt{\frac{1-x}{1+x}}, \ \ \omega_4(x) = \sqrt{1-x^2},$$

with the error functional

$$\ell_i(x) = \frac{\omega_i(x)\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{N}(-1)^\alpha C_{\alpha,i}[\beta]\delta^{(\alpha)}(x - x_\beta + 1), \ i = 1,2,3,4, \tag{14}$$

where $-1 < t < 1$, $C_{\alpha,i}[\beta]$ are the coefficients, $x_\beta - 1$ ($\in [-1,1]$) are the nodes, N is a natural number, $\varepsilon_{[-1,1]}(x)$ is the characteristic function of the interval $[-1,1]$, δ is the Dirac delta function, φ is a function of the space $L_2^{(m)}(-1,1)$. Here $L_2^{(m)}(-1,1)$ is the Sobolev space of functions with a square integrable mth generalized derivative and equipped with the norm

$$\|\varphi|L_2^{(m)}(-1,1)\| = \left\{\int_{-1}^{1}(\varphi^{(m)}(x))^2\mathrm{d}x\right\}^{1/2}$$

and $\int_{-1}^{1}(\varphi^{(m)}(x))^2\mathrm{d}x < \infty$.

Since the functional ℓ of the form (14) is defined on the space $L_2^{(m)}(-1,1)$ it is necessary to impose the following conditions (see [35])

$$(\ell_i, x^\alpha) = 0, \ \ \alpha = 0,1,2,...,m-1, \ \ i = 1,2,3,4. \tag{15}$$

Hence it is clear that for existence of the quadrature formulas of the form (13) the condition $N \geq m-1$ has to be met.

The difference

$$(\ell_i, \varphi) = \int_{-\infty}^{\infty} \ell_i(x)\varphi(x)\mathrm{d}x = \int_{0}^{1} \frac{\omega_i(x)\varphi(x)}{x-t}\mathrm{d}x - \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{N} C_{\alpha,i}[\beta]\varphi^{(\alpha)}(x_\beta) \tag{16}$$

is called *the error* of the formula (13).

By the Cauchy-Schwarz inequality

$$\left|(\ell_i, \varphi)\right| \leq \left\|\varphi|L_2^{(m)}\right\| \cdot \left\|\ell_i|L_2^{(m)*}\right\|$$

the error (16) of the formula (13) on functions of the space $L_2^{(m)}(-1,1)$ is estimated by the norm of the error functional ℓ_i in the conjugate space $L_2^{(m)*}(-1,1)$.

Obviously the norm of the error functional ℓ_i depends on the coefficients and the nodes of the quadrature formula (13). The problem of finding the minimum of the norm of the error functional ℓ_i by coefficients and by nodes is called *the S.M. Nikol'skii problem*, and the obtained formula is called *the optimal quadrature formula in the sense of Nikol'skii*. This problem was first considered by S.M. Nikol'skii [26], and continued by many authors, see e.g. [27] and references therein. Minimization of the norm of the error functional ℓ_i by coefficients when the nodes are fixed is called *Sard's problem* and the obtained formula is called *the optimal quadrature formula in the sense of Sard*. First this problem was investigated by A. Sard [28].

There are different ways to create optimal quadrature formulas according to Sard's definition. The spline method, ϕ-function method (see, e.g. [5, 21]), and Sobolev's method (see e.g.[20, 30, 31, 33–36]) are some of them. Sobolev's approach involves developing discrete versions of a linear differential operator.

The goal of my research is to develop accurate quadrature formulas in the sense of Sard [28]. Specifically, I am working on constructing formulas of the form (13) in the space $L_2^{(m)}(-1,1)$ using the Sobolev method. These formulas will be used for approximating integrals of the Cauchy type singular integral. This means to find the coefficients $C_{\alpha,i}[\beta]$ which attain the quantity

$$\|\mathring{\ell}_i|L_2^{(m)*}\| := \inf_{C_{\alpha,i}[\beta]} \|\ell_i|L_2^{(m)*}\|. \tag{17}$$

Consequently, to construct optimal quadrature formulas in the form (13) in the sense of Sard, we must solve the following problems sequentially.

Problem 1.2. *Find the norm of the error functional (14) of the quadrature formula (13) in the space* $L_2^{(m)*}(-1,1)$.

Problem 1.3. *Find the coefficients* $C_{\alpha,i}[\beta]$ *which give the minimum to the norm* $\|\ell_i\|$.

Numerous studies have addressed the problem of approximate integration of Cauchy-type singular integrals (see, for instance, [1–4, 13, 14, 22, 30–32] and references therein).

The rest of the paper is organized as follows.

In Section 2, we use the concept of extremal function to determine the norm of the error functional (14). Section 3 focuses on minimizing $\|\ell\|^2$ with respect to the coefficients $C_{\alpha,i}[\beta]$ in a successive manner. Section 4 provides definitions and known results that are used in proving the main results. In Section 5, we present the algorithm for constructing optimal quadrature formulas of the form (13). Finally, Section 6 contains numerical results that validate our theoretical findings.

2. An extremal function and the expression for the norm of the error functional

In order to solve Problem 1.2, which involves finding the norm of the error functional (14) in the space $L_2^{(m)}(-1,1)$, the concept of an extremal function is utilized [35, 36]. The function ψ_ℓ is referred to as the extremal function for the error functional (14) if the following equality holds:

$$(\ell, \psi_\ell) = \|\ell|L_2^{(m)*}\| \, \|\psi_\ell|L_2^{(m)*}\|. \tag{18}$$

In the space $L_2^{(m)}$ the extremal function ψ_ℓ of a functional ℓ is found by S.L. Sobolev [35, 36]. This extremal function has the form

$$\psi_\ell(x) = (-1)^m \ell(x) * G_m(x) + P_{m-1}(x), \tag{19}$$

where

$$G_m(x) = \frac{|x|^{2m-1}}{2 \cdot (2m-1)!} \tag{20}$$

is a solution of the equation

$$\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} G_m(x) = \delta(x), \tag{21}$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$, and $*$ is the operation of convolution and it is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y = \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y.$$

It is well known [35] that for any functional ℓ_i in $L_2^{(m)*}$ the equality

$$\|\ell_i|L_2^{(m)*}\|^2 = (\ell_i, \psi_\ell) = (\ell_i(x), (-1)^m \ell_i(x) * G_m(x)) = \int_{-\infty}^{\infty} \ell_i(x)\left((-1)^m \int_{-\infty}^{\infty} \ell_i(y) G_m(x-y)\mathrm{d}y\right)\mathrm{d}x$$

holds [35].

Applying this equality to the error functional (14) and taking into account (19) we obtain the following

$$\begin{aligned}\|\ell_i\|^2 \quad = \quad & (-1)^m\Big[\sum_{k=0}^{m-1}\sum_{\alpha=0}^{m-1}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k C_{k,i}[\gamma]C_{\alpha,i}[\beta]\frac{(h\beta-h\gamma)^{2m-1-\alpha-k}\mathrm{sgn}(h\beta-h\gamma)}{2(2m-1-\alpha-k)!} \\ & -2\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{N}(-1)^\alpha C_{\alpha,i}[\beta]\int_0^1 \frac{\omega_i(x)(x-h\beta)^{2m-1-\alpha}\mathrm{sgn}(x-h\beta)}{2(2m-1-\alpha)!(x-t)}\mathrm{d}x \\ & +\int_0^1\int_0^1 \frac{\omega_i(x)\omega_i(y)(x-y)^{2m-1}\mathrm{sgn}(x-y)}{2(2m-1)!(x-t)(y-t)}\mathrm{d}x\mathrm{d}y\Big],\end{aligned} \tag{22}$$

where $\mathrm{sgn}x$ is the signum function.

Thus Problem 1.2 is solved for quadrature formulas of the form (13) in the space $L_2^{(m)}(-1,1)$.

Further we consider Problem 1.3.

3. Minimization of the norm of the error functional $\ell(x)$

Now we consider the minimization problem of the norm (22) of the error functional ℓ under the conditions (15).

It should be noted that minimization of $\|\ell_i\|^2$ by $C_{\alpha,i}[\beta]$, $\alpha = 0,1,2,...,m-1$, $\beta = 0,1,2,...,N$, $i = 1,2,3,4$ is very hard. Here we suggest successive minimization of $\|\ell_i\|^2$ by $C_{\alpha,i}[\beta]$, i.e. first we consider the case $m = 1$ and the expression (22) of $\|\ell_i\|^2$ we minimize by $C_{0,i}[\beta]$. Further we consider the case $m = 2$, and using the obtained values for $C_{0,i}[\beta]$, the expression (22) of $\|\ell_i\|^2$ we minimize by $C_{1,i}[\beta]$. After that in the case $m = 3$, using the obtained values of $C_{0,i}[\beta]$ and $C_{1,i}[\beta]$, the expression (22) for $\|\ell_i\|^2$ we minimize by $C_{2,i}[\beta]$ and so on.

First we consider *the case* $m = 1$ then the quadrature formula (13) has the form

$$\int_0^1 \frac{\omega_i(x)\varphi(x)}{x-t}\mathrm{d}x \cong \sum_{\beta=0}^{N} C_{0,i}[\beta]\varphi(h\beta) \tag{23}$$

and $\|\ell_i\|^2$ depends only on $C_{0,i}[\beta]$ ($\beta = \overline{0,N}$).

Here we use conditional extremum of a function of several variables

$$\Phi_{1,i}(\mathbf{C_{0,i}}, \lambda_{0,i}) = \|\ell_i\|^2_{L_2^{(1)*}(-1,1)} + 2\lambda_{0,i}(\ell_i, 1),$$

where $\|\ell_i\|^2$ is defined by (22) and $\mathbf{C_{0,i}} = (C_{0,i}[0], C_{0,i}[1], ..., C_{0,i}[N])$.

Equating to zero partial derivatives of $\Phi_{1,i}(\mathbf{C_{0,i}}, \lambda_{0,i})$ by $C_{0,i}[\beta]$ and $\lambda_{0,i}$ we get the following system of linear equations

$$\sum_{\gamma=0}^{N} \mathring{C}_{0,i}[\gamma]\frac{(h\beta-h\gamma)\ \mathrm{sgn}(h\beta-h\gamma)}{2} + \lambda_{0,i} = F_{0,i}(h\beta), \tag{24}$$

$\beta = 0,1,...,N,$

$$\sum_{\gamma=0}^{N} \mathring{C}_{0,i}[\gamma] = g_{0,i}, \tag{25}$$

where

$$F_{0,i}(h\beta) = y_{0,i}(h\beta) = \int_{-1}^{1} \frac{\omega_i(x)(x-(h\beta-1))\mathtt{sign}(x-(h\beta-1))}{2(x-t)}\mathrm{d}x$$

$$g_{0,i} = \int_{-1}^{1} \frac{\omega_i(x)}{x-t}\mathrm{d}x, \quad i=1,2,3,4,$$

here

$$y_{0,1}(h\beta) = -\mathtt{arcsin}(h\beta-1) + \frac{t-(h\beta-1)}{\sqrt{1-t^2}} ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|, \tag{26}$$

$$y_{0,2}(h\beta) = \sqrt{1-(h\beta-1)^2} - \Big(2+t-h\beta\Big)\mathtt{arcsin}(h\beta-1) \tag{27}$$
$$-(t-(h\beta-1))\sqrt{\frac{1+t}{1-t}} ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|,$$

$$y_{0,3}(h\beta) = -\sqrt{1-(h\beta-1)^2} + \Big(t-h\beta\Big)\mathtt{arcsin}(h\beta-1) \tag{28}$$
$$-(t-(h\beta-1))\sqrt{\frac{1-t}{1+t}} ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{t-h\beta+1}\right|,$$

$$y_{0,4}(h\beta) = \left(\frac{h\beta-1}{2}-t\right)\sqrt{1-(h\beta-1)^2} + \left(t^2-t(h\beta-1)-\frac{1}{2}\right)\mathtt{arcsin}(h\beta-1) \tag{29}$$
$$+(t-(h\beta-1))\sqrt{1-t^2} ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|,$$

$$g_{0,1} = \int_{-1}^{1} \frac{\omega_1(x)}{x-t}\mathrm{d}x = 0, \tag{30}$$

$$g_{0,2} = \int_{-1}^{1} \frac{\omega_2(x)}{x-t}\mathrm{d}x = \pi, \tag{31}$$

$$g_{0,3} = \int_{-1}^{1} \frac{\omega_3(x)}{x-t}\mathrm{d}x = -\pi \tag{32}$$

$$g_{0,4} = \int_{-1}^{1} \frac{\omega_4(x)}{x-t}\mathrm{d}x = -\pi t. \tag{33}$$

Further, we consider *the case* $m = 2$. In this case the quadrature formula (13) takes the form

$$\int_{-1}^{1} \frac{\omega_i(x)\varphi(x)}{x-t}\mathrm{d}x \cong \sum_{\beta=0}^{N}\Big(C_{0,i}[\beta]\varphi(h\beta) + C_{1,i}[\beta]\varphi'(h\beta)\Big) \tag{34}$$

and expression (22) of $\|\ell_i\|^2$ depends on $C_{0,i}[\beta]$ and $C_{1,i}[\beta]$. Then using the solution $C_{0,i}[\beta]$ and $\lambda_{0,i}$ of the system (24)-(25), equating to zero partial derivatives of the following function by $C_{1,i}[\beta]$ and $\lambda_{1,i}$

$$\Phi_{2,i}(\mathbf{C_{1,i}},\lambda_{1,i}) = \|\ell_i\|^2_{L_2^{(2)*}(-1,1)} - 2\lambda_{1,i}(\ell_i, x),$$

where $\|\ell_i\|^2$ is defined by (22) and $\mathbf{C_{1,i}} = (C_{1,i}[0], C_{1,i}[1], ..., C_{1,i}[N])$, we get

$$\sum_{\gamma=0}^{N} \mathring{C}_{1,i}[\gamma]\frac{(h\beta - h\gamma)\ \mathtt{sgn}(h\beta - h\gamma)}{2} - \lambda_{1,i} = F_{1,i}(h\beta), \tag{35}$$

$\beta = 0, 1, ..., N,$

$$\sum_{\gamma=0}^{N}\left(\mathring{C}_{0,i}[\gamma](h\gamma - 1) + \mathring{C}_{1,i}[\gamma]\right) = g_{1,i}, \tag{36}$$

where

$$\begin{aligned} F_{1,i}(h\beta) &= -y_{1,i}(h\beta) + \sum_{\gamma=0}^{N}\mathring{C}_{0,i}[\gamma]\frac{(h\beta - h\gamma)^2\ \mathtt{sgn}(h\beta - h\gamma)}{4}, \\ y_{1,i}(h\beta) &= -\int_{-1}^{1}\frac{\omega_i(x)(x - (h\beta - 1))^2\mathtt{sgn}(x - (h\beta - 1))}{4(x - t)}\mathrm{d}x, \\ g_{1,i} &= \int_{-1}^{1}\frac{\omega_i(x)x}{x - t}\mathrm{d}x, \quad i = 1, 2, 3, 4, \end{aligned}$$

here

$$\begin{aligned} y_{1,1}(h\beta) &= -\frac{1}{2}\Big[- \sqrt{1 - (h\beta - 1)^2} + \Big(t - 2h\beta + 2\Big)\mathtt{arcsin}(h\beta - 1) - \\ &\quad - \frac{(t - (h\beta - 1))^2}{\sqrt{1 - t^2}} ln\left|\frac{1 - t(h\beta - 1) + \sqrt{(1 - t^2)(1 - (h\beta - 1)^2)}}{h\beta - 1 - t}\right|\Big], \end{aligned} \tag{37}$$

$$\begin{aligned} y_{1,2}(h\beta) &= -\frac{1}{2}\Big[\Big(2(h\beta - 1) - \frac{1}{2}(h\beta + 2t + 1)\Big)\sqrt{1 - (h\beta - 1)^2} + \\ &\quad + \Big(\frac{1}{2}(2t^2 + 2t + 1) - 2(h\beta - 1)(1 + t) + (h\beta - 1)^2\Big)\mathtt{arcsin}(h\beta - 1) + \\ &\quad + (t - (h\beta - 1))^2\sqrt{\frac{1 + t}{1 - t}} ln\left|\frac{1 - t(h\beta - 1) - \sqrt{(1 - t^2)(1 - (h\beta - 1)^2)}}{h\beta - 1 - t}\right|\Big], \end{aligned} \tag{38}$$

$$\begin{aligned} y_{1,3}(h\beta) &= \frac{1}{4}\Big[\Big(3h\beta - 2t - 1\Big)\sqrt{1 - (h\beta - 1)^2} + \Big(2(h\beta - t)^2 + 2t - 1\Big)\mathtt{arcsin}(h\beta - 1) \\ &\quad - 2(t - (h\beta - 1))^2\sqrt{\frac{1 - t}{1 + t}} ln\left|\frac{1 - t(h\beta - 1) - \sqrt{(1 - t^2)(1 - (h\beta - 1)^2)}}{t - h\beta + 1}\right|\Big], \end{aligned} \tag{39}$$

$$\begin{aligned} y_{1,4}(h\beta) &= -\frac{1}{2}\Big[\frac{1}{6}\Big(2(h\beta)^2 - h\beta(4 + 9t) + 9t + 6t^2\Big)\sqrt{1 - (h\beta - 1)^2} \\ &\quad + \Big(\frac{1}{2}(t - 2(h\beta - 1)) - t(t - (h\beta - 1))^2\Big)\mathtt{arcsin}(h\beta - 1) - \\ &\quad - (t - (h\beta - 1))^2\sqrt{1 - t^2} ln\left|\frac{1 - t(h\beta - 1) + \sqrt{(1 - t^2)(1 - (h\beta - 1)^2)}}{h\beta - 1 - t}\right|\Big], \end{aligned} \tag{40}$$

and

$$g_{1,1} = \int_{-1}^{1} \frac{\omega_1(x)x}{x-t}\mathrm{d}x = \pi, \tag{41}$$

$$g_{1,2} = \int_{-1}^{1} \frac{\omega_2(x)x}{x-t}\mathrm{d}x = \pi(1+t), \tag{42}$$

$$g_{1,3} = \int_{-1}^{1} \frac{\omega_3(x)x}{x-t}\mathrm{d}x = \pi(1-t) \tag{43}$$

$$g_{1,4} = \int_{-1}^{1} \frac{\omega_4(x)x}{x-t}\mathrm{d}x = \frac{\pi}{2}(1-2t^2). \tag{44}$$

In *the case* $m = 3$ the quadrature formula (13) has the form

$$\int_{-1}^{1} \frac{\omega_i(x)\varphi(x)}{x-t}\mathrm{d}x \cong \sum_{\beta=0}^{N}\Big(C_{0,i}[\beta]\varphi(h\beta) + C_{1,i}[\beta]\varphi'(h\beta) + C_{2,i}[\beta]\varphi''(h\beta)\Big) \tag{45}$$

and $\|\ell_i\|^2$, defined by equality (22), depends on $C_{0,i}[\beta]$, $C_{1,i}[\beta]$ and $C_{2,i}[\beta]$. Then using solutions $C_{0,i}[\beta]$ and $\lambda_{0,i}$ of system (24)-(25) and $C_{1,i}[\beta]$, $\lambda_{1,i}$ of system (35)-(36), equating to zero partial derivatives of

$$\Phi_{3,i}(\mathbf{C_{2,i}}, \lambda_{2,i}) = \|\ell_i\|^2_{L_2^{(3)*}(-1,1)} + 2\lambda_{2,i}(\ell_i, x^2),$$

by $C_{2,i}[\beta]$ and $\lambda_{2,i}$ we have, where $\|\ell_i\|^2$ is defined by (22) and $\mathbf{C_{2,i}} = (C_{2,i}[0], C_{2,i}[1], ..., C_{2,i}[N])$.

$$\sum_{\gamma=0}^{N} \mathring{C}_{2,i}[\gamma]\frac{(h\beta-h\gamma)\ \mathtt{sgn}(h\beta-h\gamma)}{2} + \lambda_{2,i} = F_{2,i}(h\beta), \tag{46}$$

$\beta = 0, 1, ..., N,$

$$\sum_{\gamma=0}^{N}\Big(\mathring{C}_{0,i}[\gamma](h\gamma-1)^2 + 2\mathring{C}_{1,i}[\gamma](h\gamma-1) + 2C_{2,i}[\gamma]\Big) = g_{2,i}, \tag{47}$$

where

$$F_{2,i}(h\beta) = y_{2,i}(h\beta) - \sum_{\gamma=0}^{N}\mathring{C}_{0,i}[\gamma]\frac{(h\beta-h\gamma)^3\ \mathtt{sgn}(h\beta-h\gamma)}{12} + \sum_{\gamma=0}^{N}\mathring{C}_{1,i}[\gamma]\frac{(h\beta-h\gamma)^2\ \mathtt{sgn}(h\beta-h\gamma)}{4},$$

$$y_{2,i}(h\beta) = \int_{-1}^{1}\frac{\omega_i(x)(x-(h\beta-1))^3\ \mathtt{sgn}(x-(h\beta-1))}{12(x-t)}\mathrm{d}x,$$

$$g_{2,i} = \int_{-1}^{1}\frac{\omega_i(x)x^2}{x-t}\mathrm{d}x, \quad i = 1, 2, 3, 4,$$

here

$$y_{2,1}(h\beta) = \frac{1}{12}\Bigg[(2t-5(h\beta-1))\sqrt{1-(h\beta-1)^2} - \Big(1+2t^2-6t(h\beta-1)+6(h\beta-1)^2\Big)\mathtt{arcsin}(h\beta-1) - + \frac{2(t-(h\beta-1))^3}{\sqrt{1-t^2}} ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg], \tag{48}$$

$$y_{2,2}(h\beta) = \frac{1}{12}\Bigg[\Big(-2(t-h\beta+1)^3 - 6(h\beta-1)^2 + 3(h\beta-1)(2t+1) - 2t^2 - t - 1\Big)\mathtt{arcsin}(h\beta-1) \tag{49}$$

$$+\Big(\frac{11}{3}(h\beta-1)^2-5(h\beta-1)(t+1)+\frac{1}{3}(6t^2+6t+4)\Big)\sqrt{1-(h\beta-1)^2}$$
$$-\frac{2(t-(h\beta-1))^3\sqrt{1+t}}{\sqrt{1-t}}ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Big],$$

$$y_{2,3}(h\beta) = \frac{1}{12}\Big[\Big(-2(h\beta-1)^3+6(h\beta-1)^2(t-1)-3(h\beta-1)(2t^2-2t+1)+2t^3-2t^2+t-1\Big)\arcsin(h\beta-1) \tag{50}$$
$$+\Big(-\frac{11}{3}(h\beta-1)^2+5(h\beta-1)(t-1)-\frac{1}{3}(6t^2-6t+4)\Big)\sqrt{1-(h\beta-1)^2}$$
$$-2(t-h\beta+1)^3\sqrt{\frac{1-t}{1+t}}ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{t-h\beta+1}\right|\Big],$$

$$y_{2,4}(h\beta) = \frac{1}{12}\Big[\Big(\frac{1}{2}(h\beta-1)^3-\frac{11}{3}(h\beta-1)^2t+5(h\beta-1)t^2-2t^3-\frac{7}{4}(h\beta-1)+\frac{2}{3}t\Big)\sqrt{1-(h\beta-1)^2} \tag{51}$$
$$+\Big(-\frac{1}{4}(-8t^4+4t^2+1)+3(h\beta-1)(t-2t^3)-3(h\beta-1)^2(1-2t^2)-2t(h\beta-1)^3\Big)\arcsin(h\beta-1)$$
$$+2(t-(h\beta-1))^3\times\sqrt{1-t^2}ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Big],$$

and

$$g_{2,1} = \int_{-1}^{1}\frac{\omega_1(x)x^2}{x-t}\mathrm{d}x=\pi t, \tag{52}$$

$$g_{2,2} = \int_{-1}^{1}\frac{\omega_2(x)x^2}{x-t}\mathrm{d}x=\frac{\pi}{2}(2t^2+2t+1), \tag{53}$$

$$g_{2,3} = \int_{-1}^{1}\frac{\omega_3(x)x^2}{x-t}\mathrm{d}x=\frac{\pi}{2}(-2t^2+2t-1) \tag{54}$$

$$g_{2,4} = \int_{-1}^{1}\frac{\omega_4(x)x^2}{x-t}\mathrm{d}x=\frac{\pi}{2}(t-2t^3). \tag{55}$$

Suppose, continuing by this way, for the cases $m=1,2,...,k-1$ we have found the coefficients $C_{0,i}[\beta], C_{1,i}[\beta], ..., C_{k-2,i}[\beta]$ and $\lambda_{0,i}, \lambda_{1,i}, ..., \lambda_{k-2,i}$. We consider *the case* $m=k$. Then square of the norm (22) of the error functional ℓ_i of quadrature formulas (13) depends on $C_{0,i}[\beta], C_{1,i}[\beta], ..., C_{k-2,i}[\beta]$ and $C_{k-1,i}[\beta]$. Further using the obtained solutions $C_{0,i}[\beta], C_{1,i}[\beta], ..., C_{k-2,i}[\beta]$ and $\lambda_{0,i}, \lambda_{1,i}, ..., \lambda_{k-2,i}$ of corresponding systems, equating to zero partial derivatives of the function

$$\Phi_{k,i}(\mathbf{C_{k-1,i}}, \lambda_{k-1,i}) = \|\ell_i\|^2_{L_2^{(m)*}(-1,1)} - 2(-1)^k\lambda_{k-1,i}(\ell_i, x^{k-1}),$$

by $C_{k-1,i}[\beta]$ and $\lambda_{k-1,i}$, here $\|\ell_i\|^2$ is defined by (22) and
$\mathbf{C_{k-1,i}} = (C_{k-1,i}[0], C_{k-1,i}[1], ..., C_{k-1,i}[N])$, we arrive to the following system of linear equations

$$\sum_{\gamma=0}^{N}\mathring{C}_{k,i}[\gamma]\frac{(h\beta-h\gamma)\mathrm{sgn}(h\beta-h\gamma)}{2}+(-1)^{k-1}(k-1)!\lambda_{k-1,i}=F_{k-1,i}[\beta], \tag{56}$$
$\beta=0,1,...,N,$
$$\sum_{\gamma=0}^{N}\mathring{C}_{k-1,i}[\gamma]=\frac{g_{k-1,i}}{(k-1)!}-\sum_{j=0}^{k-2}\sum_{\gamma=0}^{N}\mathring{C}_{j,i}[\gamma]\frac{(h\gamma)^{k-j-1}}{(k-j-1)!}. \tag{57}$$

Here

$$F_{k-1,i}[\beta] = y_{k-1,i}[\beta] - \sum_{j=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^{j+k-1}\mathring{C}_{j,i}[\gamma]\frac{(h\beta-h\gamma)^{k-j}\mathrm{sgn}(h\beta-h\gamma)}{2(k-j)!}, \tag{58}$$

$$\begin{aligned} y_{k-1,i}[\beta] &= \int_{-1}^{1}\frac{\omega_i(x)(x-h\beta+1)^k\mathrm{sgn}(x-h\beta+1)}{2(k)!(x-t)}\mathrm{d}x,\\ g_{k-1,i} &= \int_{-1}^{1}\frac{\omega_i(x)x^{k-1}}{x-t}\mathrm{d}x, \end{aligned}$$

$$\begin{aligned} y_{k-1,1}(h\beta) &= \int_{-1}^{1}\frac{(x-h\beta+1)^k\mathrm{sgn}(x-h\beta+1)}{2\cdot k!\sqrt{1-x^2}(x-t)}\mathrm{d}x = -\frac{1}{k!}\Bigg[\sum_{\alpha=1}^{k}\binom{k}{\alpha}(t-h\beta+1)^{k-\alpha}(A_1+A_2) - \frac{(t-h\beta+1)^k}{\sqrt{1-t^2}}A_3\Bigg], \qquad (59)\\ A_1 &= \sum_{j=1}^{\left[\frac{\alpha-1}{2}\right]}\binom{\alpha-1}{2j}(-t)^{\alpha-2j-1}\Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{2j}\Bigg[(h\beta-1)^{2j-1} + \sum_{l=1}^{j-1}\frac{(2j-1)(2j-3)\ldots(2j-2l+1)}{2^l(j-1)(j-2)\ldots(j-l)}(h\beta-1)^{2j-2l-1}\Bigg]\\ &\quad + \frac{(2j-1)!!}{2^j j!}\arcsin(h\beta-1)\Bigg),\\ A_2 &= \sum_{j=0}^{\left[\frac{\alpha-2}{2}\right]}\binom{\alpha-1}{2j+1}\sum_{l=0}^{j}\frac{(-t)^{\alpha-2j-2}(-1)^{l+1}}{(2l+1)}\binom{j}{l}\Big(\sqrt{1-(h\beta-1)^2}\Big)^{2l+1} + (-t)^{i-1}\arcsin(h\beta-1),\\ A_3 &= \ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|, \end{aligned}$$

$$\begin{aligned} y_{k-1,2}(h\beta) &= \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{(x-h\beta+1)^k\mathrm{sgn}(x-h\beta+1)}{2\cdot k!(x-t)}\mathrm{d}x = -\frac{1}{k!}\Bigg[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}\Bigg(\sum_{n=1}^{i-1}\binom{i-1}{n}(-1)^n \qquad (60)\\ &\quad \times(1-t)^{i-1-n}(B_1+B_2) + (1-t)^{i-1}\Big(-\sqrt{1-(h\beta-1)^2}+\arcsin(h\beta-1)\Big)\Bigg) + (t-h\beta+1)^k\Bigg(\sqrt{\frac{1+t}{1-t}}B_3+\arcsin(h\beta-1)\Bigg)\Bigg],\\ B_1 &= \sum_{j=1}^{\left[\frac{n-1}{2}\right]}\binom{n-1}{2j}\Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{4j(j+1)}\Bigg[-2j(h\beta-1)^{2j+1}+(h\beta-1)^{2j-1} + \sum_{l=1}^{j-1}\prod_{p=1}^{l}\frac{2j-2p+1}{2(j-p)}(h\beta-1)^{2j-2l-1}\Bigg]\\ &\quad + \frac{1}{2(j+1)}\prod_{p=0}^{j-1}\frac{2j-2p-1}{2(j-p)}\arcsin(h\beta-1)\Bigg),\\ B_2 &= \frac{h\beta-1}{2}\sqrt{1-(h\beta-1)^2} + \frac{1}{2}\arcsin(h\beta-1) - \sum_{j=1}^{\left[\frac{n}{2}\right]}\binom{n-1}{2j-1}\sum_{p=0}^{j-1}\binom{j-1}{p}\frac{(-1)^{p+1}}{(2p+3)}\Big(\sqrt{1-(h\beta-1)^2}\Big)^{2p+3},\\ B_3 &= \ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|, \end{aligned}$$

$$y_{k-1,3}(h\beta) = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{(x-h\beta+1)^k\mathrm{sgn}(x-h\beta+1)}{2\cdot k!(x-t)}\mathrm{d}x = -\frac{1}{k!}\Bigg[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}\Bigg[\sum_{n=1}^{i-1}\binom{i-1}{n}(-t-1)^{i-n-1}(T_1+T_2)$$

$$+(-t-1)^{i-1}\left(\sqrt{1-(h\beta-1)^2}+\arcsin(h\beta-1)\right)\Bigg]+(t-h\beta+1)^k T_3\Bigg], \tag{61}$$

$$T_1 = \sum_{j=1}^{[\frac{n-1}{2}]}\binom{n-1}{2j}\Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{4j(j+1)}\Big[-2j(h\beta-1)^{2j+1}+(h\beta-1)^{2j-1}+\sum_{l=1}^{j-1}\frac{(2j-1)(2j-3)...(2j-2l+1)}{2^l(j-1)(j-2)...(j-l)}(h\beta-1)^{2j-2l-1}\Big] + \frac{(2j-1)!!}{2^j j!}\arcsin(h\beta-1)\Bigg),$$

$$T_2 = \sum_{j=1}^{[\frac{n}{2}]}\binom{n-1}{2j-1}\sum_{p=0}^{j-1}\binom{j-1}{p}(-1)^{p+1}\frac{\left(\sqrt{1-(h\beta-1)^2}\right)^{2p+3}}{2p+3}+\frac{h\beta-1}{2}\sqrt{1-(h\beta-1)^2}+\frac{1}{2}\arcsin(h\beta-1),$$

$$T_3 = \sqrt{\frac{1-t}{1+t}}\ln\left|\frac{1-t(h\beta-1)-\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|-\arcsin(h\beta-1),$$

$$y_{k-1,4}(h\beta) = \int_{-1}^{1}\frac{\sqrt{1-x^2}(x-h\beta+1)^k\operatorname{sgn}(x-h\beta+1)}{2\cdot k!\cdot(x-t)}dx = -\frac{1}{k!}\Bigg[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}(Q_1+Q_2)+(t-h\beta+1)^k \times\left(\sqrt{1-(h\beta-1)^2}-t\arcsin(h\beta-1)-\sqrt{1-t^2}Q_3\right)\Bigg], \tag{62}$$

$$Q_1 = \sum_{j=1}^{[\frac{i-1}{2}]}\binom{i-1}{2j}(-t)^{i-2j-1}\Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{4j(j+1)}\Big[-2j(h\beta-1)^{2j+1}+(h\beta-1)^{2j-1}+\sum_{l=1}^{j-1}\prod_{p=1}^{l}\frac{(2j-2p+1)}{2(j-p)}(h\beta-1)^{2j-2l-1}\Big] + \frac{1}{2(j+1)}\prod_{p=0}^{j-1}\frac{(2j-2p-1)}{2(j-p)}\arcsin(h\beta-1)\Bigg)$$

$$Q_2 = \sum_{j=1}^{[\frac{i}{2}]}\binom{i-1}{2j-1}(-t)^{i-2j}\sum_{p=0}^{j-1}\binom{j}{p}(-1)^{p+1}\frac{\left(\sqrt{1-(h\beta-1)^2}\right)^{2p+3}}{2p+3}+(-t)^{i-1}\left(\frac{h\beta-1}{2}\sqrt{1-(h\beta-1)^2}+\frac{1}{2}\arcsin(h\beta-1)\right),$$

$$Q_3 = \ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|,$$

and

$$g_{k-1,1} = \int_{-1}^{1}\frac{x^{k-1}}{\sqrt{1-x^2}(x-t)}dx \tag{63}$$

$$= \pi\sum_{i=1}^{k-1}\binom{k-1}{i}t^{k-1-i}\left(\sum_{j=1}^{[\frac{i-1}{2}]}\binom{i-1}{2j}(-t)^{i-1-j}\frac{(2j-1)!!}{2^j j!}+(-t)^{i-1}\right),$$

$$g_{k-1,2} = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{x^{k-1}}{x-t}dx = \sum_{i=1}^{k-1}\binom{k-1}{i}t^{k-1-i}\Bigg(\sum_{j=1}^{i-1}\binom{i-1}{j}(-1)^j(1-t)^{i-1-j} \times\left[\pi\sum_{n=1}^{[\frac{j-1}{2}]}\binom{i-1}{2n}\frac{1}{2(n+1)}\prod_{p=0}^{n-1}\frac{(2n-2p-1)}{2(n-p)}+\frac{\pi}{2}\right]+(1-t)^{i-1}\pi\Bigg)+t^{k-1}\pi, \tag{64}$$

$$g_{k-1,3} = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{x^{k-1}}{x-t}dx = \sum_{i=1}^{k-1}\binom{k-1}{i}\Bigg[\sum_{p=1}^{i-1}\binom{i-1}{p}(-1-t)^{i-p-1} \tag{65}$$

$$\times\left(\frac{\pi}{2}\sum_{n=1}^{\left[\frac{p-1}{2}\right]}\binom{p-1}{2n}\frac{1}{n+1}\prod_{j=0}^{n-1}\frac{2n-2j-1}{2(n-j)}+\frac{\pi}{2}\right)+(-1-t)^{i-1}\pi\Bigg]-t^{k-1}\pi.$$

$$g_{k-1,4} = \int_{-1}^{1}\frac{x^{k-1}\sqrt{1-x^2}}{(x-t)}\mathrm{d}x=\pi\sum_{i=1}^{k-1}\binom{k-1}{i}t^{k-1-i} \tag{66}$$

$$\times\left(\sum_{j=1}^{\left[\frac{i-1}{2}\right]}\binom{i-1}{2j}(-t)^{i-1-2j}\frac{1}{2(j+1)}\prod_{p=0}^{j-1}\frac{(2j-2p+1)}{2(j-p)}+\frac{(-t)^{i-1}}{2}\right)-\pi\cdot t^k.$$

Further, we solve systems (24)-(25), (35)-(36), (46)-(47) and (56)-(57), i.e. we find optimal coefficients of quadrature formulas of the form (23), (34), (45) and (13).

4. Preliminaries

In this section we give some definitions and formulas that we need to prove the main results. Here we use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [35, 36]. For completeness we give some definitions.

Assume that the nodes x_β are equally spaced, i.e., $x_\beta = h\beta$, $h=\frac{1}{N}$, $N = 1,2,...$, functions $\varphi(x)$ and $\psi(x)$ are real-valued and defined on the real line $\mathbb{R}$.

Definition 4.1. *The function $\varphi(h\beta)$ is a function of discrete argument if it is given on some set of integer values of β.*

Definition 4.2. *The inner product of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by*

$$[\varphi(h\beta),\psi(h\beta)]=\sum_{\beta=-\infty}^{\infty}\varphi(h\beta)\cdot\psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. *The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product*

$$\varphi(h\beta)*\psi(h\beta)=[\varphi(h\gamma),\psi(h\beta-h\gamma)]=\sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\beta-h\gamma).$$

In addition,in the calculations we need a discrete analogue $D_1(h\beta)$ of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ which is defined by the following formula (see [34])

$$D_1(h\beta)=\begin{cases}0, & |\beta|\geq 2,\\ h^{-2}, & |\beta|=1,\\ -2h^{-2}, & \beta=0.\end{cases} \tag{67}$$

Here are some properties of the discrete function $D_1(h\beta)$ (see [35]):

$$D_1(h\beta)*1=0,\quad D_1(h\beta)*(h\beta)=0, \tag{68}$$

$$hD_1(h\beta)*\frac{|h\beta|}{2}=\delta_d(h\beta). \tag{69}$$

where $\delta_d(h\beta)$ is the discrete delta-function.

5. Finding optimal coefficients of the quadrature formula of the form (13)

Now we solve the system (56)-(57), which depends on m. Then we substitute instead of m the values $m = 1, 2, 3$ and it provide coefficients for quadrature formulas of (23), (34) and (45). The following theorem is valid.

Theorem 5.1. *The optimal coefficients for the quadrature formulas of the form (13) in the Sobolev space $L_2^{(m)}(-1, 1)$ are defined as follows*

$$\mathring{C}_{k-1,i}[0] = h^{-1}\Big[F_{k-1,i}[1] - F_{k-1,i}[0] + \frac{h}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right)\Big], \quad (70)$$

$$\mathring{C}_{k-1,i}[\beta] = h^{-1}\Big[F_{k-1,i}[\beta-1] - 2F_{k-1,i}[\beta] + F_{k-1,i}[\beta+1]\Big], \quad (71)$$

for $\beta = 1, ..., N-1$

$$\mathring{C}_{k-1,i}[N] = h^{-1}\Big[F_{k-1,i}[N-1] - F_{k-1,i}[N] + \frac{h}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right)\Big], \quad (72)$$

$k = 0, 1, 2, ..., m-1$, *where* $\nu_{k-2,i} = \sum\limits_{\alpha=0}^{k-2}\sum\limits_{\gamma=0}^{N} C_{\alpha,i}[\gamma]\frac{(h\gamma)^{k-\alpha-1}}{(k-\alpha-1)!}$, $F_{k-1,i}$ *and* $g_{k-1,i}$ *are defined by (58) and (59).*

Proof. Now we solve the system (56)-(57). This solution we find by the following way.
We denote

$$u_i(h\beta) = \sum_{\gamma=0}^{N} \mathring{C}_{k-1,i}[\beta]\frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} + (-1)^{k-1}(k-1)!\lambda_{k-1,i}.$$

Assume $\beta \le 0$, then using (57) we have

$$u_i(h\beta) = -\frac{h\beta}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right) - \mu_{k-1,i} + (-1)^{k-1}(k-1)!\lambda_{k-1,i}.$$

Now suppose that $\beta \ge N$, then taking into account (57), we get

$$u_i(h\beta) = \frac{h\beta}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right) + \mu_{k-1,i} + (-1)^{k-1}(k-1)!\lambda_{k-1,i}.$$

Here

$$\nu_{k-2,i} = \sum_{\alpha=0}^{k-2}\sum_{\gamma=0}^{N} C_{\alpha,i}[\gamma]\frac{(h\gamma)^{k-\alpha-1}}{(k-\alpha-1)!},$$

$$\mu_{k-1,i} = -\frac{1}{2}\sum_{\gamma=0}^{N} C_{k-1,i}[\gamma](h\gamma).$$

We denote

$$a_{k-1,i}^{-} = \mu_{k-1,i} - (k-1)!(-1)^{k-1}\lambda_{k-1,i},$$

$$a_{k-1,i}^{+} = \mu_{k-1,i} + (k-1)!(-1)^{k-1}\lambda_{k-1,i}.$$

Then we obtain that

$$u_i(h\beta) = \begin{cases} -\frac{h\beta}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right) - a_{k-1,i}^{-}, & \beta \le 0, \\ F_{k-1,i}[\beta], & 0 \le \beta \le N, \\ \frac{h\beta}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right) + a_{k-1,i}^{+}, & \beta \ge N, \end{cases} \quad (73)$$

where $a_{k-1,i}^{-}, a_{k-1,i}^{+}$ are unknowns.
Hence, taking into account the values of the function $u_i(h\beta)$ at the points $\beta = 0$ and $\beta = N$, we get

$$a_{k-1,i}^{-} = F_{k-1,i}[0],$$

$$a^{+}_{k-1,i} = F_{k-1,i}[N] - \frac{1}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right).$$

Using properties (69) of the operator $D_1(h\beta)$ from (67) and (73), for the optimal coefficients $\mathring{C}_{k-1,i}[\beta]$ when $0 \le \beta \le N$ we get the following

$$\mathring{C}_{k-1,i}[\beta] = D_1[\beta] * u_i(h\beta)$$

or

$$\mathring{C}_{k-1,i}[\beta] = h\sum_{\gamma=-\infty}^{\infty} D_1[\beta-\gamma]u_i(h\gamma) = h\Bigg[\sum_{\gamma=0}^{N} D_1[\beta-\gamma]F_{k-1,i}[\gamma] + \sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]\Big(\frac{h\gamma}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right) - a^{-}_{k-1,i}\Big)$$
$$+\sum_{\gamma=1}^{\infty} D_1[N+\gamma-\beta]\Big(\frac{1+h\gamma}{2}\left(\frac{g_{k-1,i}}{(k-1)!} - \nu_{k-2,i}\right) + a^{+}_{k-1,i}\Big)\Bigg].$$

Hence, using (67), (68) and formula (69), taking into account (58) and (59), after some calculations, we arrive at the expressions for the coefficients $C_{k-1,i}[\beta], \beta = 0,1,2,...,N$ which are given in the statement of Theorem 5.1. Theorem 5.1 is proved.

From Theorem 5.1, we get the following results by recalculating the cases when $m = 1,2$ and $m = 3$. These results are convenient in use.

Corollary 5.2. *In the space $L_2^{(1)}(-1,1)$ when $\omega_1(x) = \frac{1}{\sqrt{1-x^2}}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (23) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{0,1}[0] &= h^{-1}\Big(y_{0,1}[1] - \tfrac{\pi}{2}\Big),\\
\overset{\circ}{C}_{0,1}[\beta] &= h^{-1}\Big(y_{0,1}[\beta-1] - 2y_{0,1}[\beta] + y_{0,1}[\beta+1]\Big),\\
&\quad \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{0,1}[N] &= h^{-1}\Big(y_{0,1}[N-1] + \tfrac{\pi}{2}\Big),
\end{aligned}$$

where $y_{0,1}$ is defined by (26).

Corollary 5.3. *In the space $L_2^{(1)}(-1,1)$ when $\omega_2(x) = \sqrt{\frac{1+x}{1-x}}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (23) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{0,2}[0] &= h^{-1}\Big(y_{0,2}[1] - \tfrac{\pi}{2}(2+t-h)\Big),\\
\overset{\circ}{C}_{0,2}[\beta] &= h^{-1}\Big(y_{0,2}[\beta-1] - 2y_{0,2}[\beta] + y_{0,2}[\beta+1]\Big),\\
&\quad \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{0,2}[N] &= h^{-1}\Big(y_{0,2}[N-1] + \tfrac{\pi}{2}(t+h)\Big),
\end{aligned}$$

where $y_{0,2}$ is defined by (27).

Corollary 5.4. *In the space $L_2^{(1)}(-1,1)$ when $\omega_3(x) = \sqrt{\frac{1-x}{1+x}}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (23) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{0,3}[0] &= h^{-1}\Big(y_{0,3}[1] + \tfrac{\pi}{2}(t-h)\Big),\\
\overset{\circ}{C}_{0,3}[\beta] &= h^{-1}\Big(y_{0,3}[\beta-1] - 2y_{0,3}[\beta] + y_{0,3}[\beta+1]\Big),\\
&\quad \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{0,3}[N] &= h^{-1}\Big(y_{0,3}[N-1] + \tfrac{\pi}{2}(2-t-h)\Big),
\end{aligned}$$

where $y_{0,3}$ is defined by (28).

Corollary 5.5. *In the space $L_2^{(1)}(-1,1)$ when $\omega_4(x) = \sqrt{1-x^2}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (23) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{0,4}[0] &= h^{-1}\Big(y_{0,4}[1] + \tfrac{\pi}{4}(2t^2+2t-1) - \tfrac{h}{2}\pi t\Big),\\
\overset{\circ}{C}_{0,4}[\beta] &= h^{-1}\Big(y_{0,4}[\beta-1] - 2y_{0,4}[\beta] + y_{0,4}[\beta+1]\Big),\\
&\quad \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{0,4}[N] &= h^{-1}\Big(y_{0,4}[N-1] - \tfrac{\pi}{4}(2t^2-2t-1) - \tfrac{h}{2}\pi t\Big),
\end{aligned}$$

where $y_{0,4}$ is defined by (29).

Corollary 5.6. *In the space $L_2^{(2)}(-1,1)$ when $\omega_1(x) = \frac{1}{\sqrt{1-x^2}}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (34) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{1,1}[0] &= h^{-1}\Big(y_{1,1}[1] + \tfrac{h}{2}y_{0,1}[1] + \tfrac{h}{4}\pi - \tfrac{\pi}{4}(t+2)\Big),\\
\overset{\circ}{C}_{1,1}[\beta] &= h^{-1}\Big(y_{1,1}[\beta-1] - 2y_{1,1}[\beta] + y_{1,1}[\beta+1] - \tfrac{h}{2}\Big(y_{0,1}[\beta-1] - 2y_{0,1}[\beta] + y_{0,1}[\beta+1]\Big) + h^2\sum_{\gamma=0}^{\beta} C_{0,1}[\gamma]\Big),\\
&\quad \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{1,1}[N] &= h^{-1}\Big(y_{1,1}[N-1] - \tfrac{h}{2}y_{0,1}[N-1] + \tfrac{h}{4}\pi + \tfrac{\pi}{4}(t-2)\Big),
\end{aligned}$$

where $y_{0,1}$,$y_{1,1}$ are defined by (26),(37) respectively.

Corollary 5.7. *In the space $L_2^{(2)}(-1,1)$ when $\omega_2(x) = \sqrt{\frac{1+x}{1-x}}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (34) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{1,2}[0] &= h^{-1}\Big(y_{1,2}[1] + \tfrac{h}{2}y_{0,2}[1] + \tfrac{h}{4}\pi(t+2) - \tfrac{\pi}{4}(t^2+3t+\tfrac{7}{2})\Big),\\
\overset{\circ}{C}_{1,2}[\beta] &= h^{-1}\Big(y_{1,2}[\beta-1] - 2y_{1,2}[\beta] + y_{1,2}[\beta+1] - \tfrac{h}{2}\Big(y_{0,2}[\beta-1] - 2y_{0,2}[\beta] +\\
&\quad + y_{0,2}[\beta+1]\Big) + h^2\sum_{\gamma=0}^{\beta} C_{0,2}[\gamma] - \tfrac{h^2}{2}\pi\Big),\ \ \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{1,2}[N] &= h^{-1}\Big(y_{1,2}[N-1] - \tfrac{h}{2}y_{0,2}[N-1] + \tfrac{h}{4}\pi t + \tfrac{\pi}{4}(t^2 - t - \tfrac{1}{2})\Big),
\end{aligned}$$

where $y_{0,2}$, $y_{1,2}$ are defined by (27) (38) respectively.

Corollary 5.8. *In the space $L_2^{(2)}(-1,1)$ when $\omega_3(x) = \sqrt{\frac{1-x}{1+x}}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (34) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{1,3}[0] &= h^{-1}\Big(y_{1,3}[1] + \tfrac{h}{2}y_{0,3}[1] - \tfrac{h}{4}\pi t + \tfrac{\pi}{8}(2t^2+2t-1)\Big),\\
\overset{\circ}{C}_{1,3}[\beta] &= h^{-1}\Big(y_{1,3}[\beta-1] - 2y_{1,3}[\beta] + y_{1,3}[\beta+1] - \tfrac{h}{2}\Big(y_{0,3}[\beta-1] - 2y_{0,3}[\beta] +\\
&\quad + y_{0,3}[\beta+1]\Big) + h^2\sum_{\gamma=0}^{\beta} C_{0,3}[\gamma] + \tfrac{h^2}{2}\pi\Big),\ \ \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{1,3}[N] &= h^{-1}\Big(y_{1,3}[N-1] - \tfrac{h}{2}y_{0,3}[N-1] + \tfrac{h}{4}\pi(2-t) - \tfrac{\pi}{8}(2(2-t)^2+2t-1)\Big),
\end{aligned}$$

where $y_{0,3}$, $y_{1,3}$ are defined by (28) (39) respectively.

Corollary 5.9. *In the space $L_2^{(2)}(-1,1)$ when $\omega_4(x)=\sqrt{1-x^2}$, $t\neq h\beta-1$, $\beta=0,1,...,N$, the coefficients of optimal quadrature formulas of the form (34) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{1,4}[0] &= h^{-1}\Big(y_{1,4}[1]+\tfrac{h}{2}y_{0,4}[1]-\tfrac{h}{8}\pi(2t^2+2t-1)+\tfrac{\pi}{8}(2t^3+4t^2+t-2)\Big),\\
\overset{\circ}{C}_{1,4}[\beta] &= h^{-1}\Big(y_{1,4}[\beta-1]-2y_{1,4}[\beta]+y_{1,4}[\beta+1]-2\Big(y_{0,4}[\beta-1]-2y_{0,4}[\beta]+\\
&\quad +y_{0,4}[\beta+1]\Big)+h^2\sum_{\gamma=0}^{\beta}C_{0,4}[\gamma]+\tfrac{h^2}{2}\pi t\Big),\ \ \beta=1,2,...,N-1,\\
\overset{\circ}{C}_{1,4}[N] &= h^{-1}\Big(y_{1,4}[N-1]-\tfrac{h}{2}y_{0,4}[N-1]-\tfrac{h}{8}\pi(2t^2-2t-1)-\tfrac{\pi}{8}(2t^3-4t^2+t+2)\Big),
\end{aligned}$$

where $y_{0,4}$, $y_{1,4}$ are defined by (29), (40) respectively.

Corollary 5.10. *In the space $L_2^{(3)}(-1,1)$ when $\omega_1(x)=\frac{1}{\sqrt{1-x^2}}$, $t\neq h\beta-1$, $\beta=0,1,...,N$, the coefficients of optimal quadrature formulas of the form (45) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{2,1}[0] &= h^{-1}\Big(y_{2,1}[1]+\tfrac{h}{2}y_{1,1}[1]+\tfrac{h^2}{12}y_{0,1}[1]-\tfrac{h^2}{24}\pi+\tfrac{h}{8}\pi(t+2)-\tfrac{\pi}{24}(2t^2+6t+7)\Big),\\
\overset{\circ}{C}_{2,1}[\beta] &= h^{-1}\Big[y_{2,1}[\beta-1]-2y_{2,1}[\beta]+y_{2,1}[\beta+1]-\tfrac{h}{2}\Big(y_{1,1}[\beta-1]-2y_{1,1}[\beta]+\\
&\quad +y_{1,1}[\beta+1]\Big)+\tfrac{h^2}{12}\Big(y_{0,1}[\beta-1]-2y_{0,1}[\beta]+y_{0,1}[\beta+1]\Big)+\\
&\quad \tfrac{h^2}{2}\Big(2\sum_{\gamma=0}^{\beta}C_{1,1}[\gamma]-\pi\Big)+\tfrac{h^3}{2}\sum_{\gamma=0}^{\beta}C_{0,1}[\gamma](2\gamma-2\beta-1)\Big],\ \ \beta=1,2,...,N-1,\\
\overset{\circ}{C}_{2,1}[N] &= h^{-1}\Big[y_{2,1}[N-1]-\tfrac{h}{2}y_{1,1}[N-1]+\tfrac{h^2}{12}y_{0,1}[N-1]+\tfrac{h^2}{24}\pi+\tfrac{h}{8}\pi(t-2)+\tfrac{\pi}{24}(2t^2-6t+7)\Big],
\end{aligned}$$

where $y_{0,1}$, $y_{1,1}$ and $y_{2,1}$ are defined by (26),(37) and (48) respectively.

Corollary 5.11. *In the space $L_2^{(3)}(-1,1)$ when $\omega_2(x)=\sqrt{\frac{1+x}{1-x}}$, $t\neq h\beta-1$, $\beta=0,1,...,N$, the coefficients of optimal quadrature formulas of the form (45)are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{2,2}[0] &= h^{-1}\Big[y_{2,2}[1]+\tfrac{h}{2}y_{1,2}[1]+\tfrac{h^2}{12}y_{0,2}[1]-\tfrac{h^2}{24}\pi(t+2)+\tfrac{h}{8}\pi(t^2+3t+\tfrac{7}{2})-\tfrac{\pi}{24}(2t^3+8t^2+13t+12)\Big],\\
\overset{\circ}{C}_{2,2}[\beta] &= h^{-1}\Big[y_{2,2}[\beta-1]-2y_{2,2}[\beta]+y_{2,2}[\beta+1]-\tfrac{h}{2}\Big(y_{1,2}[\beta-1]-2y_{1,2}[\beta]+y_{1,2}[\beta+1]\Big)+\\
&\quad +\tfrac{h^2}{12}\Big(y_{0,2}[\beta-1]-2y_{0,2}[\beta]+y_{0,2}[\beta+1]\Big)+\tfrac{h^2}{2}\Big(2\sum_{\gamma=0}^{\beta}C_{1,2}[\gamma]-\pi(2+t)\Big)\\
&\quad +\tfrac{h^3}{4}\Big(2\sum_{\gamma=0}^{\beta}C_{0,2}[\gamma](2\gamma-2\beta-1)+\pi(1+2\beta)\Big)\Big],\ \ \beta=1,2,...,N-1,\\
\overset{\circ}{C}_{2,2}[N] &= h^{-1}\Big[y_{2,2}[N-1]-\tfrac{h}{2}y_{1,2}[N-1]+\tfrac{h^2}{12}y_{0,2}[N-1]+\tfrac{h^2}{24}\pi t+\tfrac{h}{8}\pi(t^2-t-\tfrac{1}{2})+\tfrac{\pi}{24}(2t^3-4t^2+t+2)\Big],
\end{aligned}$$

where $y_{0,2}$, $y_{1,2}$ and $y_{2,2}$ are defined by (27) (38) and (49) respectively.

Corollary 5.12. *In the space $L_2^{(3)}(-1,1)$ when $\omega_3(x)=\sqrt{\frac{1-x}{1+x}}$, $t\neq h\beta-1$, $\beta=0,1,...,N$, the coefficients of optimal quadrature formulas of the form (45) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{2,3}[0] &= h^{-1}\Big[y_{2,3}[1]+\tfrac{h}{2}y_{1,3}[1]+\tfrac{h^2}{12}y_{0,3}[1]+\tfrac{h^2}{24}\pi t-\tfrac{h}{16}\pi(2t^2+2t-1)+\tfrac{\pi}{24}(2t^3+4t^2+t-2)\Big],\\
\overset{\circ}{C}_{2,3}[\beta] &= h^{-1}\Big[y_{2,3}[\beta-1]-2y_{2,3}[\beta]+y_{2,3}[\beta+1]-\tfrac{h}{2}\Big(y_{1,3}[\beta-1]-2y_{1,3}[\beta]+y_{1,3}[\beta+1]\Big)+\\
&\quad +\tfrac{h^2}{12}\Big(y_{0,3}[\beta-1]-2y_{0,3}([\beta]-1,t)+y_{0,3}([\beta+1]-1,t)\Big)+\tfrac{h^2}{2}\Big(2\sum_{\gamma=0}^{\beta}C_{1,3}[\gamma]+\pi t\Big)\\
&\quad +\tfrac{h^3}{4}\Big(2\sum_{\gamma=0}^{\beta}C_{0,3}[\gamma](2\gamma-2\beta-1)-\pi(2\beta+1)\Big)\Big],\ \ \beta=1,2,...,N-1,\\
\overset{\circ}{C}_{2,3}[N] &= h^{-1}\Big[y_{2,3}[N-1]-\tfrac{h}{2}y_{1,3}[N-1]+\tfrac{h^2}{12}y_{0,3}[N-1]+\tfrac{h^2}{24}\pi(2-t)-\tfrac{h}{16}\pi(2t^2-6t+7)-\tfrac{\pi}{24}(2t^3-8t^2+13t-12)\Big],
\end{aligned}$$

where $y_{0,3}$, $y_{1,3}$ and $y_{2,3}$ are defined by (28) (39) and (50)respectively.

Corollary 5.13. *In the space $L_2^{(3)}(-1,1)$ when $\omega_4(x) = \sqrt{1-x^2}$, $t \neq h\beta - 1$, $\beta = 0,1,...,N$, the coefficients of optimal quadrature formulas of the form (45) are defined as follows*

$$\begin{aligned}
\overset{\circ}{C}_{2,4}[0] &= h^{-1}\Big[y_{2,4}[1] + \tfrac{h}{2}y_{1,4}[1] + \tfrac{h^2}{12}y_{0,4}[1] + \tfrac{h^2}{48}\pi(2t^2+2t-1) - \tfrac{h}{16}\pi(2t^3+4t^2+t-2) + \tfrac{\pi}{24}(2t^4+6t^3+5t^2-t-\tfrac{13}{4})\Big],\\
\overset{\circ}{C}_{2,4}[\beta] &= h^{-1}\Big[y_{2,4}[\beta-1] - 2y_{2,4}[\beta] + y_{2,4}[\beta+1] - \tfrac{h}{2}\Big(y_{1,4}[\beta-1] - 2y_{1,4}[\beta] + y_{1,4}[\beta+1]\Big)\\
&\quad + \tfrac{h^2}{12}\Big(y_{0,4}[\beta-1] - 2y_{0,4}[\beta] + y_{0,4}[\beta+1]\Big) + \tfrac{h^2}{4}\Big(\pi(2t^2+2t-1) + 4\sum_{\gamma=0}^{\beta} C_{1,4}[\gamma]\Big)\\
&\quad + \tfrac{h^3}{4}\Big(-\pi t(1+2\beta) + \sum_{\gamma=0}^{\beta} C_{0,4}[\gamma](4\gamma-4\beta-2)\Big)\Big],\ \ \beta = 1,2,...,N-1,\\
\overset{\circ}{C}_{2,4}[N] &= h^{-1}\Big[y_{2,4}[N-1] - \tfrac{h}{2}y_{1,4}[N-1] + \tfrac{h^2}{12}y_{0,4}[N-1] - \tfrac{h^2}{48}\pi(2t^2-2t-1) - \tfrac{h}{16}\pi(2t^3-4t^2+t+2)\\
&\quad - \tfrac{\pi}{24}(t^4-6t^3+5t^2+t-\tfrac{13}{4})\Big],
\end{aligned}$$

where $y_{0,4}$, $y_{1,4}$ and $y_{2,4}$ are defined by (29), (40) and (51), respectively.

In the next section, we will obtain an approximate solution to equation (7). Equation (8) is derived from equation (7) by multiplying it by 2 in this work. Since the solution to equation (8) has been approximated in many scientific studies, we will also use the optimal quadrature formula to approximate the solution to equation (8).

6. Numerical results

In this section, previous work [16] found the approximate roots of equation (8) when the right side was $\varphi(t) = t^5 + 5t^3 + 20$ and $\varphi(t) = \frac{10t+2}{1+t^2}$. To compare with their findings, we also find the approximate solution of equation (8) when the right side is $\varphi(t) = t^5 + 5t^3 + 20$ and $\varphi(t) = \frac{10t+2}{1+t^2}$ using the optimal quadrature formula. Dividing the resulting approximate solution by 2, we obtained the approximate solution of equation (7). This approximate solution will serve as an estimate for the problem in section 1.1 in this work.

The formula for determining the error between the exact solution of the singular integral equation (8) and its approximate value obtained using the optimal quadrature formula form (13) is given below

$$R_{N,m-1}(\phi_i) = \phi_i(t) - \frac{1}{\pi\omega_i(t)}\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{N} C_{\alpha,i}[\beta]\phi^{(\alpha)}(x_\beta),\ i = 1,2,3,4. \tag{74}$$

Example 1. Let us consider a singular integral equation of the form (8) and with the right-hand side $\varphi(t) = t^5+5t^3+20$, we have

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(x)}{x-t}\mathrm{d}x = \varphi(t),\ \ t \in (-1,1). \tag{75}$$

Here are the solutions for the integral equation, along with their corresponding cases

$$\begin{aligned}
CaseI: &\quad \phi_1(x) = \sqrt{1-x^2}\Big(x^4 + 5\frac{1}{2}x^2 + 2\frac{7}{8}\Big),\\
CaseII: &\quad \phi_2(x) = \sqrt{1-x^2}\Big(x^4 + 5\frac{1}{2}x^2 + 2\frac{7}{8}\Big) + 20\sqrt{\frac{1-x}{1+x}},\\
CaseIII: &\quad \phi_3(x) = \sqrt{1-x^2}\Big(x^4 + 5\frac{1}{2}x^2 + 2\frac{7}{8}\Big) - 20\sqrt{\frac{1+x}{1-x}},\\
CaseIV: &\quad \phi_4(x) = \sqrt{1-x^2}\Big(x^4 + 5\frac{1}{2}x^2 + 2\frac{7}{8}\Big) - \frac{5(4x+7/16)}{\sqrt{1-x^2}}.
\end{aligned}$$

We use our optimal quadrature formula to approximate the solutions of the integral equation (75), utilizing Corollaries 5.2-5.13.

Example 2. Let φ in (8) be the following rational functions $\varphi(t) = \frac{10t+2}{1+t^2}$.

Here are the exact solutions for the integral equation of (8), along with their corresponding cases

$$\begin{aligned}
CaseI: &\quad \phi_1(x) = \sqrt{2}(5-x)\frac{\sqrt{1-x^2}}{1+x^2},\\
CaseII: &\quad \phi_2(x) = \sqrt{2}(5-x)\frac{\sqrt{1-x^2}}{1+x^2} + \sqrt{2}\sqrt{\frac{1-x}{1+x}},\\
CaseIII: &\quad \phi_3(x) = \sqrt{2}(5-x)\frac{\sqrt{1-x^2}}{1+x^2} - \sqrt{2}\sqrt{\frac{1+x}{1-x}},\\
CaseIV: &\quad \phi_4(x) = \sqrt{2}(5-x)\frac{\sqrt{1-x^2}}{1+x^2} - \frac{5(2-\sqrt{2})+\sqrt{2}x}{\sqrt{1-x^2}}.
\end{aligned}$$

For examples 1 and 2, the following results are obtained. The $R_{N,m-1}(\phi_i),\ i = 1, 2, 3, 4$ error values at specific points of the singular point t are presented in Tables 1-8. These tables demonstrate that the results obtained when $R_{N,0}(\phi_i)$ (i.e., $m = 1$, $N = 40$, $i = 1, 2, 3, 4$) are better than those obtained when $N = 40$ in the work [16]. The values of $R_{N,1}(\phi_i)$ and $R_{N,2}(\phi_i)$ are also shown in Tables 1-8, which allows us to calculate the solutions of the given singular integral equation with higher accuracy. Furthermore, Figures 1-32 provide complete information about $R_{N,m-1}(\phi_i)$ errors.

7. Conclusion

In this article, the problem of aerodynamics leading to Fredholm's I kind Cauchy type singular integral equation was studied. As a result, it was determined that singular integrals represent the solutions of the integral equation. We then constructed an optimal quadrature formula using the Sobolev method to approximate these solutions. Using the established optimal quadrature formula, the solutions of the above two integral equations were approximated. Approximate results are given in Tables and Figures. The results in the tables and graphs show that by increasing the value of N and m, the roots of Fredholm's I kind Cauchy type singular integral equation can be approximated with high accuracy.

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Table 1: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_1)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_1)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_1)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_1(t) = \frac{1}{\sqrt{1-t^2}}$ in the reference [16].

i=1		N=40		N=40	N=200
t	$R_{N,0}(\phi_1)$	$R_{N,1}(\phi_1)$	$R_{N,2}(\phi_1)$	Error in the	work [16]
-0.997	0.005957819	0.000031029	0.000000564	0.006003030	0.000549209
-0.993	0.006994973	0.000005340	0.000000912	0.007077908	0.000374527
-0.875	0.004555772	0.000078994	0.000000394	0.026921662	0.007629757
-0.685	0.004172268	0.000027913	0.000000088	0.085630364	0.004636713
-0.485	0.005939075	0.000021023	0.000000427	0.126267059	0.013167619
-0.185	0.007296553	0.000015543	0.000000808	0.129480728	0.019433972
0.185	0.007296553	0.000015543	0.000000808	0.129478578	0.019436590
0.485	0.005939075	0.000021023	0.000000427	0.126267036	0.013162577
0.685	0.004172268	0.000027913	0.000000088	0.085636840	0.004660408
0.875	0.000405421	0.000039918	0.000000397	0.002692523	0.007637017
0.993	0.006994973	0.000005340	0.000000912	0.007077417	0.000373201
0.997	0.005957819	0.000031029	0.000000564	0.006002708	0.000548340

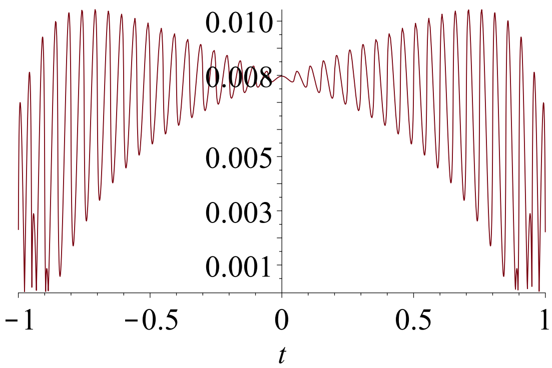

Figure 1: Value of $R_{N,m-1}(\phi_1)$ when $m = 1, N = 40$

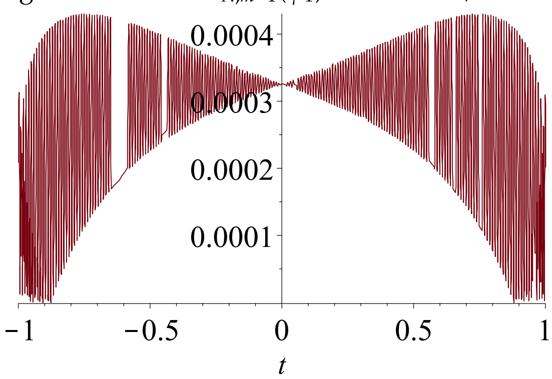

Figure 2: Value of $R_{N,m-1}(\phi_1)$ when $m = 1, N = 200$

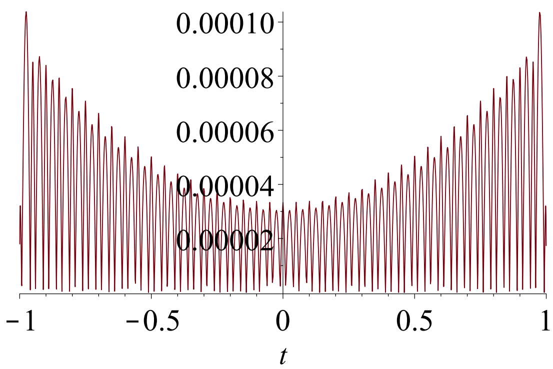

Figure 3: Value of $R_{N,m-1}(\phi_1)$ when $m = 2, N = 40$

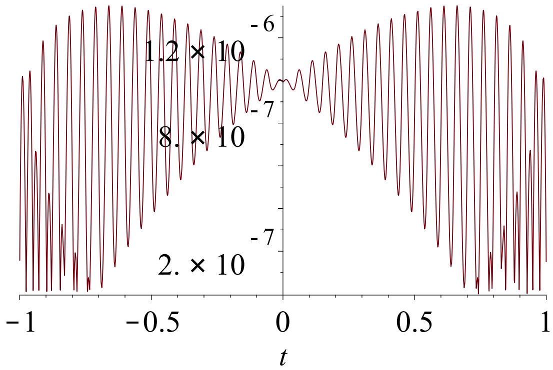

Figure 4: Value of $R_{N,m-1}(\phi_1)$ when $m = 3, N = 40$

Table 2: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_2)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_2)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_2)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_2(t) = \sqrt{\frac{1+t}{1-t}}$ in the reference [16].

i=2		N=40		N=40	N=200
t	$R_{N,0}(\phi_2)$	$R_{N,1}(\phi_2)$	$R_{N,2}(\phi_2)$	Error in the	work [16]
-0.997	0.005957819	0.000031029	0.000000564	0.006600493	0.000552332
-0.993	0.006994973	0.000005340	0.000000912	0.007768718	0.004935831
-0.875	0.004555772	0.000078994	0.000000394	0.027011257	0.008676590
-0.685	0.004172268	0.000027913	0.000000088	0.085576885	0.004011676
-0.485	0.005939075	0.000021023	0.000000427	0.126227641	0.012708680
-0.185	0.007296553	0.000015543	0.000000808	0.129452858	0.019107945
0.185	0.007296553	0.000015543	0.000000808	0.129459336	0.019212361
0.485	0.005939075	0.000021023	0.000000427	0.126253464	0.013003395
0.685	0.004172268	0.000027913	0.000000088	0.085636840	0.004660408
0.875	0.004555772	0.000078994	0.000000394	0.026931131	0.007706881
0.993	0.006994973	0.000005340	0.000000912	0.007078533	0.000388372
0.997	0.005957819	0.000031029	0.000000564	0.006003393	0.000558550

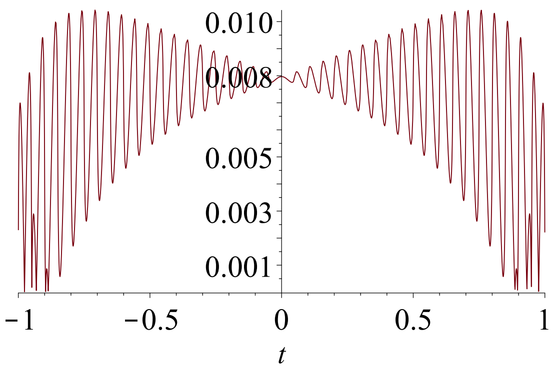

Figure 5: Value of $R_{N,m-1}(\phi_2)$ when $m = 1, N = 40$

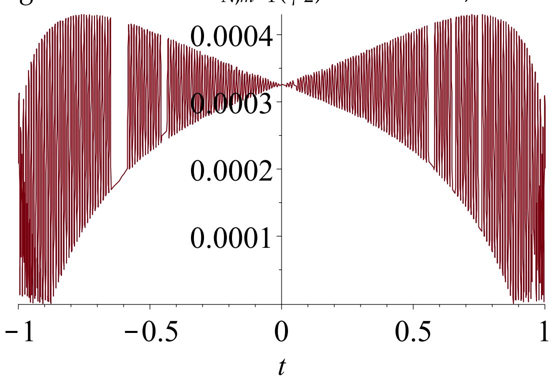

Figure 6: Value of $R_{N,m-1}(\phi_2)$ when $m = 1, N = 200$

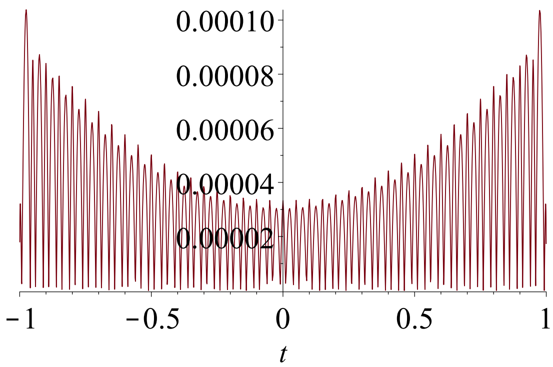

Figure 7: Value of $R_{N,m-1}(\phi_2)$ when $m = 2, N = 40$

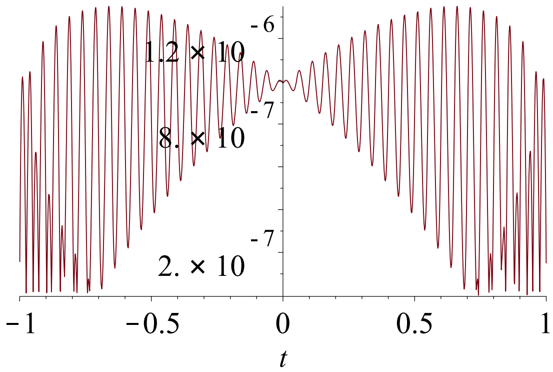

Figure 8: Value of $R_{N,m-1}(\phi_2)$ when $m = 3, N = 40$

Table 3: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_3)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_3)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_3)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_3(t) = \sqrt{\frac{1-t}{1+t}}$ in the reference [16].

i=3		N=40		N=40	N=200
t	$R_{N,0}(\phi_3)$	$R_{N,1}(\phi_3)$	$R_{N,2}(\phi_3)$	Error in the	work [16]
-0.997	0.005957819	0.000031029	0.000000564	0.006002167	0.000539051
-0.993	0.006994973	0.000005340	0.000000912	0.007076629	0.000358970
-0.875	0.004555772	0.000078994	0.000000394	0.026915548	0.007559893
-0.685	0.004172268	0.000027913	0.000000088	0.085640393	0.031126637
-0.485	0.005939075	0.000021023	0.000000427	0.126280633	0.013326970
-0.185	0.007296553	0.000015543	0.000000808	0.129500006	0.019657706
0.185	0.007296553	0.000015543	0.000000808	0.129506499	0.019763381
0.485	0.005939075	0.000021023	0.000000427	0.126305801	0.013621551
0.685	0.004172268	0.000027913	0.000000088	0.085688461	0.005286086
0.875	0.004555772	0.000078994	0.000000394	0.026836727	0.006588314
0.993	0.006994973	0.000005340	0.000000912	0.006697705	0.004197069
0.997	0.005957819	0.000031029	0.000000564	0.005407755	0.005133064

Table 4: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_4)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_4)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_4)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_4(t) = \sqrt{1-t^2}$ in the reference [16].

i=4		N=40		N=40	N=200
t	$R_{N,0}(\phi_4)$	$R_{N,1}(\phi_4)$	$R_{N,2}(\phi_4)$	Error in the	work [16]
-0.997	0.060829040	0.000084316	0.000029273	0.106981360	0.007219397
-0.993	0.042952649	0.000040260	0.000012413	0.073248012	0.004736049
-0.875	0.004217035	0.000070475	0.000024873	0.043060333	0.008628876
-0.685	0.001657335	0.000022252	0.000003001	0.074911861	0.004044323
-0.485	0.001082529	0.000016307	0.000002921	0.1173427797	0.01273519
-0.185	0.002974837	0.000011346	0.000007872	0.121546630	0.019132022
0.185	0.002974837	0.000011346	0.000007872	0.121553123	0.019236266
0.485	0.001082529	0.000016307	0.000002921	0.117368424	0.013030732
0.685	0.001657335	0.000022252	0.000003001	0.074962313	0.004575426
0.875	0.004217035	0.000070475	0.000024873	0.042979724	0.007659442
0.993	0.042952649	0.000040260	0.000012413	0.072854470	0.000192117
0.997	0.060829040	0.000084316	0.000029273	0.106397181	0.000257045

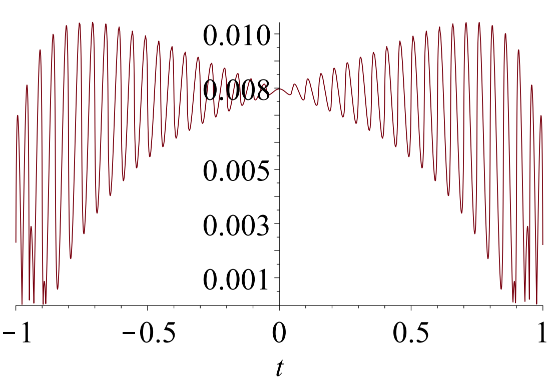

Figure 9: Value of $R_{N,m-1}(\phi_3)$ when $m = 1, N = 40$

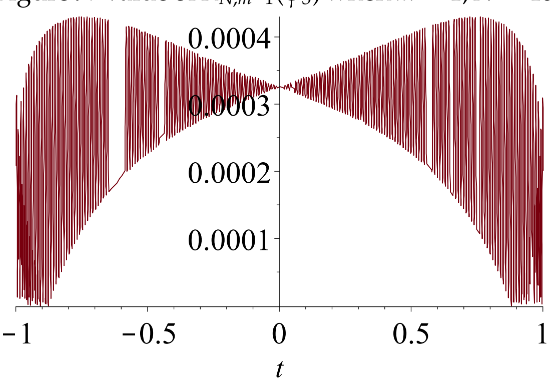

Figure 10: Value of $R_{N,m-1}(\phi_3)$ when $m = 1, N = 200$

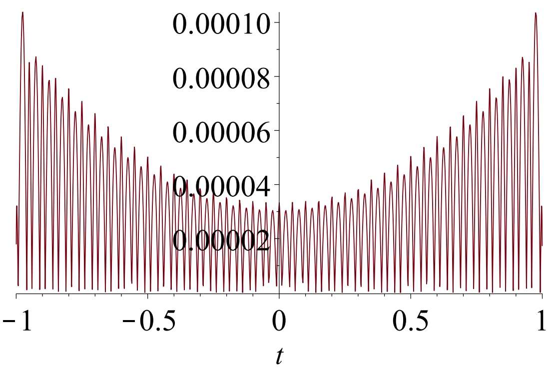

Figure 11: Value of $R_{N,m-1}(\phi_3)$ when $m = 2, N = 40$

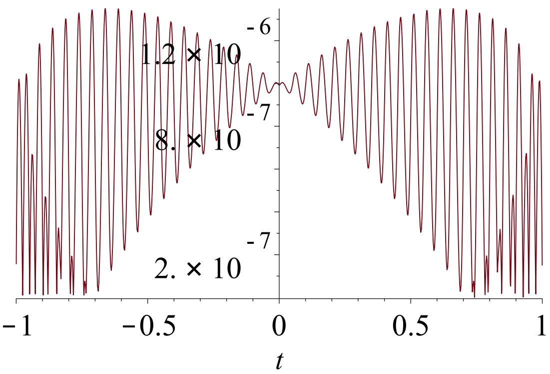

Figure 12: Value of $R_{N,m-1}(\phi_3)$ when $m = 3, N = 40$

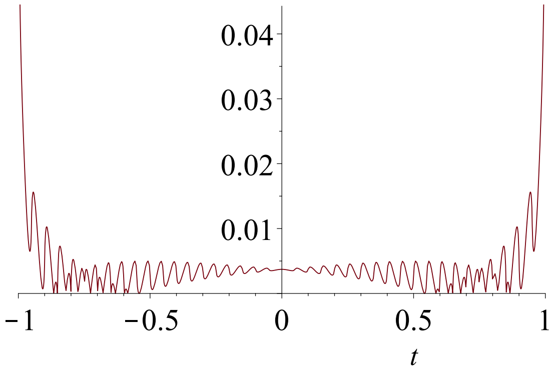

Figure 13: Value of $R_{N,m-1}(\phi_4)$ when $m = 1, N = 40$

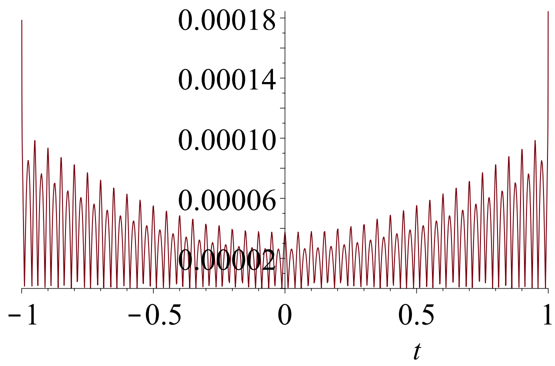

Figure 15: Value of $R_{N,m-1}(\phi_4)$ when $m = 2, N = 40$

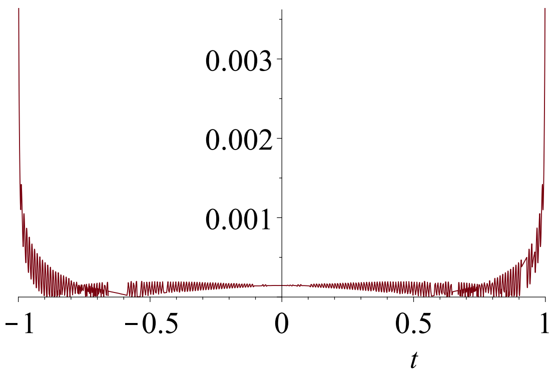

Figure 14: Value of $R_{N,m-1}(\phi_4)$ when $m = 1, N = 200$

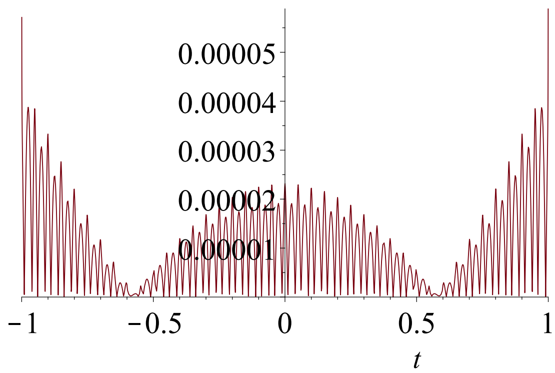

Figure 16: Value of $R_{N,m-1}(\phi_4)$ when $m = 3, N = 40$

Table 5: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_1)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_1)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_1)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_1(t) = \frac{1}{\sqrt{1-t^2}}$ in the reference [16].

i=1		N=20		N=20	N=100
t	$R_{N,0}(\phi_1)$	$R_{N,1}(\phi_1)$	$R_{N,2}(\phi_1)$	Error in the	work [16]
-0.998	0.002266247	0.000041821	0.000000529	0.002024041	0.000234694
-0.978	0.004145320	0.000043481	0.000001754	0.003340472	0.000589171
-0.893	0.003957352	0.000105556	0.000003506	0.002216034	0.000891600
-0.773	0.003709544	0.000064744	0.000008723	0.009017059	0.008687685
-0.568	0.000042179	0.000068073	0.000019368	0.062942750	0.002993924
-0.368	0.006560796	0.000062021	0.000017583	0.123760122	0.014045816
-0.035	0.017905786	0.000349827	0.000033634	0.187487254	0.010014735
0.035	0.019755505	0.000326825	0.000040459	0.148838406	0.065850842
0.368	0.000986770	0.000027036	0.000025782	0.105055092	0.013285722
0.568	0.003934099	0.000123260	0.000018859	0.032789464	0.025239776
0.773	0.005343446	0.000073686	0.000005050	0.018374471	0.008639889
0.893	0.004467696	0.000123301	0.000001510	0.005113167	0.003758796
0.978	0.003421113	0.000042108	0.000004167	0.003402971	0.000961914
0.998	0.001725892	0.000044508	0.000001156	0.001707279	0.000271750

Table 6: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_2)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_2)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_2)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_2(t) = \sqrt{\frac{1+t}{1-t}}$ in the reference [16].

i=2		N=20		N=20	N=100
t	$R_{N,0}(\phi_2)$	$R_{N,1}(\phi_2)$	$R_{N,2}(\phi_2)$	Error in the	work [16]
-0.998	0.008603270	0.000012186	0.000012894	0.008885592	0.000404501
-0.978	0.000884481	0.000052372	0.000002272	0.000567553	0.000397386
-0.893	0.002510877	0.000101612	0.000005292	0.001064237	0.000806539
-0.773	0.002748443	0.000067364	0.000009910	0.008981736	0.008744210
-0.568	0.000612997	0.000069859	0.000020178	0.063600348	0.002955408
-0.368	0.007066752	0.000060641	0.000018208	0.124267947	0.014075567
-0.035	0.018261937	0.000348856	0.000033194	0.187844720	0.010035676
0.035	0.020087568	0.000325919	0.000040049	0.149171693	0.065870367
0.368	0.001220516	0.000027673	0.000026071	0.105289691	0.013271973
0.568	0.003753591	0.000123752	0.000019082	0.032970634	0.025229167
0.773	0.005220394	0.000074021	0.000005202	0.018250953	0.008647105
0.893	0.004385935	0.000123078	0.000001611	0.005031050	0.003753993
0.978	0.003384845	0.000042207	0.000004123	0.003366496	0.000959753
0.998	0.001715012	0.000044479	0.000001142	0.001696376	0.000271203

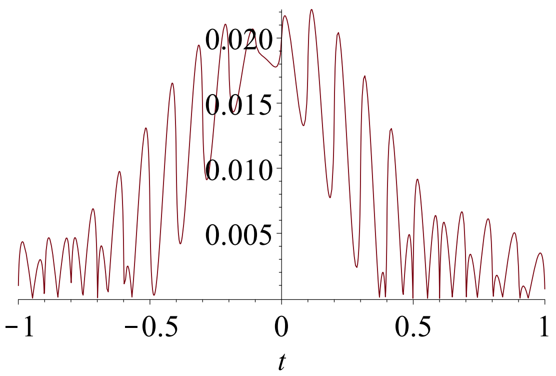

Figure 17: Value of $R_{N,m-1}(\phi_1)$ when $m = 1, N = 20$

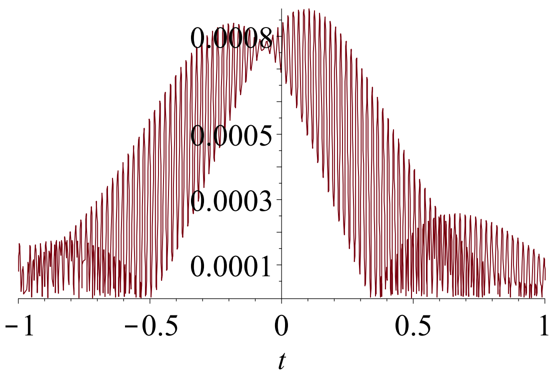

Figure 18: Value of $R_{N,m-1}(\phi_1)$ when $m = 1, N = 100$

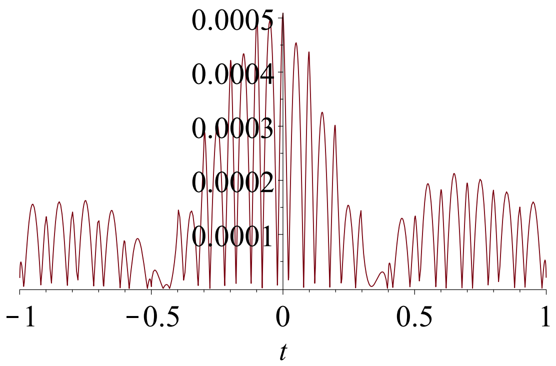

Figure 19: Value of $R_{N,m-1}(\phi_1)$ when $m = 2, N = 20$

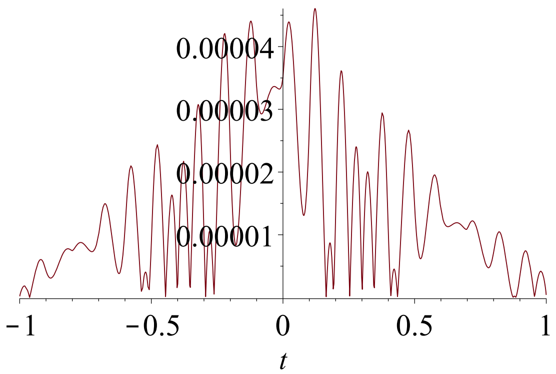

Figure 20: Value of $R_{N,m-1}(\phi_1)$ when $m = 3, N = 20$

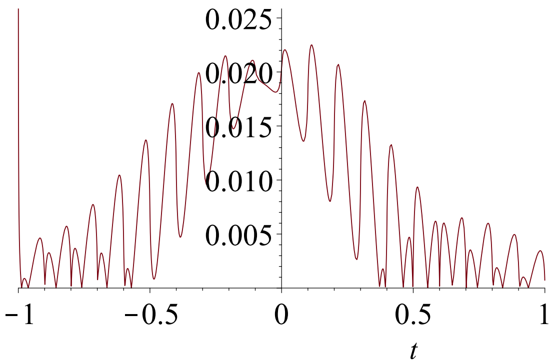

Figure 21: Value of $R_{N,m-1}(\phi_2)$ when $m = 1, N = 20$

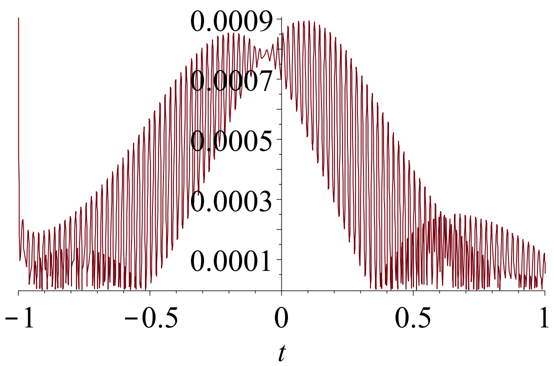

Figure 22: Value of $R_{N,m-1}(\phi_2)$ when $m = 1, N = 100$

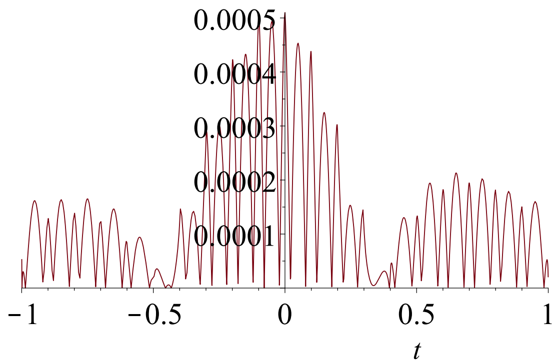

Figure 23: Value of $R_{N,m-1}(\phi_2)$ when $m = 2, N = 20$

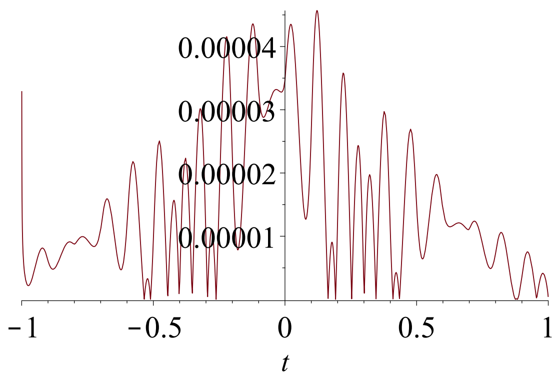

Figure 24: Value of $R_{N,m-1}(\phi_2)$ when $m = 3, N = 20$

Table 7: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_3)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_3)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_3)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_3(t) = \sqrt{\frac{1-t}{1+t}}$ in the reference [16].

i=3		N=20		N=20	N=100
t	$R_{N,0}(\phi_3)$	$R_{N,1}(\phi_3)$	$R_{N,2}(\phi_3)$	Error in the	work [16]
-0.998	0.002277127	0.000041851	0.000000542	0.002034972	0.000235374
-0.978	0.004181588	0.000043383	0.000001799	0.003376837	0.000591274
-0.893	0.004039113	0.000105779	0.000003405	0.002298069	0.000896406
-0.773	0.003832596	0.000064408	0.000008571	0.007893472	0.006680492
-0.568	0.000222687	0.000067581	0.000019145	0.062761641	0.003004693
-0.368	0.006327051	0.000062658	0.000017295	0.123525676	0.014031944
-0.035	0.017573723	0.000350733	0.000034044	0.187154011	0.010995179
0.035	0.019399354	0.000327796	0.000040890	0.148481025	0.065829887
0.368	0.000480815	0.000025656	0.000025157	0.104547327	0.013315374
0.568	0.004589276	0.000121474	0.000018050	0.032131903	0.025278336
0.773	0.006304547	0.000071065	0.000003863	0.019339077	0.008583314
0.893	0.005914171	0.000127245	0.000000276	0.006564749	0.003843916
0.978	0.006681952	0.000033217	0.000008195	0.006675945	0.001153217
0.998	0.012595410	0.000074144	0.000014580	0.012615531	0.000912040

Table 8: The table displays the errors between the exact solution of singular integral equation (8) and its approximate value calculated by the optimal quadrature formula form (13). The second, third, and fourth columns show the errors for $m = 1$, $m = 2$, and $m = 3$, respectively. The error for $m = 1$ is defined as $R_{N,0}(\phi_4)$, the error for $m = 2$ is defined as $R_{N,1}(\phi_4)$, and the error for $m = 3$ is defined as $R_{N,2}(\phi_4)$. The fifth and sixth columns of the table show the error corresponding to the weight $\omega_4(t) = \sqrt{1-t^2}$ in the reference [16].

i=4		N=20		N=20	N=100
t	$R_{N,0}(\phi_4)$	$R_{N,1}(\phi_4)$	$R_{N,2}(\phi_4)$	Error in the	work [16]
-0.998	0.063255976	0.000146957	0.000025398	0.030602557	0.001015765
-0.948	0.012487161	0.000133726	0.000007764	0.008609104	0.000466156
-0.887	0.013040699	0.000072617	0.000006613	0.001905165	0.005949221
-0.778	0.010586117	0.000011641	0.000011046	0.014159086	0.030178490
-0.567	0.004649756	0.000062470	0.000020769	0.065268322	0.002908423
-0.355	0.005838531	0.000140411	0.000004319	0.125744508	0.014117167
-0.015	0.013706566	0.000179528	0.000032007	0.189218380	0.010074475
0.015	0.017475564	0.000116692	0.000041532	0.150545394	0.065908944
0.355	0.000952665	0.000006798	0.000013168	0.106766158	0.013230389
0.567	0.009014790	0.000120981	0.000019863	0.034638644	0.025182245
0.778	0.012958572	0.000006354	0.000006178	0.009854598	0.045014099
0.887	0.014743527	0.000088918	0.000002162	0.002708428	0.006516463
0.948	0.015293560	0.000132532	0.000004866	0.003836374	0.000809926
0.998	0.073574260	0.000179250	0.000011360	0.020018720	0.000342511

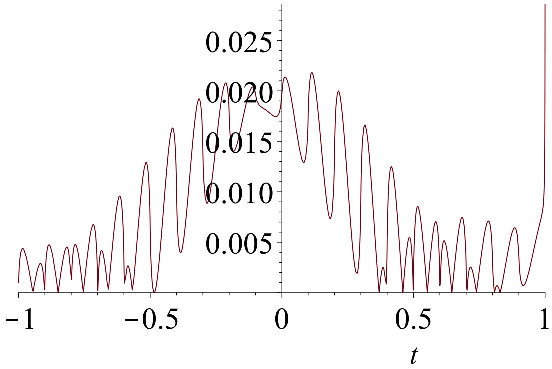

Figure 25: Value of $R_{N,m-1}(\phi_3)$ when $m = 1, N = 20$

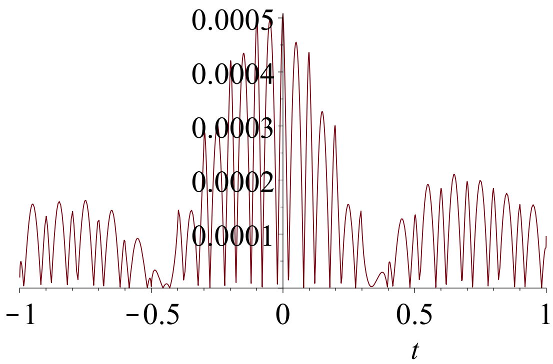

Figure 27: Value of $R_{N,m-1}(\phi_3)$ when $m = 2, N = 20$

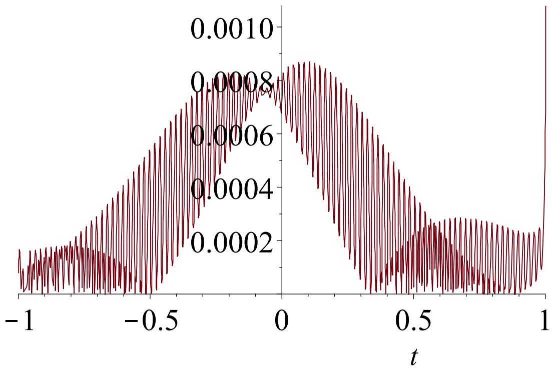

Figure 26: Value of $R_{N,m-1}(\phi_3)$ when $m = 1, N = 100$

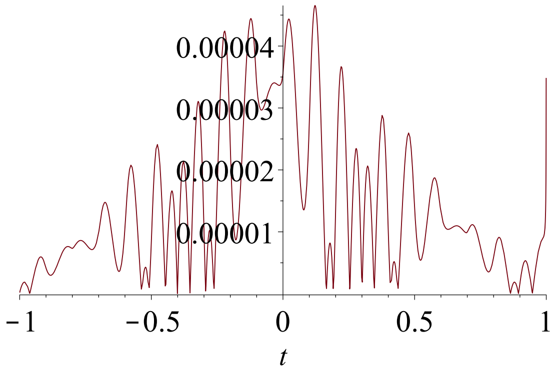

Figure 28: Value of $R_{N,m-1}(\phi_3)$ when $m = 3, N = 20$

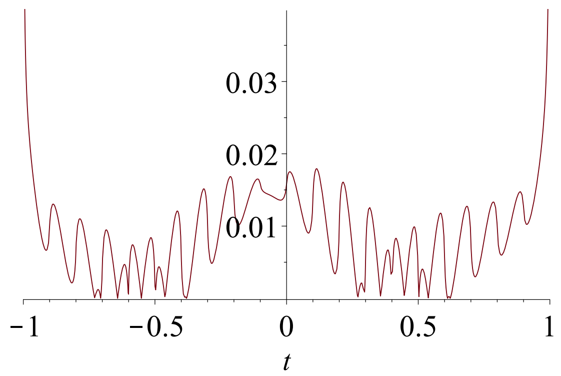

Figure 29: Value of $R_{N,m-1}(\phi_4)$ when $m = 1, N = 20$

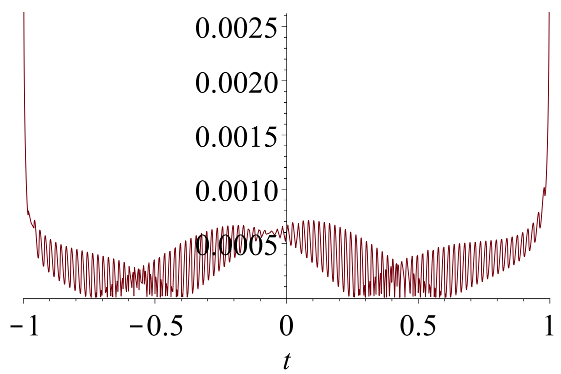

Figure 30: Value of $R_{N,m-1}(\phi_4)$ when $m = 1, N = 100$

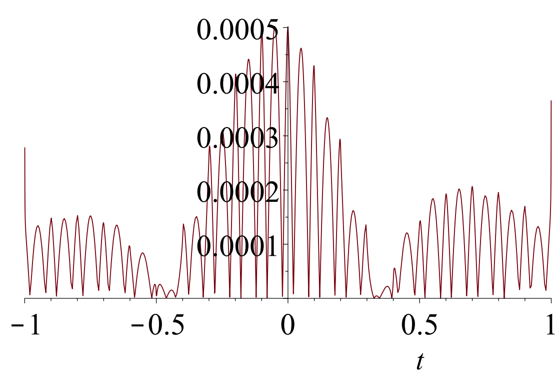

Figure 31: Value of $R_{N,m-1}(\phi_4)$ when $m = 2, N = 20$

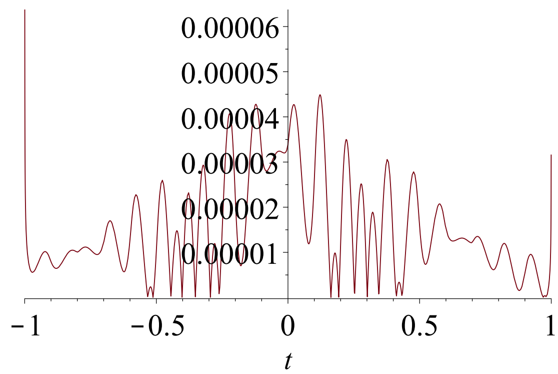

Figure 32: Value of $R_{N,m-1}(\phi_4)$ when $m = 3, N = 20$