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The answers to some questions on H-sober spaces

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Abstract. In this paper, we consider and solve several open problems posed by Xu in [11, 14]. Those open questions concern different categorical constructions of H-sober spaces and hyper-sober spaces. First, for an irreducible subset system H and a T_0 space X, we prove that H automatically satisfies property M, which was unknown before, hence we deduce that X is super H-sober iff X is H-sober and H satisfies property Q in the sense of [14]. Beyond the aforementioned work, many questions asked by Xu in [14] are also solved in the paper. Second, we derive the concrete forms of coequalizers in **H-Sob**. Finally, we obtain that the finite product of hyper-sober spaces is hyper-sober, which gives a positive answer to a question posed in [11].

1. Introduction

Sobriety, monotone convergence and well-filteredness are three of the most important and useful properties in non-Hausdorff topological spaces and domain theory (see [3],[4],[5] and [13]). In recent years, sober spaces, monotone convergence spaces (shortly called d-spaces), well-filtered spaces and their related structures have been introduced and investigated.

In [14], Xu provided a uniform approach to sober spaces, d-spaces and well-filtered spaces and developed a general frame for dealing with all these spaces. The concepts of irreducible subset systems (R-subset systems for short), H-sober spaces and super H-sober spaces for an R-subset system H were proposed. Let $\mathbf{Top}_0$ be the category of all T_0 spaces and **Sob** the category of all sober spaces. The category of all H-sober spaces (resp., super H-sober spaces) with continuous mappings is denoted by **H-Sob** (resp., **SH-Sob**). For any R-subset system H, it has been proven that **H-Sob** is a full subcategory of $\mathbf{Top}_0$ containing **Sob** and is closed with respect to homeomorphisms. Moreover, **H-Sob** is adequate (see Theorem 7.9 in [14]). Conversely, for a full subcategory **K** of $\mathbf{Top}_0$ containing **Sob**, suppose that **K** is adequate and closed with respect to homeomorphisms. There is a natural question: Is there an R-subset system H such that **K** = **H-Sob**? In this paper, we will prove the answer is positive. In [14], Xu also proposed an R-subset system $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ is said to satisfy property M if for any T_0 space X, $\mathcal{K} \in \mathrm{H}(P_S(X))$ and $A \in M(\mathcal{K})$, then $\{\uparrow(K \cap A) \mid K \in \mathcal{K}\} \in \mathrm{H}(P_S(X))$. Furthermore, he obtained some results under the assumption that H has property M. Then he posed the following problems:

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Problem 1. Let H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be an R-subset system (may not have property M) and $\{X_i \mid i \in I\}$ a family of super H-sober spaces. Is the product space $\prod_{i \in I} X_i$ super H-sober?

Problem 2. Let H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be an R-subset system, X a T_0 space and Y a super H-sober space. Is the function space $TOP(X, Y)$ equipped with the topology of pointwise convergence super H-sober?

Problem 3. Let H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be an R-subset system, X a super H-sober space and Y a T_0 space. For a pair of continuous mappings $f, g : X \to Y$, is the equalizer $E(f, g) = \{x \in X \mid f(x) = g(x)\}$ (as a subspace of X) super H-sober?

Problem 4. For an R-subset system H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$, is H **SH-Sob** complete?

Problem 5. If an R-subset system H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$ has property M, do the induced R-subset systems H^d, H^R and H^D have property M?

Problem 6. For an R-subset system H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$, is **SH-Sob** reflective in $\mathbf{Top}_0$? Or equivalently, for any T_0 space X, does the super H-sobrification of X exist?

Problem 7. Let H : $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be an R-subset system having property M. Is $\mathrm{R} : \mathcal{H} \to \mathcal{H}$, $\mathrm{H} \mapsto \mathrm{H}^R$ a closure operator?

In [15], Zhao and Ho introduced a new variant of sobriety, called hyper-sobriety. A topological space X is called hyper-sober if for any irreducible set F, there is a unique $x \in F$ such that $F \subseteq \mathrm{cl}(\{x\})$. Clearly, every hyper-sober space is sober. Later, Wen and Xu discussed some basic properties of hyper-sober spaces in [11]. Moreover, they posed a question:

Problem 8. Is the product space of two hyper-sober spaces again a hyper-sober space?

Let $\mathcal{A}$ and $\mathcal{B}$ be categories, $F : \mathcal{A} \to \mathcal{B}$ be left adjoint to $U : \mathcal{B} \to \mathcal{A}$. We know that U preserves limits and F preserves colimits (see [9]). Xu has proven that **H-Sob** is adequate (see Theorem 7.9 in [14]), in other words, it is reflective in $\mathbf{Top}_0$. In fact that $\mathbf{Top}_0$ is also reflective in **Top**, so **H-Sob** is reflective in **Top**. Since **Top** is complete and cocomplete (that is, limits and colimits all exist), we get that **H-Sob** is also complete and cocomplete. This implies that coequalizers in **H-Sob** exist. But we do not know its concrete forms. In Section 4, we investigate the coequalizers in **H-Sob**.

In Section 5, we find that property M mentioned above naturally holds for each R-subset system H. Based on this, for a T_0 space X, we deduce that X is super H-sober iff X is H-sober and H satisfies property Q in the sense of [14]. Additionally, we give positive answers to Problem 1 ∼ Problem 6. Furthermore, we prove Problem 7 holds.

In Section 6, we will give a positive answer to Problem 8.

2. Preliminaries

In this section, we briefly recall some standard definitions and notations to be used in this paper. For further details, refer to [3–6, 12].

Let P be a poset and $A \subseteq P$. We denote $\downarrow A = \{x \in P \mid x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P \mid x \geq a \text{ for some } a \in A\}$. For any $a \in P$, we denote $\downarrow\{a\} = \downarrow a = \{x \in P \mid x \leq a\}$ and $\uparrow\{a\} = \uparrow a = \{x \in P \mid x \geq a\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$).

The category of all topological spaces with continuous mappings is denoted by **Top**. For a topological space X, let $\mathcal{O}(X)$ (resp., $\Gamma(X)$) be the set of all open subsets (resp., closed subsets) of X. For a subset A of X, the closure of A is denoted by $\mathrm{cl}(A)$ or $\overline{A}$. We use $\leq_X$ to represent the specialization quasi-order of X, that is, $x \leq_X y$ iff $x \in \overline{\{y\}}$. A subset B of X is called *saturated* if B equals the intersection of all open sets containing it (equivalently, B is an upper set in the specialization quasi-order). Let $S(X) = \{\{x\} \mid x \in X\}$ and $S_c(X) = \{\overline{\{x\}} \mid x \in X\}$.

The category of all T_0 spaces with continuous mappings is denoted by $\mathbf{Top}_0$. For a T_0 space X and a nonempty subset A of X, A is called *irreducible* if for any $B, C \in \Gamma(X)$, $A \subseteq B \cup C$ implies $A \subseteq B$ or $A \subseteq C$. Denote by $\mathrm{Irr}(X)$ the set of all irreducible subsets of X and $\mathrm{Irr}_c(X)$ the set of all closed irreducible subsets of X. A topological space X is called *sober*, if for any $A \in \mathrm{Irr}_c(X)$, there is a unique point $a \in X$ such that $A = \overline{\{a\}}$. The category of all sober spaces with continuous mappings is denoted by **Sob**.

For a topological space X, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, let $\Box_{\mathcal{G}}(A) = \{G \in \mathcal{G} \mid G \subseteq A\}$ and $\Diamond_{\mathcal{G}}(A) = \{G \in \mathcal{G} \mid G \cap A \neq \emptyset\}$. The symbols $\Box_{\mathcal{G}}(A)$ and $\Diamond_{\mathcal{G}}(A)$ will be simply written as $\Box A$ and $\Diamond A$ respectively if there is no confusion.

The *upper Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Box U \mid U \in \mathcal{O}(X)\}$ as a base, and the resulting space is denoted by $P_S(\mathcal{G})$. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U \mid U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathrm{Irr}(X)$, then $\{\Diamond U \mid U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$.

We shall use $K(X)$ to denote the set of all nonempty compact saturated subsets of X ordered by reverse inclusion, that is, for $K_1, K_2 \in K(X)$, $K_1 \leq_{K(X)} K_2$ iff $K_2 \subseteq K_1$. The space $P_S(K(X))$ denoted shortly by $P_S(X)$, is called the *Smyth power space* or *upper space* of X ([6]).

Lemma 2.1. *(Topological Rudin Lemma) [6] Let X be a T_0 space, C a closed subset of X and $\mathcal{K}$ an irreducible subset of the Smyth power space $P_S(X)$. If C intersects all members of $\mathcal{K}$, then there exists a minimal (irreducible) closed subset A of C that intersects all members of $\mathcal{K}$.*

Definition 2.2. *[1] Let P be a poset. A map $c : P \to P$ is called a* closure operator *(on P) if, for all $x, y \in P$,*

(1) $x \leq y \Longrightarrow c(x) \leq c(y)$ (c is monotone),

(2) $x \leq c(x)$ (c is expansive),

(3) $c(c(x)) = c(x)$ (c is idempotent).

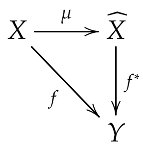
Lemma 2.3. *[7, 10] Let X be an T_0 space. If $\mathcal{K} \in K(P_S(X))$, then $\bigcup \mathcal{K} \in K(X)$.*

Corollary 2.4. *[7, 10] For a T_0 space X, the mapping $\bigcup : P_S(P_S(X)) \to P_S(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.*

3. The K-category and H-sober spaces

In what follows, The category **K** always refers to a full subcategory $\mathbf{Top}_0$ containing **Sob**, the objects of **K** are called **K**-*spaces*.

Definition 3.1. *[12] Let X be a T_0 space. A* **K**-reflection *of X is a pair $\langle \widehat{X}, \mu \rangle$ consisting of a **K**-space $\widehat{X}$ and a continuous mapping $\mu : X \to \widehat{X}$ satisfying that for any continuous mapping $f : X \to Y$ to a **K**-space, there exists a unique continuous mapping $f^* : \widehat{X} \to Y$ such that $f^* \circ \mu = f$, that is, the following diagram commutes.*



By a standard argument, **K**-reflections, if they exist, are unique to homeomorphisms. We shall use X^k to indicate the space of the **K**-reflection of X if it exists.

Definition 3.2. *[12] Let (X, τ) be a T_0 space. A nonempty subset A of X is called* **K**-determined *provided for any continuous mapping $f : X \to Y$ to a **K**-space Y, there exists a unique $y_A \in Y$ such that $\mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(\{y_A\})$. Clearly, a subset A of a space X is a **K**-determined set iff $\mathrm{cl}(A)$ is a K-determined set. Denote by $\mathrm{K}^d(X)$ the set of all **K**-determined subsets of X and $\mathrm{K}^d_c(X)$ the set of all closed **K**-determined subsets of X.*

Lemma 3.3. *Let X, Y be two T_0 spaces. If $f : X \to Y$ is a continuous mapping and $A \in \mathrm{K}^d(X)$, then $f(A) \in \mathrm{K}^d(Y)$.*

Proof. Let Z be a **K**-space and $g : Y \to Z$ is continuous. Note that $g \circ f : X \to Z$ is continuous and $A \in \mathrm{K}^d(X)$, then there exists an element $z \in Z$ such that $\overline{g \circ f(A)} = \overline{g(f(A))} = \overline{\{z\}}$. So $f(A) \in \mathrm{K}^d(Y)$. □

Definition 3.4. *[12] A full subcategory* **K** *of* $\mathbf{Top}_0$ *is said to be* closed with respect to homeomorphisms *if homeomorphic copies of* **K**-*spaces are* **K**-*spaces.*

Definition 3.5. *[12]* **K** *is called* adequate *if for any T_0 space X, $P_H(\mathrm{K}^d_c(X))$ is a* **K**-*space.*

Corollary 3.6. *[12] Assume that* **K** *is adequate and closed with respect to homeomorphisms. Then for any T_0 space X, the following conditions are equivalent:*

(1) X is a **K***-space.*

(2) $\mathrm{K}_c^d(X) = S_c(X)$, that is, for each $A \in \mathrm{K}_c^d(X)$, there exists an $x \in X$ such that $A = \overline{\{x\}}$.

(3) $X \cong X^k$.

The category of all sets with mappings is denoted by **Set**.

Definition 3.7. *[14] A covariant functor* $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *is called a* subset system on $\mathbf{Top}_0$ *provided that the following two conditions are satisfied:*

(1) $S(X) \subseteq \mathrm{H}(X) \subseteq 2^X$ (the set of all subsets of X) for each $X \in ob(\mathbf{Top}_0)$.

(2) For any continuous mapping $f : X \to Y$ in $\mathbf{Top}_0$, $\mathrm{H}(f)(A) = f(A) \in \mathrm{H}(Y)$ for all $A \in \mathrm{H}(X)$.

For a subset system $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *and a T_0 space X, let $\mathrm{H}_c(X) = \{\overline{A} \mid A \in \mathrm{H}(X)\}$. We call $A \subseteq X$ an* H-set *if $A \in \mathrm{H}(X)$. The sets in $\mathrm{H}_c(X)$ are called* closed H-sets.

Definition 3.8. *[14] A subset system* $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *is called an* irreducible subset system*, or an* R-subset system *for short, if $\mathrm{H}(X) \subseteq \mathrm{Irr}(X)$ for all $X \in ob(\mathbf{Top}_0)$. The family of all R-subset systems is denoted by $\mathcal{H}$. Define a partial order $\leq$ on $\mathcal{H}$ by $\mathrm{H}_1 \leq \mathrm{H}_2$ iff $\mathrm{H}_1(X) \subseteq \mathrm{H}_2(X)$ for all $X \in ob(\mathbf{Top}_0)$.*

Definition 3.9. *[14] Let* $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *be an R-subset system. A T_0 space X is called* H-sober *if for any $A \in \mathrm{H}(X)$, there is a (unique) point $x \in X$ such that $\overline{A} = \overline{\{x\}}$ or, equivalently, if $\mathrm{H}_c(X) = S_c(X)$. The category of all* H*-sober spaces with continuous mappings is denoted by* **H-Sob***.*

It is not difficult to verify that **Sob** $\subseteq$ **H-Sob**. By Definition 3.1, for **K** = **H-Sob**, the **K**-reflection of X is called the H-*sober reflection* of X, or the H-*sobrification* of X and X^h to denote the space of H-sobrification of X if it exists. Moreover, a **K**-determined set of X in Definition 3.2 is called a H-*sober determined set* of X, Denote by $\mathrm{H}^d(X)$ the set of all H-sober determined subsets of X. The set of all closed H-sober determined subsets of X is denoted by $\mathrm{H}_c^d(X)$.

Theorem 3.10. *[14] Let* $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *be an R-subset system. Then* **H-Sob** *is adequate. Therefore, for any T_0 space X, $X^h = P_H(\mathrm{H}_c^d(X))$ with the canonical topological embedding η_X^h: $X \to X^h$ is the* H*-sobrification of X, where $\eta_X^h(x) = \overline{\{x\}}$ for all $x \in X$.*

Remark 3.11. *If X is only a topological space, then $\langle P_H(\mathrm{H}_c^d(X)), \eta_X^h \rangle$ is the* H*-sobrification of X, where $\eta_X^h(x) = \overline{\{x\}}$ for all $x \in X$.*

Let $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be an R-subset system. In [14], it has been shown that **H-Sob** is a full subcategory of $\mathbf{Top}_0$ containing **Sob** and is closed with respect to homeomorphisms. Moreover, **H-Sob** is adequate by Theorem 3.10.

Conversely, suppose that **K** is adequate and closed with respect to homeomorphisms. Is there an R-subset system H such that **K** = **H-Sob**? The answer is positive, we will investigate it in Section 5.

4. Coequalizers in H-Sob

For any topological space X, since the H-sobrification of X exists, we have that **H-Sob** is reflective in **Top**. The fact that **Top** is complete and cocomplete (that is, limits and colimits all exist), so **H-Sob** is also complete and cocomplete. This implies that coequalizers in **H-Sob** exist. But we do not know its concrete forms. In this section, inspired by the construction of coequalizers in **Dcpo** (see [2] and [8]), we will investigate coequalizers in **H-Sob**.

Definition 4.1. *[9] Let C be a category and $f, g : A \to B$ be a pair of parallel arrows. A* coequalizer *of f and g is an object C together with an arrow $h : B \to C$ with the following properties:*

(1) $h \circ f = h \circ g$;

(2) for each arrow $k : B \to D$ such that $k \circ f = k \circ g$, there is a unique arrow $l : C \to D$ such that $l \circ h = k$.

In order to give a specific characterization of coequalizers in **H-Sob**, first, we introduce the concept of R-topology on a topological space.

Definition 4.2. *[4] Let X be a set and R an equivalence relation on X. A subset A of X is* R-saturated *if, for every $a \in A$, for every $x \in X$ such that $a\ R\ x$, x is also in A. Equivalently, A is R-saturated iff it is a union of equivalence classes for R.*

Let (X, τ) be a topological space and R an equivalence relation on X. We call a subset U of X *R-open* if it is an R-saturated open set. It is easy to see that $\emptyset, X$ are R-open, the intersection of any finite family of R-open sets is R-open and the union of any family of R-open sets is R-open. Then the family of all R-open sets in X precisely forms a new topology on X. We call it an *R-topology* and denote it by τ_R. Note that $\tau_R \subseteq \tau$. Dually, an R-closed subset A of X is the complement $X \backslash U$ of an R-open subset U of X. For a subset A of X, the closure of A in (X, τ_R) is denoted by $\mathrm{cl_R}(A)$.

For any $(X, \tau), (Y, \mu) \in$ **H-Sob** and continuous maps $f, g : (X, \tau) \to (Y, \mu)$, define

$$\mathcal{K} = \{k \mid \text{there exists an H-sober space } (Z, \nu) \text{ such that } k : (Y, \mu) \to (Z, \nu) \text{ is continuous and } k \circ f = k \circ g\}$$

and

$$R = \{(x, y) \in Y \times Y \mid \text{for any } k \in \mathcal{K},\ k(x) = k(y)\}.$$

Then R is an equivalence relation on Y. It is not difficult to verify that $\{(f(x), g(x)) \mid x \in X\} \subseteq R$.

Let $\mathrm{H}_c^R(Y) = \{A \subseteq Y \mid A$ is R-closed and for any continuous map $k : (Y, \mu) \to (Z, \nu) \in \mathcal{K}$, there exists a unique $z \in Z$ such that $\overline{k(A)} = \overline{\{z\}}\}$. For the topological space (Y, μ_R), we have the following lemma:

Lemma 4.3. *For (Y, μ_R), $Y^h = P_H(\mathrm{H}_c^R(Y))$ with the canonical continuous map η_Y^h: $(Y, \mu_R) \to Y^h$ is the* H-sobrification *of (Y, μ_R), where $\eta_Y^h(y) = \mathrm{cl_R}(\{y\})$ for all $y \in Y$.*

Proof. By Remark 3.11, we only need to prove that $\mathrm{H}_c^R(Y) = \mathrm{H}_c^d(Y)$. Define

$$\mathcal{L} = \{l \mid \text{there exists an H-sober space } (Z, \nu) \text{ such that } l : (Y, \mu_R) \to (Z, \nu) \text{ is continuous}\}.$$

We claim that $\mathcal{K} = \mathcal{L}$. Since $\mu_R \subseteq \mu$ and $l : (Y, \mu_R) \to (Z, \nu)$ is continuous, this implies that $l : (Y, \mu) \to (Z, \nu)$ is continuous. Moreover, for any $x \in X$, we have $lf(x) = lg(x)$. Suppose not, there exists an element $x \in X$ such that $lf(x) \neq lg(x)$. Without loss of generality, assume that $lf(x) \nleq lg(x)$, then $lf(x) \in Z \backslash {\downarrow} lg(x)$. So $f(x) \in l^{-1}(Z \backslash {\downarrow} lg(x))$. Since Z is an H-sober space, it is T_0, so $Z \backslash {\downarrow} lg(x)$ is an open subset in Z. Because $l : (Y, \mu_R) \to (Z, \nu)$ is continuous, we have that $l^{-1}(Z \backslash {\downarrow} lg(x))$ is R-open. As $(f(x), g(x)) \in R$ and $f(x) \in l^{-1}(Z \backslash {\downarrow} lg(x))$, we infer that $g(x) \in l^{-1}(Z \backslash {\downarrow} lg(x))$. Thus $lg(x) \in Z \backslash {\downarrow} lg(x)$, which is a contradiction. Therefore, $lf(x) = lg(x)$. We conclude that $\mathcal{L} \subseteq \mathcal{K}$. Conversely, suppose that $k \in \mathcal{K}$. Let $U \in \nu$. Then $k^{-1}(U)$ is open in Y. It remains to show that $k^{-1}(U)$ is R-saturated. For every x in $k^{-1}(U)$ and $x\ R\ a$, as $k \in \mathcal{K}$, we have $k(x) = k(a)$ and $k(x) \in U$. Hence, $k(a) \in U$, that is $a \in k^{-1}(U)$. Thus $k : (Y, \mu_R) \to (Z, \nu)$ is continuous. So $\mathcal{K} \subseteq \mathcal{L}$. Therefore, $\mathcal{K} = \mathcal{L}$. By the definitions of $\mathrm{H}_c^R(Y)$ and $\mathrm{H}_c^d(Y)$, we could get $\mathrm{H}_c^R(Y) = \mathrm{H}_c^d(Y)$. □

By the above lemma, we could see $P_H(\mathrm{H}_c^R(Y))$ is H-sober and η_Y^h: $(Y, \mu_R) \to P_H(\mathrm{H}_c^R(Y))$ is continuous. Since $\mu_R \subseteq \mu$, we have η_Y^h: $(Y, \mu) \to P_H(\mathrm{H}_c^R(Y))$ is also continuous.

Theorem 4.4. *For $(X, \tau), (Y, \mu) \in$ **H-Sob** and continuous maps $f, g : (X, \tau) \to (Y, \mu)$, $P_H(\mathrm{H}_c^R(Y))$ together with an arrow η_Y^h: $(Y, \mu) \to P_H(\mathrm{H}_c^R(Y))$ is a coequalizer of f and g.*

Proof. Let $x \in X$. Since $(f(x), g(x)) \in R$, $\mathrm{cl}_R(\{f(x)\}) = \mathrm{cl}_R(\{g(x)\})$. This implies $\eta^h_Y f(x) = \eta^h_Y g(x)$. That is, $\eta^h_Y \circ f = \eta^h_Y \circ g$. For any H-sober space (Z, ν) and any continuous map $k : (Y, \mu) \to (Z, \nu)$ such that $k \circ f = k \circ g$, from the proof of Lemma 4.3, we could see $k : (Y, \mu_R) \to (Z, \nu)$ is continuous. Because $P_H(\mathrm{H}^R_c(Y))$ is the H-sobrification of (Y, μ_R), there exists a unique continuous mapping $k^* : P_H(\mathrm{H}^R_c(Y)) \to (Z, \nu)$ such that $k^* \circ \eta^h_Y = k$. So $P_H(\mathrm{H}^R_c(Y))$ together with an arrow η^h_Y: $(Y, \mu) \to P_H(\mathrm{H}^R_c(Y))$ is the coequalizer of f and g. □

5. On some problems about super H-sober spaces

In this section, we will prove that for each irreducible subset system (R-subset system for short) H, property M mentioned in [14] naturally holds. Based on this result, we get that Problem 1∼Problem 7 hold. Furthermore, we generalize some results in [14].

Lemma 5.1. *For a T_0 space X, $S(X) \subseteq \mathrm{K}^d(X) \subseteq \mathrm{Irr}(X)$.*

Proof. Clearly, $S(X) \subseteq \mathrm{K}^d(X)$. Suppose that $A \in \mathrm{K}^d(X)$. Consider the sobrification X^s $(= P_H(\mathrm{Irr}_c(X)))$ of X and the canonical topological embedding $\eta_X : X \to X^s$ defined by $\eta_X(x) = \overline{\{x\}}$. Then there is a $B \in \mathrm{Irr}_c(X)$ such that $\overline{\eta_X(A)} = \overline{\{B\}}$. It is easy to check that $\overline{A} = B$. Hence, $A \in \mathrm{Irr}(X)$. □

Theorem 5.2. *Suppose that* **K** *is adequate and closed with respect to homeomorphisms. Then for a covariant functor* $\mathrm{K} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *defined by* $\mathrm{K}^d(X)$, $\forall\, X \in ob(\mathbf{Top}_0)$, *the following statements hold:*

(1) $\mathrm{K} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *is an R-subset system.*

(2) **K=K-sober**, *that is for each* **K***-space X and any* $A \in \mathrm{K}^d(X)$, *there exists a (unique) element* $x \in X$ *such that* $\overline{A} = \overline{\{x\}}$.

Proof. These statements directly follow from Lemma 3.3, Lemma 5.1 and Corollary 3.6. □

For a T_0 space X and $\mathcal{K} \subseteq K(X)$, let $M(\mathcal{K}) = \{A \in \Gamma(X) \mid A \cap K \neq \emptyset$ for all $K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \diamond A$) and $m(\mathcal{K}) = \{A \in \Gamma(X) \mid A$ is a minimal member of $M(\mathcal{K})\}$ ([12]).

In [14], Xu proposed that an R-subset system $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ is said to satisfy property M if for any T_0 space X, $\mathcal{K} \in \mathrm{H}(P_S(X))$ and $A \in M(\mathcal{K})$, then $\{\uparrow(K \cap A) \mid K \in \mathcal{K}\} \in \mathrm{H}(P_S(X))$. Furthermore, he proved some conclusions under the assumption that H has property M. In the following, we find that property M naturally holds for each R-subset system H.

Proposition 5.3. *Let* $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *be an R-subset system. Then property M holds.*

Proof. Suppose that X is a T_0 space, $\mathcal{K} \in \mathrm{H}(P_S(X))$ and $A \in M(\mathcal{K})$. Take a map $f : P_S(X) \to P_S(X)$ defined by

$$f(K) = \uparrow(K \cap A)$$

for any $K \in K(X)$. It is straightforward to check that f is well-defined.

Claim: f is continuous.

For any $U \in \mathcal{O}(X)$, $f^{-1}(\Box U) = \{K \in K(X) \mid \uparrow(K \cap A) \in \Box U\} = \{K \in K(X) \mid \uparrow(K \cap A) \subseteq U\} = \Box((X \backslash A) \cup U)$. Hence, $f^{-1}(\Box U)$ is open in $K(X)$. So f is continuous.

Since H is an R-subset system and $\mathcal{K} \in \mathrm{H}(P_S(X))$, we have that $f(\mathcal{K}) = \{\uparrow(K \cap A) \mid K \in \mathcal{K}\} \in \mathrm{H}(P_S(X))$, that is, property M holds. □

By Theorem 6.19, Theorem 6.20, Proposition 6.21, Corollary 6.22, Theorem 7.16 in [14] and Proposition 5.3, we get that Problem 1 ∼ Problem 6 hold.

Definition 5.4. *[14] Let* $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *be an R-subset system. A T_0 space X is called* super H-sober *provided its Smyth power space $P_S(X)$ is* H*-sober. The category of all super* H*-sober spaces with continuous mappings is denoted by* **SH-Sob**.

Definition 5.5. *[14] Let* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *be an R-subset system and X a T_0 space. A nonempty subset A of X is said to have* H-Rudin property, *if there exists $\mathcal{K} \in$ H$(P_S(X))$ such that $\overline{A} \in m(\mathcal{K})$, that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$. Let* $\mathrm{H}^R(X) = \{A \subseteq X \mid A$ *has* H*-Rudin property*$\}$. *The sets in* $\mathrm{H}^R(X)$ *will also be called* H*-Rudin sets.*

Lemma 5.6. *[14] Let* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *be an R-subset system and X a T_0 space. Then* $\mathrm{H}(X) \subseteq \mathrm{H}^R(X) \subseteq \mathrm{Irr}(X)$.

Lemma 5.7. *[14] Let* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *be an R-subset system. Then* $\mathrm{H}^R : \mathbf{Top}_0 \longrightarrow$ **Set** *is an R-subset system, where for any continuous mapping $f : X \to Y$ in $\mathbf{Top}_0$, $\mathrm{H}^R(f) : \mathrm{H}^R(X) \to \mathrm{H}^R(Y)$ is defined by $\mathrm{H}^R(f)(A) = f(A)$ for each $A \in \mathrm{H}^R(X)$.*

Theorem 5.8. *[14] Let* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *be an R-subset system and X a T_0 space. Then the following conditions are equivalent:*

(1) X is super H*-sober.*

(2) X is H^R*-sober.*

Definition 5.9. *[14] An R-subset system* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *is said to satisfy* property *Q if for any $\mathcal{K} \in$ H$(P_S(X))$ and any $A \in M(\mathcal{K})$, A contains a closed* H*-set C such that $C \in M(\mathcal{K})$.*

In [14], Theorem 5.12 pointed out that if X is H-sober and H has property Q, then X is super H-sober. The following Theorem will show that the converse also holds.

Theorem 5.10. *Let* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *be an R-subset system. For a T_0 space X, the following two conditions are equivalent:*

(1) X is H*-sober and* H *has property Q.*

(2) X is super H*-sober.*

Proof. (1) ⇒ (2): Follows directly from Theorem 5.12 in [14].

(2) ⇒ (1): We only need to show that H has property Q. For any $\mathcal{K} \in$ H$(P_S(X))$ and any $A \in M(\mathcal{K})$, by Rudin Lemma, we have that there exists a minimal closed subset $B \subseteq A$ such that $B \in M(\mathcal{K})$. Thus $B \in \mathrm{H}^R(X)$. Since X is super H-sober, by Theorem 5.8, we have that X is H^R-sober. Therefore, there exists $x \in X$ such that $B = \overline{\{x\}}$. This implies that B is a closed H-set. Thus H has property Q. □

In the following, we will prove that for an R-subset system H : $\mathbf{Top}_0 \longrightarrow$ **Set**, R : $\mathcal{H} \to \mathcal{H}$, defined by H $\mapsto \mathrm{H}^R$ is a closure operator.

Theorem 5.11. *Let* H : $\mathbf{Top}_0 \longrightarrow$ **Set** *be an R-subset system. Then* R : $\mathcal{H} \to \mathcal{H}$*, defined by* H $\mapsto \mathrm{H}^R$ *is a closure operator.*

Proof. For any R-subset system H, by Lemma 5.7, we have that H^R is also an R-subset system. So R is well-defined.

Claim 1: R is monotone, that is, for $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{H}$ and $\mathrm{H}_1 \leq \mathrm{H}_2$, $\mathrm{H}_1^R \leq \mathrm{H}_2^R$.

Assume that X is a T_0 space. For any $A \in \mathrm{H}_1^R(X)$, by Definition 5.5, there exists $\mathcal{K} \in \mathrm{H}_1(P_S(X))$ such that $\overline{A} \in m(\mathcal{K})$. Since $\mathrm{H}_1 \leq \mathrm{H}_2$, that is, $\mathrm{H}_1(P_S(X)) \subseteq \mathrm{H}_2(P_S(X))$, we have that $\mathcal{K} \in \mathrm{H}_2(P_S(X))$ and hence, $A \in \mathrm{H}_2^R(X)$. So $\mathrm{H}_1^R(X) \subseteq \mathrm{H}_2^R(X)$ for any T_0 space X, which implies that $\mathrm{H}_1^R \leq \mathrm{H}_2^R$.

Claim 2: R is expansive, that is, for any H $\in \mathcal{H}$, H $\leq \mathrm{H}^R$.

Follows directly from Lemma 5.6.

Claim 3: R is idempotent, that is, for any H $\in \mathcal{H}$, $(\mathrm{H}^R)^R = \mathrm{H}^R$.

Since R is expansive, we have that $\mathrm{H}^R \leq (\mathrm{H}^R)^R$. Conversely, we only need to show that $(\mathrm{H}^R)^R(X) \subseteq \mathrm{H}^R(X)$ for any T_0 space X. Suppose that $A \in (\mathrm{H}^R)^R(X)$. By Definition 5.5, there exists $\mathcal{A} \in \mathrm{H}^R(P_S(X))$ such that $\overline{A} \in m(\mathcal{A})$. For $\mathcal{A}$, again by Definition 5.5, there exists $\mathcal{K} = \{\mathcal{K}_i\}_{i \in I} \in$ H$(P_S(P_S(X)))$ such that $\overline{\mathcal{A}} \in m(\mathcal{K})$. For any $i \in I$, let $K_i = \bigcup \uparrow_{K(X)}(\mathcal{K}_i \bigcap \overline{\mathcal{A}}) = \bigcup(\mathcal{K}_i \bigcap \overline{\mathcal{A}})$. Then by Lemma 2.3, Corollary 2.4 and property M of H, we have $\{K_i\}_{i\in I} \in \mathrm{H}(P_S(X))$ and $K_i \in \overline{\mathcal{A}}$ for each $i \in I$ since $\overline{\mathcal{A}}$ is a lower set.

Claim 3.1: $\overline{A} \in M(\{K_i\}_{i\in I})$.

Suppose not, there exists $i \in I$ such that $\overline{A} \cap K_i = \emptyset$. That is $K_i \subseteq X \backslash \overline{A}$. So $K_i \in \Box(X\backslash\overline{A})$. Since $K_i \in \overline{\mathcal{A}}$, there exists $K \in \Box(X\backslash\overline{A}) \bigcap \mathcal{A}$, which contradicts the fact that $\overline{A} \in m(\mathcal{A})$.

Claim 3.2: $\overline{A} \in m(\{K_i\}_{i\in I})$.

Let $B \subseteq \overline{A}$ be a closed subset and $B \in M(\{K_i\}_{i\in I})$. Then for any $i \in I$, there exists an element $N \in \mathcal{K}_i \bigcap \overline{\mathcal{A}}$ such that $B \cap N \neq \emptyset$, it is $\Diamond B \bigcap \mathcal{K}_i \bigcap \overline{\mathcal{A}} \neq \emptyset$. By the minimality of $\overline{\mathcal{A}}$, we have $\overline{\mathcal{A}} = \Diamond B \bigcap \overline{\mathcal{A}}$, and consequently, $\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq \Diamond B$. So $B \in M(\mathcal{A})$. By the minimality of $\overline{A}$, we have $\overline{A} = B$. Thus $\overline{A} \in m(\{K_i\}_{i\in I})$.

Therefore, for A, there exists $\{K_i\}_{i\in I} \in \mathrm{H}(P_S(X))$ such that $A \in m(\{K_i\}_{i\in I})$, which means $A \in \mathrm{H}^R(X)$. □

6. The finite product of hyper-sober spaces

The concept of hyper-sober spaces was introduced in [15]. And in [11], Wen and Xu gave a counterexample to show that the product of a countable infinite family of hyper-sober spaces is not hyper-sober in general. Meantime, they posed the following question:

Is the product space of two hyper-sober spaces again a hyper-sober space?

In this section, we will give a positive answer to the above question.

Definition 6.1. *([15]) A topological space X is called* hyper-sober *if for any irreducible set F, there is a unique $x \in F$ such that $F \subseteq \mathrm{cl}(\{x\})$.*

Lemma 6.2. *([14]) Let X be a space. Then the following conditions are equivalent for a subset $A \subseteq X$:*

(1) A is an irreducible subset of X.

(2) $\mathrm{cl}_X(A)$ is an irreducible subset of X.

Lemma 6.3. *([14]) If $f : X \to Y$ is continuous and $A \in \mathrm{Irr}(X)$, then $f(A) \in \mathrm{Irr}(Y)$.*

Corollary 6.4. *([14]) Let $\{X_i\}_{i\in I}$ be a family of T_0 spaces and $X = \prod_{i\in I} X_i$ the product space. If $A \in \mathrm{Irr}_c(X)$, then $A = \prod_{i\in I} p_i(A)$ and $p_i(A) \in \mathrm{Irr}_c(X_i)$ for each $i \in I$.*

Theorem 6.5. *Let X and Y be two hyper-sober spaces. Then the product space $X \times Y$ is also hyper-sober.*

Proof. Let A be an irreducible subset in $X \times Y$. Suppose $P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ are projections, respectively. Note that P_X and P_Y are continuous. By Lemma 6.3, $P_X(A) \in \mathrm{Irr}(X)$ and $P_Y(A) \in \mathrm{Irr}(Y)$. Since X and Y are hyper-sober, there exist $x \in P_X(A)$ and $y \in P_Y(A)$ such that $P_X(A) \subseteq \mathrm{cl}(\{x\})$ and $P_Y(A) \subseteq \mathrm{cl}(\{y\})$. This implies that $A \subseteq P_X(A) \times P_Y(A) \subseteq \mathrm{cl}(\{x\}) \times \mathrm{cl}(\{y\}) = \downarrow(x, y)$, and $(x, y) \in P_X(A) \times P_Y(A) \subseteq P_X(\overline{A}) \times P_Y(\overline{A}) = \overline{A}$ by Corollary 6.4. It is sufficient to prove that $(x, y) \in A$.

Claim: $x \notin \overline{P_X(A)\backslash\{x\}}$ and $y \notin \overline{P_Y(A)\backslash\{y\}}$.

Suppose not, $x \in \overline{P_X(A)\backslash\{x\}}$. One can directly get $\overline{P_X(A)\backslash\{x\}} = \downarrow x$. Then $P_X(A)\backslash\{x\} \in \mathrm{Irr}(X)$ by Lemma 6.2. Again since X is hyper-sober, there is an element $a \in P_X(A)\backslash\{x\}$ such that $P_X(A)\backslash\{x\} \subseteq \downarrow a$. This implies that $x \in X \setminus \downarrow a$. Thus $(P_X(A)\backslash\{x\}) \cap (X \setminus \downarrow a) \neq \emptyset$, which contradicts $P_X(A)\backslash\{x\} \subseteq \downarrow a$. So $x \notin \overline{P_X(A)\backslash\{x\}}$. For $y \notin \overline{P_Y(A)\backslash\{y\}}$, the proof is similar to that the case $x \notin \overline{P_X(A)\backslash\{x\}}$.

Therefore, there exist open neighborhoods U of x and V of y such that $U \cap (P_X(A)\backslash\{x\}) = \emptyset$ and $V \cap (P_Y(A)\backslash\{y\}) = \emptyset$, respectively. Since $(x, y) \in U \times V$ and $(x, y) \in \mathrm{cl}(A)$, there exists $(b, c) \in (U \times V) \cap A$. This implies that $b \in U \cap P_X(A)$ and $c \in V \cap P_Y(A)$. So $b = x$ and $c = y$, and hence, $(x, y) \in A$. □

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