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The dual notion of Baer modules and related topics

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Abstract. The aim of this paper is to introduce and investigate the dual notions of σ-submodules of modules, normal modules, and Baer modules over a commutative ring. Furthermore, we obtain some results about the relations between them.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. For a submodule N of an R-module M, a non-empty subset K of M, and a non-empty subset J of R, the residuals of N by K and J are defined as $(N :_R K) = \{a \in R : aK \subseteq N\}$ and $(N :_M J) = \{m \in M : Jm \subseteq N\}$, respectively. In particular, we use $Ann_R(M)$ to denote $(0 :_R M)$.

Let M be an R-module. A submodule N of M is said to be a *σ-submodule* of M if $m \in N$ implies that $Ann_R(m)+(N :_R M) = R$. In particular, an ideal I of R is called a *σ-ideal* if I is a σ-submodule of the R-module R [15]. A non-zero submodule N of M is said to be *second* if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero [22]. A second submodule N of M is said to be a *maximal second submodule* of M, if there does not exist a second submodule K of M such that $N \subset K \subset M$ [4]. M is called a *reduced module* if $rm = 0$ implies that $rM \cap Rm = 0$, where $r \in R$ and $m \in M$ [20]. R is said to be a *normal ring* if it is reduced (without nilpotent elements) and every two distinct minimal prime ideals are comaximal [13]. A finitely generated reduced R-module M is said to be a *normal module* if every two distinct minimal prime submodules are comaximal [16]. R is said to be a *Baer ring* if for each $a \in R$, the annihilator $Ann_R(a) = \{r \in R : ra = 0\}$ of a is generated by an idempotent element $e \in R$ [17]. An element $e \in R$ is said to be a *weak idempotent element* if $em = e^2m$ for each $m \in M$, or equivalently, $e - e^2 \in Ann_R(M)$. Also, M is said to be a *Baer module* if for each $m \in M$ there exists a weak idempotent element $e \in R$ such that $Ann_R(m)M = eM$ [15].

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. Every submodule of M is an intersection of completely irreducible submodules of M. Thus the intersection of all completely irreducible submodules of M is zero [12]. M is said to be *coreduced module* if $(L :_M r) = M$ implies that $L + (0 :_M r) = M$, where $r \in R$ and L is a completely irreducible submodule of M [6]. A submodule N of M is said to be a *co-Baer submodule* if for each completely irreducible submodule L of M with $N \subseteq L$, we have $N \subseteq (L :_R M)M$ [11].

In Section 2 of this paper, we define the dual notion of σ-submodules of an R-module M and obtain some results about $d\sigma$-submodules and co-Baer submodules. We say that a submodule N of an R-module

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M is a *dual of σ-submodule* ($d\sigma$-submodule in short) if for each completely irreducible submodule L of M with $N \subseteq L$ we have $Ann_R(N) + (L :_R M) = R$ (see Definition 2.1). Among various results, we show that if every submodule of M is a $d\sigma$-submodule of M, then M is a comultiplication R-module (see Proposition 2.3 (b)). Also, it is shown that every $d\sigma$-submodule of M is a copure submodule of M, but the converse is not true in general (see Proposition 2.5 (c) and Example 2.6). Moreover, we prove that if M is an R-module such that $M/I^M_P(M)$ is a finitely cogenerated R-module for each maximal ideal P of R, then a submodule N of M is a $d\sigma$-submodule if and only if $N = \sum_Q I^M_Q(M)$, where Q denotes any maximal ideal of R that containing $Ann_R(N)$ (see Theorem 2.7).

In Section 3, we introduce and investigate the dual notions of Baer R-modules and normal R-modules. Also, we obtain some results about the relations of $d\sigma$-submodules, co-Baer submodules, normal R-modules with co-Baer R-modules. We say that an R-module M is a *co-Baer module* if for each completely irreducible submodule L of M there exists a weak idempotent element $e \in R$ such that $(0 :_M (L :_R M)) = (0 :_M e)$ (see Definition 3.1). Among the other results in this section, we show that every finitely generated comultiplication co-Baer R-module is a coreduced R-module (see Theorem 3.8). Furthermore, it is proved that if M is a finitely generated comultiplication co-Baer R-module, then $R/Ann_R(M)$ is a Baer ring (see Theorem 3.13). Also, we determine the condition under which if $R/Ann_R(M)$ is a Baer ring, then M is a co-Baer R-module. In particular, the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a co-Baer $\mathbb{Z}$-module if n is square free (Theorem 3.14 and Example 3.15). Let M be a comultiplication R-module. Then we show that M is a co-Baer module if and only if every co-Baer submodule of M is a $d\sigma$-submodule of M (see Theorem 3.20). It is shown that if M is a comultiplication R-module, then M is a co-Baer R-module if and only if for every submodule N of M we have M/N is a co-Baer R-module (see Theorem 3.21). We say that a finitely cogenerated coreduced R-module M is a *conormal module* if for any two distinct maximal second submodules S_1 and S_2 of M we have $S_1 \cap S_2 = 0$ (see Definition 3.24). Every finitely generated comultiplication co-Baer R-module is a conormal R-module (see Corollary 3.26). It is proved that if M is a finitely generated comultiplication R-module, then M is a conormal module if and only if $R/Ann_R(M)$ is a normal ring (see Theorem 3.27). Finally we determine the conditions under which M is a normal R-module if and only if M is a conormal R-module (see Corollary 3.28).

2. $d\sigma$-submodules and co-Baer submodules

Definition 2.1. *We say that a submodule N of an R-module M is a dual of σ-submodule ($d\sigma$-submodule in short) if for each completely irreducible submodule L of M with $N \subseteq L$ we have $Ann_R(N) + (L :_R M) = R$.*

Remark 2.2. *(See [5].) Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.*

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. For more information about comultiplication R-modules, we refer the reader to [2].

Proposition 2.3. *Let N be a $d\sigma$-submodule of an R-module M. Then we have the following.*

(a) If K is a submodule of M such that $N \subseteq K$ and M/K is a finitely cogenerated R-module, then $Ann_R(N) + (K :_R M) = R$.

(b) $N = (0 :_M Ann_R(N))$. In particular, if every submodule of M is a $d\sigma$-submodule of M, then M is a comultiplication R-module.

Proof. (a) Let K be a submodule of M such that $N \subseteq K$ and M/K be a finitely cogenerated R-module. Then there exist completely irreducible submodules L_i for $i = 1, 2, \ldots, n$ such that $K = \cap^n_{i=1} L_i$. By assumption, $Ann_R(N) + (L_i :_R M) = R$ for $i = 1, 2, \ldots, n$. Hence by [19, Proposition 3.59], $Ann_R(N) + \cap^n_{i=1}(L_i :_R M) = R$. Thus $Ann_R(N) + (K :_R M) = R$.

(b) Always we have $N \subseteq (0 :_M Ann_R(N))$. Now let L be a completely irreducible submodule of M such that $N \subseteq L$. By assumption, $Ann_R(N) + (L :_R M) = R$. Thus we have

$$L = (L :_M R) = (L :_M Ann_R(N) + (L :_R M)) =$$

$$(L :_M Ann_R(N)) \cap (L :_M (L :_R M)) =$$

$$(L :_M Ann_R(N)) \cap M \supseteq (0 :_M Ann_R(N)).$$

Now the result follows from Remark 2.2. □

Let M be an R-module. A submodule N of M is said to be *pure* if $IN = N \cap IM$ for each ideal I of R [1]. A submodule N of M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for each ideal I of R [2].

Proposition 2.4. *Let M be a multiplication R-module. Then we have the following.*

(a) If N is a pure submodule of M, then N is σ-submodule of M.

(b) If N is a copure submodule of M, then N is σ-submodule of M.

Proof. (a) Let N be a pure submodule of M and $x \in N$. Since M is a multiplication R-module, there exist ideals I and J of R such that $Rx = IM$ and $N = JM$. As N is pure, we get that $Rx = IM = IM \cap JM = IJM = JRx$. Thus by [8, Corollary 2.5], $Ann_R(Rx) + J = R$. Since $J \subseteq (JM :_R M)$, we have $Ann_R(Rx) + (JM :_R M) = R$.

(b) Let N be a copure submodule of M and $x \in N$. Since M is a multiplication R-module, there exist ideals I and J of R such that $Rx = IM$ and $N = JM$. As $IM \subseteq JM$, we have $M = (JM :_R I)$. Now since N is copure, $M = JM + (0 :_M I)$. Thus $Rx = IM = IJM = JRx$ and so $Ann_R(Rx) + J = R$ by [8, Corollary 2.5]. Now the result follows from the fact that $J \subseteq (JM :_R M)$. □

A family $\{N_i\}_{i\in I}$ of submodules of an R-module M is said to be an *inverse family* of submodules of M if the intersection of two of its submodules contains a module in $\{N_i\}_{i\in I}$. Also, M satisfies *Grothendieck's condition* $AB5^*$ (the property $AB5^*$ in short) if for every submodule K of M and every inverse family $\{N_i\}_{i\in I}$ of submodules of M, $K + \cap_{i\in I} N_i = \cap_{i\in I}(N_i + K)$[21, p.435].

An R-module M satisfies the *double annihilator conditions* (DAC for short) if for each ideal I of R we have $I = Ann_R(0 :_M I)$ [9].

An R-module M is said to be a *strong comultiplication module* if M is a comultiplication R-module and satisfies the DAC conditions [2].

Proposition 2.5. *Let M be an R-module. Then we have the following.*

(a) If $N_1, N_2, .., N_k$ are $d\sigma$-submodules of M, then $\sum_{i=1}^k N_i$ is a $d\sigma$-submodule of M.

(b) If M satisfies the property $AB5^*$ and $\{N_i\}_{i\in I}$ is a inverse family of $d\sigma$-submodules of M, then $\cap_{i\in I} N_i$ is a $d\sigma$-submodule of M.

(c) Every $d\sigma$-submodule of M is a copure submodule of M.

(d) If N is a pure submodule of a strong comultiplication R-module M, then N is $d\sigma$-submodule of M.

Proof. (a) Let $N_1, N_2, .., N_k$ be $d\sigma$-submodules of M and let L be a completely irreducible submodule of M such that $\sum_{i=1}^k N_i \subseteq L$. Then $N_i \subseteq L$ for $i = 1, 2, .., k$. So by assumption, $Ann_R(N_i) + (L :_R M) = R$ for $i = 1, 2, .., k$. This implies that $Ann_R(\sum_{i=1}^k N_i) + (L :_R M) = \cap_{i=1}^k Ann_R(N_i) + (L :_R M) = R$ by [19, Proposition 3.59]. Hence $\sum_{i=1}^k N_i$ is a $d\sigma$-submodule of M.

(b) Let M satisfies the property $AB5^*$ and $\{N_i\}_{i\in I}$ be a inverse family of $d\sigma$-submodules of M. Let L be a completely irreducible submodule of M such that $\cap_{i\in I} N_i \subseteq L$. Then $\cap_{i\in I} N_i + L = L$. Since M satisfies the property $AB5^*$, we have $\cap_{i\in I}(N_i + L) = L$. Thus $N_i \subseteq L$ for some $i \in I$ because L is completely irreducible submodule of M. So by assumption, $Ann_R(N_i) + (L :_R M) = R$. This implies that $Ann_R(\cap_{i\in I} N_i) + (L :_R M) = R$, as needed.

(c) Let N be a $d\sigma$-submodule of M and I be an ideal of R. Clearly $N + (0 :_M I) \subseteq (N :_M I)$. Assume that L is a completely irreducible submodule of M such that $N + (0 :_M I) \subseteq L$. By Remark 2.2, it is enough to show that $(N :_M I) \subseteq L$. As N is a $d\sigma$-submodule of M, we have $Ann_R(N) + (L :_R M) = R$. Thus $1 = s + t$ for some $s \in Ann_R(N)$ and $t \in (L :_R M)$. Let $x \in (N :_M I)$. Then $xI \subseteq N$. We have $x = 1x = sx + tx$. Since $Isx \subseteq sN = 0$, we have $sx \in (0 :_M I)$. Thus $x \in (0 :_M I) + L = L$, as needed.

(d) Let N be a pure submodule of a strong comultiplication R-module M. Suppose that L is a completely irreducible submodule of M such that $N \subseteq L$. As M is a comultiplication R-module, there exist ideals I and J of R such that $L = (0 :_M I)$ and $N = (0 :_M J)$. As $(0 :_M J) \subseteq (0 :_M I)$, we have $I(0 :_M J) = 0$. Now since N is a pure submodule of M, $(0 :_M J) \cap IM = 0$. This implies that $(0 :_M J + Ann_R(IM)) = (0 :_M R)$. It follows that $J + Ann_R(IM) = R$ because M is a strong comultiplication R-module. Hence $Ann_R((0 :_M J)) + Ann_R(IM) = R$, as needed. □

The following example shows that a copure submodule of an R-module M is not a $d\sigma$-submodule of M in general.

Example 2.6. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_{p^\infty}$ and $N = 0 \oplus \mathbb{Z}_{p^\infty}$. Then N is a copure submodule of M but N is not a $d\sigma$-submodule of M.*

Let M be an R-module and P be a prime ideal of R [3, 4, 7]. Then

$$I^M_P(M) = \cap\{L \mid L \textit{ is a completely irreducible submodule of } M \textit{ and}$$

$$(L :_R M) \not\subseteq P\}.$$

Theorem 2.7. *Let M be an R-module such that $M/I^M_P(M)$ is a finitely cogenerated R-module for each maximal ideal P of R. Then a submodule N of M is a $d\sigma$-submodule if and only if $N = \sum_Q I^M_Q(M)$, where Q denotes any maximal ideal of R that containing $Ann_R(N)$.*

Proof. ⇒: Let L be a completely irreducible submodule of M such that $N \subseteq L$. Since N is a $d\sigma$-submodule of M, We have that $Ann_R(N) + (L :_R M) = R$. Let P be a maximal ideal of R containing $Ann_R(N)$. Then $(L :_R M) \not\subseteq P$. Thus $I^M_P(M) \subseteq L$. This implies that $\sum_Q I^M_Q(M) \subseteq L$ and so $\sum_Q I^M_Q(M) \subseteq N$ by Remark 2.2. Conversely, assume that L is a completely irreducible submodule of M such that $\sum_Q I^M_Q(M) \subseteq L$, where Q is any maximal ideal containing $Ann_R(N)$. Let P be a maximal ideal of R. If P contains $Ann_R(N)$, then $I^M_P(M) \subseteq L$ and so $(L :_R M) \not\subseteq P$ by using [7, Lemma 2.3]. This implies that $(L :_R M)_P = R_P$. Otherwise, we would have $Ann_R(N) \not\subseteq P$ and so $Ann_R(N)_P = R_P$. In both cases we have $Ann_R(N)_P + (L :_R M)_P = R_P$ for all maximal ideal P of R. This implies that $Ann_R(N) + (L :_R M) = R$. Hence $Ann_R(N)N + (L :_R M)N = N$ and so $N \subseteq (L :_R M)M \subseteq L$. Therefore, $N \subseteq \sum_Q I^M_Q(M)$ by Remark 2.2.

⇐: Let $N = \sum_Q I^M_Q(M)$, where Q denotes any maximal ideal of R that containing $Ann_R(N)$. We show that N is a $d\sigma$-submodule of M. Let L be a completely irreducible submodule of M such that $N \subseteq L$. Assume contrary that $Ann_R(N) + (L :_R M) \neq R$. Then there exists a maximal ideal $\acute{Q}$ of R such that $Ann_R(N) + (L :_R M) \subseteq \acute{Q}$. Since $I^M_{\acute{Q}}(M) \subseteq \sum_Q I^M_Q(M) = N \subseteq L$, we have $(L :_R M) \not\subseteq \acute{Q}$ by using [7, Lemma 2.3]. Which is a contradiction. □

Proposition 2.8. *Let N be a submodule of an R-module M. Then the following statements are equivalent:*

(a) N is a co-Baer submodule of M;

(b) $(L :_R M) \subseteq (\acute{L} :_R M)$ with $L, \acute{L}$ are completely irreducible submodules of M and $N \subseteq L$ implies that $N \subseteq \acute{L}$;

Proof. $(a) \Rightarrow (b)$ Let N be a co-Baer submodule of M and $(L :_R M) \subseteq (\acute{L} :_R M)$ with $L, \acute{L}$ are completely irreducible submodules of M and $N \subseteq L$. Then

$$N \subseteq (L :_R M)M \subseteq (\acute{L} :_R M)M \subseteq \acute{L}.$$

$(b) \Rightarrow (a)$ Let N be a submodule of M and L be a completely irreducible submodule of M with $N \subseteq L$. Assume that $\acute{L}$ is a completely irreducible submodule of M such that $(L :_R M)M \subseteq \acute{L}$. Then $(L :_R M) \subseteq (\acute{L} :_R M)$. Thus by part (a), $N \subseteq \acute{L}$. Thus $N \subseteq (L :_R M)M$ by Remark 2.2. □

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2, \dots, n$. Let $R = R_1 \times R_2 \times \cdots \times R_n$. Then $M = M_1 \times M_2 \times \cdots \times M_n$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2 \times \cdots \times N_n$ for some submodules N_i of M_i for $i = 1, 2, \dots, n$.

Proposition 2.9. *Let M_i be an R_i-module for each $i = 1, 2$. Set $M := M_1 \times M_2$, $R := R_1 \times R_2$, and $N := N_1 \times N_2$ a submodule of M. If N is a co-Baer submodule of M, then N_i is a co-Baer submodule of M_i for $i = 1, 2$.*

Proof. Let L_1 be a completely irreducible submodule of M_1 such that $N_1 \subseteq L_1$. Then $L_1 \times M_2$ is a completely irreducible submodule of M with $N \subseteq L_1 \times M_2$ and so by assumption, $N_1 \times N_2 \subseteq (L_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)(M_1 \times M_2)$. Now the result follows form the fact that

$$(L_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2) = (L_1 :_{R_1} M_1) \times (M_2 :_{R_2} M_2).$$

□

Proposition 2.10. *Let $f : M \to \acute{M}$ be an epimorphism of R-modules. If N is a co-Baer submodule of M, then $f(N)$ is a co-Baer submodule of $\acute{M}$.*

Proof. Let N be a co-Baer submodule of M and $\acute{L}$ be a completely irreducible submodule of $\acute{M}$ such that $f(N) \subseteq \acute{L}$. Then by [11, Lemma 2.8.(b)], $f^{-1}(\acute{L})$ is a completely irreducible submodule of M. Thus $N \subseteq f^{-1}(f(N)) \subseteq f^{-1}(\acute{L})$ implies that $N \subseteq (f^{-1}(\acute{L}) :_R M)M$ since N is a co-Baer submodule of M. Hence $f(N) \subseteq (f^{-1}(\acute{L}) :_R M)\acute{M}$. Now $(f^{-1}(\acute{L}) :_R M) \subseteq (\acute{L} :_R \acute{M})$ implies that $f(N) \subseteq (\acute{L} :_R \acute{M})\acute{M}$, as needed. □

Corollary 2.11. *Let M be an R-module and N, K be two submodules of M such that $N \subseteq K \subseteq M$. If K is a co-Baer submodule of M, then K/N is a co-Baer submodule of M/N.*

Proposition 2.12. *Let M be an R-module. Then every submodule of M is a co-Baer submodule if and only if every completely irreducible submodule of M is a co-Baer submodule.*

Proof. Let N be a submodule of M and L be a completely irreducible submodule of M with $N \subseteq L$. Let $\acute{L}$ be a completely irreducible submodule of M such that $(L :_R M)M \subseteq \acute{L}$. Then $(L :_R M) \subseteq (\acute{L} :_R M)$. Now since L is a co-Baer submodule of M and $L \subseteq L$, we have $L \subseteq \acute{L}$ by Proposition 2.8. This implies that $N \subseteq \acute{L}$. Now the result follows from Remark 2.2.

The converse is clear. □

3. co-Baer modules and conormal modules

Definition 3.1. *We say that an R-module M is a co-Baer module if for each completely irreducible submodule L of M there exists a weak idempotent element $e \in R$ such that $(0 :_M (L :_R M)) = (0 :_M e)$.*

Example 3.2. *(a) Every prime R-module is a co-Baer module*

(b) Every simple R-module is a co-Baer module.

Definition 3.3. *We say that an R-module M is a μ_0-module if for each finite number of ideals $I_1, I_2, \dots, I_n$ of R we have*

$$\sum_{i=1}^{n}(0 :_M I_i) = (0 :_M \bigcap_{i=1}^{n}(I_i + Ann_R(M))).$$

Example 3.4. *By [2, Theorem 115 (c)], every strong comultiplication R-module is a μ_0-module.*

Proposition 3.5. *Let an R-module M be a μ_0-module. The following statements are equivalent:*

(a) M is a Baer R-module;

(b) *For any submodule N of M with M/N is a finitely cogenerated R-module,* $(0 :_M (N :_R M)) = (0 :_M e)$ *for some weak idempotent* $e \in R$.

Proof. $(a) \Rightarrow (b)$. Let N be a submodule of M with M/N is a finitely cogenerated R-module. Then there exist completely irreducible submodules $L_1, L_2, \ldots, L_n$ of M such that $N = \cap_{i=1}^n L_i$. As M is co-Baer module, there exist weak idempotents $e_i \in R$ such that $(0 :_M (L_i :_R M)) = (0 :_M e_i)$ for each $i = 1, 2, \ldots, n$. Since M is a μ_0-module, we have

$$\sum_{i=1}^{n}(0 :_M e_i) = \sum_{i=1}^{n}(0 :_M (L_i :_R M)) = (0 :_M \bigcap_{i=1}^{n}((L_i :_R M) + Ann_R(M))).$$

Now the result follows from the fact that $\sum_{i=1}^n (0 :_M e_i) = (0 :_M e_1e_2 \cdots e_n)$ and $e_1e_2 \cdots e_n$ is a weak idempotent element of R.

$(a) \Rightarrow (b)$. This is clear. □

Let R be an integral domain. A submodule N of an R-module M is said to be a *cotorsion-free submodule of M* (the dual of torsion-free) if $I_0^M(N) = N$. Also, M said to be a *cotorsion-free module* if M is a cotorsion-free submodule of itself [3].

Example 3.6. *Every cotorsion free R-module is a co-Baer R-module.*

Example 3.7. *Let M be an R-module with* $Ann_R(M)$ *be a maximal ideal of R and let L be a completely irreducible submodule of M. As* $Ann_R(M) \subseteq (L :_R M)$*, we have that either* $(L :_R M) = Ann_R(M)$ *or* $(L :_R M) = R$*. Thus* $(0 :_M (L :_R M)) = (0 :_M Ann_R(M)) = M = (0 :_M 0)$ *or* $(0 :_M (L :_R M)) = (0 :_M R) = 0 = (0 :_M 1)$*. Hence M is a co-Baer R-module. For instance, for a prime number p, the* $\mathbb{Z}$*-module* $\mathbb{Z}_p \times \mathbb{Z}_p$ *is a co-Baer* $\mathbb{Z}$*-module, and also it is neither a simple nor a cotorsion free* $\mathbb{Z}$*-module.*

Theorem 3.8. *Every finitely generated comultiplication co-Baer R-module is a coreduced R-module.*

Proof. Let M be a finitely generated comultiplication co-Baer R-module. Assume that L is a completely irreducible submodule of M and $a \in R$ such that $(L :_M a^2) = M$. Then $a \in ((L :_M a) :_R M)$. By [7, Lemma 2.1], $(L :_M a)$ is a completely irreducible submodule of M. Thus as M is a co-Baer R-module, there exists a weak idempotent element $e \in R$ such that

$$(0 :_M (L :_R aM)) = (0 :_M ((L :_M a) :_R M)) = (0 :_M e).$$

On the other hand, $e(1 - e)M = 0$ implies that $(1 - e)M \subseteq (0 :_M e) \subseteq (0 :_M a)$. Thus $a(1 - e)M = 0 \subseteq L$. It follows that $(1 - e) \in (L :_R aM)$ and so $(0 :_M (L :_R aM)) \subseteq (0 :_M 1 - e)$. Therefore $(0 :_M e) \subseteq (0 :_M 1 - e)$. This implies that $(0 :_M e) = (0 :_M e) \cap (0 :_M 1 - e) = 0$. Therefore, $(0 :_M (L :_R aM)) = 0$. Now we will show that $(L :_R aM) = R$. Assume to the contrary that $(L :_R aM) \neq R$. Then there exists a maximal ideal Q of R which containing $(L :_R aM)$. As $(0 :_M (L :_R aM)) = 0$, we get $(0 :_M Q) = 0$. So by [2, Theorem 7 (d)], there exists $q \in Q$ such that $(1 - q)M = 0$. Since $Ann_R(M) \subseteq Q$, we have that $1 \in Q$, which is a contradiction. Therefore $(L :_R aM) = R$ and hence M is a coreduced R-module. □

Proposition 3.9. *Let N be a copure submodule of a co-Baer R-module M. Then M/N is a co-Baer R-module.*

Proof. Let L/N be a completely irreducible submodule of M/N. Then by using $(M/N)/(L/N) \cong M/L$ and [12, Remark 1.1], we have L is a completely irreducible submodule of M. Thus as M is a co-Baer R-module, we get that $(0 :_M (L :_R M)) = (0 :_M e)$ for some weak idempotent $e \in R$. Now since N is copure, we have

$$(0 :_{M/N} (L/N :_R M/N)) = (N :_M (L :_R M))/N = (N + (0 :_M (L :_R M))/N$$

$$= (N + (0 :_M e))/N = (0 :_{M/N} e).$$

Thus M/N is a co-Baer R-module. □

Definition 3.10. *We say that an R-module M is an annihilator comultiplication module if for each completely irreducible submodule L of M there exists a finitely generated ideal I of R such that $(L :_R M) = Ann_R(IM)$.*

Lemma 3.11. *Every comultiplication R-module M is an annihilator comultiplication module.*

Proof. Let L be a completely irreducible submodule of M. Then by assumption, $L = (0 :_M Ann_R(L))$. Thus $L = \cap_{a \in Ann_R(L)}(0 :_M Ra)$. This implies that $L = (0 :_M Ra)$ for some $a \in Ann_R(L)$ since L is a completely irreducible submodule of M. Hence M is an annihilator comultiplication R-module. □

An R-module M is said to satisfy the *condition* $(\star)$ if $(0 :_M Q) = 0$ implies that $(1 - q)M = 0$ for some $q \in Q$, where Q is a maximal ideal of R.

Proposition 3.12. *Let M be a finitely generated co-Baer R-module satisfying the condition $(\star)$. Then M is an annihilator comultiplication R-module.*

Proof. Assume that L is a completely irreducible submodule of M. Then there exists a weak idempotent $e \in R$ such that $((0 :_M (L :_R M)) = (0 :_M e)$. It follows that

$$0 = (0 :_M (L :_R M)) \cap (0 :_M 1 - e) = (0 :_M (L :_R M) + R(1 - e)).$$

Now we will show that $(L :_R M) + R(1 - e) = R$. Assume to the contrary that $(L :_R M) + R(1 - e) \neq R$. Then there exists a maximal ideal Q of R that containing $(L :_R M) + R(1 - e)$. This implies that $(0 :_M Q) = 0$. As M satisfying the condition $(\star)$, we get $(1 - r)M = 0$ for some $r \in Q$. Since $1 - r \in Ann_R(M) \subseteq Q$, we get $1 \in Q$, which is a contradiction. Thus we have $(L :_R M) + R(1 - e) = R$. Then $r_1 + (1 - e)x = 1$ for some $r_1 \in (L :_R M)$ and $x \in R$. This implies that $e = er_1 + e(1 - e)x$ and so $e \in (L :_R M) + Ann_R(M) = (L :_R M)$. Therefore, $Re \subseteq (L :_R M)$. Now let $r_2 \in (L :_R M)$. Then

$$M = (0 :_M (L :_R M)(1 - e)) \subseteq (0 :_M r_2(1 - e))$$

and hence $r_2(1 - e) \in Ann_R(M)$. This implies that $r_2 = r_2e + r_2(1 - e) \in Re + Ann_R(M)$. Thus we have $(L :_R M) = Re + Ann_R(M)$. Since $Ann_R((1 - e)M) = Re + Ann_R(M)$, we get $(L :_R M) = Ann_R((1 - e)M)$ as needed. □

Theorem 3.13. *Let M be a finitely generated comultiplication co-Baer R-module. Then $R/Ann_R(M)$ is a Baer ring.*

Proof. Let $\bar{R} = R/Ann_R(M)$ and $a + Ann_R(M) \in \bar{R}$. It is enough to show that $Ann_{\bar{R}}(a + Ann_R(M)) = (e + Ann_R(M))\bar{R}$ for some idempotent $e + Ann_R(M) \in \bar{R}$. Clearly, $Ann_{\bar{R}}(a + Ann_R(M)) = Ann_R(aM)/Ann_R(M)$. By using [2, Theorem 24], M is a finitely cogenerated R-module. Thus there exist completely irreducible submodules L_i of M for $i = 1, 2, \ldots, n$ such that $\cap_{i=1}^n L_i = 0$. Thus $(0 :_M a) = \cap_{i=1}^n (L_i :_M a)$. It follows that $Ann_R(aM) = \cap_{i=1}^n ((L_i :_M a) :_R M)$. As M is a finitely generated comultiplication co-Baer R-module, a similar argument as in the proof of Proposition 3.12 shows that $((L_i :_M a) :_R M) = Re_i + Ann_R(M)$ for some weak idempotent $e_i \in R$. This implies that $((L_i :_M a) :_R M)/Ann_R(M) = (e_i + Ann_R(M))\bar{R}$. Then note that $e_i + Ann_R(M)$ is an idempotent of $\bar{R}$ and hence

$$Ann_R(aM)/Ann_R(M) = \cap_{i=1}^n ((L_i :_M a) :_R M)/Ann_R(M) =$$

$$\cap_{i=1}^n (((L_i :_M a) :_R M)/Ann_R(M)) = \cap_{i=1}^n ((e_i + Ann_R(M))\bar{R})$$

$$= (e_1e_2 \cdots e_n + Ann_R(M)) = (e + Ann_R(M))\bar{R},$$

where $e = e_1e_2 \cdots e_n$ and $e + Ann_R(M)$ is an idempotent of $\bar{R}$. □

Theorem 3.14. *Let M be an annihilator comultiplication R-module and let $R/Ann_R(M)$ be a Baer ring. Then M is a co-Baer R-module.*

Proof. Let L be a completely irreducible submodule of M As M is an annihilator comultiplication R-module, we have $(L :_R M) = Ann_R(IM)$ for some finitely generated ideal $I = R(a_1, a_2, ..., a_n)$ of R. This implies that $(L :_R M) = Ann_R(a_1M + \; + a_nM) = \cap_{i=1}^{n} Ann_R(a_iM)$. Since $R/Ann_R(M)$ is a Baer ring, for each $a_i \in R$ we have

$$Ann_{R/Ann_R(M)}(a_i + Ann_R(M)) = Ann_R(a_iM)/Ann_R(M)$$

$$= ((e_i)R + Ann_R(M))/Ann_R(M)$$

for some weak idempotent $e_i \in R$. This implies that $Ann_R(a_iM) = (e_i)R + Ann_R(M)$. Then we have

$$(L :_R M) = \cap_{i=1}^{n} Ann_R(a_iM) = \cap_{i=1}^{n}((e_i)R + Ann_R(M)).$$

Since $\cap_{i=1}^{n}((e_i)R + Ann_R(M)) = (e)R + Ann_R(M)$, where $e = e_1e_2 \cdots e_n$ is a weak idempotent of R, we conclude that $(0 :_M (L :_R M)) = (0 :_M e)$, which completes the proof. □

Example 3.15. *By Theorem 3.14, the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a co-Baer $\mathbb{Z}$-module if n is square free.*

Proposition 3.16. *Let $\{M_i\}_{i\in I}$ be a family of R-modules. Consider the following cases:*

(a) $\prod_{i\in I} M_i$ is a co-Baer R-module.

(b) $\bigoplus_{i\in I} M_i$ is a co-Baer R-module.

(c) M_i is a co-Baer R-module for each $i \in I$.

Then $(a) \Rightarrow (b) \Rightarrow (c)$ always holds.

Proof. The proof is straightforward. □

The following example shows that in Proposition 3.16 the implication $(c) \Rightarrow (a)$ is not true in general.

Example 3.17. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. For completely irreducible submodule $L = 0 \oplus \mathbb{Z}_3$ of M we have $(0 :_M (L :_R M)) \neq (0 :_M 0)$ and $(0 :_M (L :_R M)) \neq (0 :_M 1)$. Thus we have M is not a co-Baer $\mathbb{Z}$-module since the only weak idempotents of $\mathbb{Z}$ are 0 and 1. But $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are co-Baer $\mathbb{Z}$-modules by Example 3.15.*

Lemma 3.18. *Let L be a completely irreducible submodule of an R-module M. Then $(L :_R M)M$ is a co-Baer submodule of M.*

Proof. Let $\acute{L}$ be a completely irreducible submodule of M such that $(L :_R M)M \subseteq \acute{L}$. Then $(L :_R M) \subseteq (\acute{L} :_R M)$. Thus $(L :_R M)M \subseteq (\acute{L} :_R M)M$ as needed. □

Lemma 3.19. *Let N be a submodule of a comultiplication R-module M and e be a weak idempotent of R such that $(0 :_M (N :_R M)) = (0 :_M e)$. Then $(N :_R M) = Re + Ann_R(M)$.*

Proof. As M is a comultiplication R-module, $(0 :_M (N :_R M)) = (0 :_M e)$ implies that $(N :_R M)M = eM$. Thus $eM \subseteq N$ and so $Re \subseteq (N :_R M)$. Hence $Re + Ann_R(M) \subseteq (N :_R M)$. On the other hand $(1 - e)M \subseteq (0 :_M e)$. Thus $(1 - e)(N :_R M)M = 0$. Hence

$$(N :_R M) \subseteq Ann_R((1 - e)M) \subseteq Re + Ann_R(M).$$

□

Theorem 3.20. *Let M be a comultiplication R-module. Then M is a co-Baer module if and only if every co-Baer submodule of M is a dσ-submodule of M.*

Proof. Let M be a co-Baer R-module and N be a co-Baer submodule of M. Assume that L is a completely irreducible submodule of M such that $N \subseteq L$. As N is a co-Baer submodule of M, we have $N \subseteq (L :_R M)M$. Since M is a co-Baer R-module, $(0 :_M (L :_R M)) = (0 :_M e)$ for some weak idempotent $e \in R$. Thus $(L :_R M) = Re + Ann_R(M)$ by Lemma 3.19. As we can see in the proof of Lemma 3.19, $(L :_R M)(1 - e)M = 0$ and $N \subseteq (L :_R M)M = eM$. These in turn imply that $R = Re + Ann_R(M) + R(1 - e) \subseteq Ann_R(N) + (L :_R M)$. Therefore, $Ann_R(N) + (L :_R M) = R$ as needed. For the converse, assume that every co-Baer submodule of M is a $d\sigma$-submodule of M and L is a completely irreducible submodule of M. By Lemma 3.18, $(L :_R M)M$ is a co-Baer submodule of M. Thus by assumption, $(L :_R M)M$ is a $d\sigma$-submodule of M. Hence, $Ann_R((L :_R M)M) + (L :_R M) = R$ because $(L :_R M)M \subseteq L$. Now since $Ann_R((L :_R M)M)(L :_R M) \subseteq Ann_R(M)$, we have $(L :_R M) + Ann_R(M) = Re + Ann_R(M)$ for some weak idempotent $e \in R$ by [14, Lemma 2 (i)]. It follows that $(0 :_M (L :_R M)) = (0 :_M e)$ for some weak idempotent $e \in R$ and so M is a co-Baer R-module. □

Theorem 3.21. *Let M be a comultiplication R-module. Then M is a co-Baer R-module if and only if for every submodule N of M we have M/N is a co-Baer R-module.*

Proof. The "only if" part is clear. Now let M be a co-Baer R-module and N be a submodule of M. Assume that L/N is a completely irreducible submodule of M/N. Then L is a completely irreducible submodule of M. Thus $(0 :_M (L :_R M)) = (0 :_M e)$ for some weak idempotent $e \in R$. So, $(L :_R M) = Re + Ann_R(M)$ by Lemma 3.19. Therefore,

$$(0 :_{M/N} (L/N :_R M/N)) = (N :_M (L :_R M))/N =$$
$$((N :_M e) \cap (N :_M Ann_R(M)))/N = (N :_M e)/N = (0 :_{M/N} e),$$

as desired □

Proposition 3.22. *Let M be a second R-module. Then M is a co-Baer R-module.*

Proof. Let L be a completely irreducible submodule of M. As M is second, $(L :_R M)M = 0$ or $M = (L :_R M)M \subseteq L$. Hence $(0 :_M (L :_R M)) = M = (0 :_M 0)$ or $(0 :_M (L :_R M)) = (0 :_M (M :_R M)) = (0 :_M 1)$, as needed. □

The following example shows that the converse of Proposition 3.22 is not true in general.

Example 3.23. *By Example 3.15, the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a co-Baer $\mathbb{Z}$-module. But the $\mathbb{Z}$-module $\mathbb{Z}_6$ is not a second $\mathbb{Z}$-module.*

Definition 3.24. *We say that a finitely cogenerated coreduced R-module M is a conormal module if for any two distinct maximal second submodules S_1 and S_2 of M we have $S_1 \cap S_2 = 0$.*

Proposition 3.25. *Let M be a finitely generated comultiplication co-Baer R-module. Then for any two distinct maximal second submodules S_1 and S_2 of M we have $S_1 \cap S_2 = 0$.*

Proof. First note that minimal submodules and hence second submodules and maximal second submodules exist in comultiplication R-modules [2, Theorem 7 (e)]. Let S_1 and S_2 be two distinct maximal second submodules of M. Then $Ann_R(S_1)$ and $Ann_R(S_2)$ are prime ideals of R containing $Ann_R(M)$. Thus $Ann_R(S_1)$ contains P_1 and $Ann_R(S_2)$ contains P_2, where P_1 and P_2 are prime ideals which are minimal over $Ann_R(M)$. Therefore $P_1/Ann_R(M)$ and $P_2/Ann_R(M)$ are minimal prime ideals of $R/Ann_R(M)$. By Theorem 3.13, $R/Ann_R(M)$ is a Baer ring. Hence $P_1/Ann_R(M) + P_2/Ann_R(M) = R/Ann_R(M)$, which implies that $P_1 + P_2 = R$. It follows that $Ann_R(S_1) + Ann_R(S_2) = R$ and so $S_1 \cap S_2 = 0$. □

Corollary 3.26. *Every finitely generated comultiplication co-Baer R-module is a conormal R-module.*

Proof. By using [2, Theorem 24], M is a finitely cogenerated R-module. Now the result follows from Theorem 3.8 and Proposition 3.25. □

Theorem 3.27. *Let M be a finitely generated comultiplication R-module. Then M is a conormal module if and only if $R/Ann_R(M)$ is a normal ring.*

Proof. ⇒: Let M be a conormal R-module. Since M is a coreduced R-module, we have $R/Ann_R(M)$ is reduced ring by [10, Proposition 2.16 (b)]. Take two distinct minimal prime ideals $P_1/Ann_R(M)$ and $P_2/Ann_R(M)$ of $R/Ann_R(M)$. Then P_1 and P_2 are two distinct minimal prime ideals of $Ann_R(M)$. By [10, Proposition 2.1 (a), (d)], $(0 :_M P_1)$ and $(0 :_M P_2)$ are two distinct maximal second submodules of M. Therefore $(0 :_M P_1) \cap (0 :_M P_2) = 0$ and so $(0 :_M P_1 + P_2) = 0$. Thus as M is a comultiplication R-module, $(P_1 + P_2)M = M$. Hence since M is finitely generated and $Ann_R(M) \subseteq P_1 + P_2$, we have $P_1 + P_2 = R$ by [8, Corollary 2.5]. Thus $P_1/Ann_R(M) + P_2/Ann_R(M) = R/Ann_R(M)$ and so $R/Ann_R(M)$ is a normal ring.

⇐: Let $R/Ann_R(M)$ be a normal ring. Then as $R/Ann_R(M)$ is a reduced ring, we have M is a coreduced module by [10, Proposition 2.16 (a)]. Now let S_1 and S_2 be two distinct maximal second submodules of M. Then $Ann_R(S_1)$ and $Ann_R(S_2)$ are two distinct minimal prime ideals of R containing $Ann_R(M)$ by [10, Proposition 2.1 (c)]. Thus $Ann_R(S_1)/Ann_R(M)$ and $Ann_R(S_2)/Ann_R(M)$ are two distinct minimal prime ideals of $R/Ann_R(M)$. Hence

$$Ann_R(S_1)/Ann_R(M) + Ann_R(S_2)/Ann_R(M) = R/Ann_R(M),$$

which implies that $Ann_R(S_1) + Ann_R(S_2) = R$. It follows that $S_1 \cap S_2 = 0$, as needed. □

Corollary 3.28. *Let M be a finitely generated multiplication and comultiplication R-module. Then M is a normal R-module if and only if M is a conormal R-module.*

Proof. This follows from [16, Proposition 2.12] and Theorem 3.27. □

Proposition 3.29. *Let M_i be an R_i-module for each $i = 1, 2, \dots, n$. Set $M := M_1 \times M_2 \times \cdots \times M_n$ and $R := R_1 \times R_2 \times \cdots \times R_n$. Then the following statements are equivalent:*

(a) M is a conormal R-module;

(a) M_i is a conormal R_i-module for each $i = 1, 2, \dots, n$.

Proof. First note that M is a finitely cogenerated coreduced R-module if and only if each M_i is a finitely cogenerated coreduced R_i-module for $i = 1, 2, \dots, n$.

$(a) \Rightarrow (b)$ Let M be a conormal R-module and choose $i \in \{1, 2, \dots, n\}$. Assume that S_{i1} and S_{i2} are two distinct maximal second submodules of M_i. Then $N_1 = 0\times 0\times \cdots \times S_{i1} \times \cdots \times 0$ and $N_2 = 0\times 0\times \cdots \times S_{i2} \times \cdots \times 0$ are two distinct maximal second submodules of M. As M is a conormal R-module, we have $N_1 \cap N_2 = 0$ which implies that $S_{i1} \cap S_{i2} = 0$. Therefore, M_i is a conormal R_i-module.

$(b) \Rightarrow (a)$ Let M_i be a conormal R_i-module for each $i = 1, 2, \dots, n$. Suppose that S and K are two maximal second submodules of M. Then $S = 0 \times 0 \times \times S_t \times \cdots \times 0$ and $K = 0 \times 0 \times \cdots \times K_j \times \cdots \times 0$ for some maximal second submodules S_t of M_t and K_j of M_j. If $j \neq t$, we have $S \cap K = 0$. So assume that $j = t$. Since $K \neq S$ and M_t is conormal, we have $S_t \neq K_t$ so that $S_t \cap K_t = 0$ which implies that $S \cap K = 0$. Therefore, M is a conormal R-module. □

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