Filomat 38:30 (2024), 10519–10527
https://doi.org/10.2298/FIL2430519Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The solvability conditions for the inverse problem of Re-nnd and Re-pd matrices on a linear manifold

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Abstract. Let the linear manifold $\mathcal{S}$ be

$$\mathcal{S} = \left\{ A \in \mathbb{C}^{n\times n} \middle| f(A) = \|AX_1 - Z\|^2 + \|X_1^* A - W^*\|^2 = min, X_1, Z, W \in \mathbb{C}^{n\times k} \right\},$$

where $\|.\|$ is Frobenius norm. We consider the following problem:
Problem P. Given $Y_2 \in \mathbb{C}^{n\times m}$, $X_2 \in \mathbb{C}^{n\times p}$, $D \in \mathbb{C}^{m\times p}$, find $A \in \mathbb{C}^{n\times n}_{\geq 0}(\mathcal{S})$ $(\mathbb{C}^{n\times n}_{>0}(\mathcal{S}))$ such that

$$Y_2^* A X_2 = D,$$

where $\mathbb{C}^{n\times n}_{\geq 0}(\mathcal{S}) = \left\{ A \in \mathbb{C}^{n\times n} \middle| \frac{1}{2}(A + A^*) \geq 0, \forall A \in \mathcal{S} \right\} \left(\mathbb{C}^{n\times n}_{>0}(\mathcal{S}) = \left\{ A \in \mathbb{C}^{n\times n} \middle| \frac{1}{2}(A + A^*) > 0, \forall A \in \mathcal{S} \right\} \right)$.

In this paper, we consider a class of inverse problems for Re-nnd and Re-pd matrices on a linear manifold. Utilizing variable substitution, the generalized inverses, and some matrix decompositions, the solvability conditions are obtained and the representation of the general solution for Problem P are given.

1. Introduction

Throughout this paper, we will adopt the following terminology, $\mathbb{C}^{m\times n}$ denote the set of all complex $m \times n$ matrix, M^*, $\mathcal{R}(M)$, $\mathcal{N}(M)$ and $M^\dagger$ denote the conjugate transpose, the range space, the null space and the Moore-Penrose inverse (MPI) of a matrix $M \in \mathbb{C}^{m\times n}$, respectively. Also, the symbols E_M and F_M stand for the two orthogonal projectors (OPs) $E_M = I_m - MM^\dagger$, $F_M = I_n - M^\dagger M$ induced by $M \in \mathbb{C}^{m\times n}$. Let the largest singular value of the matrix M be σ, if $\sigma \leq 1$ (< 1), we say that M is a (strict) contraction matrix (CM (SCM)). For matrices $M = (\alpha_{ij}) \in \mathbb{C}^{m\times n}, N = (\beta_{ij}) \in \mathbb{C}^{m\times n}$, $M * N$ is used to define the Hadamard product of M and N , that is, $M * N = (\alpha_{ij}\beta_{ij}) \in \mathbb{C}^{m\times n}$.

Re-pd matrices play a crucial role in the context of stability analysis within the realms of control theory. If given matrix $M \in \mathbb{C}^{n\times n}$, Duan et al. [4, 5] propose a new criterion that if $-M < 0$, then the system is Hurwitz stable. The result has been widely applied to solve some problems in Cyber–security control,

2020 *Mathematics Subject Classification*. Primary 15A39; Secondary 15A24.
Keywords. variable substitution; inverse problem; Re-nnd (Re-pd); generalized singular value decomposition.
Received: 29 June 2024; Accepted: 10 September 2024
Communicated by Dragana Cvetković Ilić
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Pinning control, Spatial quasi-neutral dynamics, DC microgrid, control of power networks, Cyber-Physical Systems [8, 11, 15, 18, 20, 26]. Due to its significant applications, there has been considerable research on the Re-nnd and Re-pd (see [24, p.1, Definition1.1]) solutions of the matrix equations (MEs) [12, 23, 24].

As a kind of classical ME, the ME

$$AXB = D \tag{1}$$

has certainly received extensive attention. In 2008, Dragana [2] discussed the Re-nnd solutions of (1) with the constraints that A, B are nonnegative definite by g–inverses. Different from [2], Yuan and Zuo [25] considered the Re-nnd and Re-pd solutions to (1) without any constraints on coefficient matrices by utilizing the MPIs and OPs. Setting $B = I$ in (1), we have $AX = D$. In the early 20th century, Wu et al. [10, 21, 22] discussed the Re-pd and Re-nnd solutions to the matrix inverse problem (IP) $AX = D$. However, it appears that there is little research on the IPs of Re-pd and Re-nnd to (1). Concretely, given $Y \in \mathbb{C}^{n\times m}$, $X \in \mathbb{C}^{n\times p}$, $D \in \mathbb{C}^{m\times p}$, seeking $A \in \mathbb{C}^{n\times n}_{\geq 0}(\mathbb{C}^{n\times n}_{>0})$ ($\mathbb{C}^{n\times n}_{\geq 0}(\mathbb{C}^{n\times n}_{>0})$ is Re-nnd (Re-pd) set) such that

$$Y^{*}AX = D. \tag{2}$$

In particular, when $Y^{*} = I, D = X\Lambda, \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ or $X = I, D = \Omega Y^{*}, \Omega = \mathrm{diag}(\omega_1, \omega_2, ..., \omega_m)$, it becomes an inverse eigenvalue problem (IEP) of Re-nnd and Re-pd matrices. When it comes to the IEPs, there are many significant applications in various applied sciences and engineering technologies. In 1997, Li first put forward in [13] that the matrix IEP can be converted to a pole assignment problem in control theory. Besides, the IEPs have been encountered in the Spring-Mass Systems [7, 17] and vibration absorption [16]. However, we are aware of the kind of IPs, especially in the context of linear manifold, seems to be rarely considered. In this paper, we will find the solvability conditions for the IP of Re-nnd and Re-pd to (1) on a linear manifold, the details are as follows.
Let the linear manifold $\mathcal{S}$ be

$$\mathcal{S} = \left\{ A \in \mathbb{C}^{n\times n} \middle| f(A) = \|AX_1 - Z\|^2 + \|X_1^{*}A - W^{*}\|^2 = min, X_1, Z, W \in \mathbb{C}^{n\times k} \right\},$$

where $\|.\|$ is Frobenius norm. We consider the following problem:
Problem P. Given $Y_2 \in \mathbb{C}^{n\times m}$, $X_2 \in \mathbb{C}^{n\times p}$, $D \in \mathbb{C}^{m\times p}$, find $A \in \mathbb{C}^{n\times n}_{\geq 0}(\mathcal{S})$ ($\mathbb{C}^{n\times n}_{>0}(\mathcal{S})$) ($\mathbb{C}^{n\times n}_{\geq 0}(\mathcal{S})$ ($\mathbb{C}^{n\times n}_{>0}(\mathcal{S})$) is the Re-nnd (Re-pd) set on the linear manifold $\mathcal{S}$) such that

$$Y_2^{*}AX_2 = D. \tag{3}$$

The solvability conditions and the expression of the general solution (GS) for Problems P are provided by using some matrix decompositions and MPIs.

2. Preliminaries

In order to solve Problem P, we introduce the following lemmas.

Lemma 2.1. *[27] Let the singular value decomposition (SVD) of X_1 be*

$$X_1 = P\begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix}Q^{*}, \tag{4}$$

where $\Omega = diag(\omega_1, \omega_2, ..., \omega_r), r = rank(X_1)$, $P = [P_1, P_2] \in \mathbb{C}^{n\times n}, Q = [Q_1, Q_2] \in \mathbb{C}^{k\times k}$ are unitary matrices (UMs) with $P_1 \in \mathbb{C}^{n\times r}, Q_1 \in \mathbb{C}^{k\times r}$, the partition of $P^{*}AP$ be

$$P^{*}AP = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}.$$

Then, the linear manifold $\mathcal{S}$ can be expressed as

$$\mathcal{S} = \left\{ P \begin{bmatrix} \Phi * (P_1^* Z Q_1 \Omega + \Omega Q_1^* W^* P_1) & \Omega^{-1} Q_1^* W^* P_2 \\ P_2^* Z Q_1 \Omega^{-1} & \tilde{A}_{22} \end{bmatrix} P^* \middle| \tilde{A}_{22} \in \mathbb{C}^{(n-r)\times(n-r)} \right\},$$

where $\Phi = \phi_{ij} \in \mathbb{C}^{r\times r}, \phi_{ij} = \frac{1}{\gamma_i^2+\sigma_j^2}, 1 \le i \le r, 1 \le j \le r$. Let

$$A_1 = P \begin{bmatrix} \Phi * (P_1^* Z Q_1 \Omega + \Omega Q_1^* W^* P_1) & \Omega^{-1} Q_1^* W^* P_2 \\ P_2^* Z Q_1 \Omega^{-1} & 0 \end{bmatrix} P^*, \tag{5}$$

then the linear manifold $\mathcal{S}$ can be expressed as

$$\mathcal{S} = \left\{ A_1 + P_2 \tilde{A}_{22} P_2^* \middle| \tilde{A}_{22} \in \mathbb{C}^{(n-r)\times(n-r)} \right\}.$$

From Lemma 2.1, we can see that

$$A = A_1 + P_2 \tilde{A}_{22} P_2^*,$$

where $\tilde{A}_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ is arbitrary. To facilitate the subsequent solutions of Problem P, we have made a trick transformation as follows.

Note that $\tilde{A}_{22}$ is arbitrary, if take $\tilde{A}_{22} = P_2^* A_{22} P_2$, then we have $A = A_1 + P_2 P_2^* A_{22} P_2 P_2^*$. On the other hand, according to (4) *we know that $X_1 = P_1 \Omega Q_1^*, X_1^\dagger = Q_1 \Omega^{-1} P_1^*$, which implies that $P_2 P_2^* = I - P_1 P_1^* = I - X_1 X_1^\dagger = E_{X_1}$, namely*

$$A = A_1 + E_{X_1} A_{22} E_{X_1}, \tag{6}$$

where $A_{22} \in \mathbb{C}^{n\times n}$ is arbitrary. Eq. (6) *will play a crucial role in the process of solving Problem P.*

Lemma 2.2. *[1] Let $G \in \mathbb{C}^{m\times p}, H \in \mathbb{C}^{q\times n}, C \in \mathbb{C}^{m\times n}$. Then the ME $GXH = C$ have a solution if and only if*

$$GG^\dagger C H^\dagger H = C,$$

or equivalently,

$$E_G C = 0, \; C F_H = 0.$$

In this case, the GS is

$$X = G^\dagger C H^\dagger + F_G V_1 + V_2 E_H,$$

where $V_1, V_2 \in \mathbb{C}^{p\times q}$ are arbitrary matrices.

Lemma 2.3. *[1] Let $P_\mathcal{T}$ and $P_\mathcal{S}$ be the OPs on the subspaces $\mathcal{T}$ and $\mathcal{S}$, respectively. Then $P_\mathcal{T} P_\mathcal{S}$ is an OP if and only if $P_\mathcal{T} P_\mathcal{S} = P_\mathcal{S} P_\mathcal{T}$. In this case, $P_\mathcal{T} P_\mathcal{S} = P_{\mathcal{T}\cap\mathcal{S}}$.*

Lemma 2.4. *[3] Suppose that $G \in \mathbb{C}^{m\times p}$ and $\mathcal{T}$ is a subspace of $\mathbb{C}^p$. Let $\tilde{\mathcal{T}} = \mathcal{R}(P_\mathcal{T} G^*) = P_\mathcal{T} \mathcal{R}(G^*)$, then*

$$\tilde{\mathcal{T}} = \mathcal{T} \cap (\mathcal{T} \cap \mathcal{N}(G))^\perp, \quad \tilde{\mathcal{T}}^\perp = \mathcal{T}^\perp \overset{\perp}{\oplus} (\mathcal{T} \cap \mathcal{N}(G)).$$

Lemma 2.5. *[9, 14] Let $G \in \mathbb{C}^{m\times p}, H \in \mathbb{C}^{q\times m}$ and $D = D^* \in \mathbb{C}^{m\times m}$. Then the ME*

$$GXH + (GXH)^* = D$$

has a solution if and only if

$$E_G D E_G = 0, \;\; F_H D F_H = 0, \; [G, H^*][G, H^*]^\dagger D = D.$$

In this case, the GSs $X \in \mathbb{C}^{p\times q}$ gives

$$X = G^\dagger \left(\tilde{\Theta} + F_L S_X F_L G G^\dagger \right) H^\dagger + M - G^\dagger G M H H^\dagger,$$

where $S_X \in \mathbb{C}^{m\times m}$ is skew-Hermitian, and $M \in \mathbb{C}^{p\times q}$ is arbitrary (If $S_X = -S_X^*$, then S_X is skew-Hermitian), and $\tilde{\Theta}$ is given by $\tilde{\Theta} = \frac{1}{2} D(2I - GG^\dagger) + \frac{1}{2}(\Psi - \Psi^*) G G^\dagger$ with $\Psi = 2 L^\dagger F_H D + \left(I_m - L^\dagger F_H \right) D L^\dagger L$, $L = F_H G G^\dagger$.

Lemma 2.6. *[6, 28] Let $G \in \mathbb{C}^{m\times p}$, $D \in \mathbb{C}^{m\times p}$, and the SVD of G be $G = P\begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix}Q^*$, where $\Omega = diag(\omega_1, \omega_2, ..., \omega_r)$, $r = rank(G)$, $P = [P_1, P_2] \in \mathbb{C}^{m\times m}$, $Q = [Q_1, Q_2] \in \mathbb{C}^{p\times p}$ are UMs with $P_1 \in \mathbb{C}^{m\times r}$, $Q_1 \in \mathbb{C}^{p\times r}$. Then:*
(a). The ME $GX = D$ has a $X \in \mathbb{C}_{\mathrm{H}}^{p\times p} \geq 0$ ($\mathbb{C}_{\mathrm{H}}^{p\times p} \geq 0$ is the Hermitian nonnegative definite set) if and only if $DG^* \geq 0$, $\mathcal{R}(D) = \mathcal{R}(DG^*)$, in this case, the GSs are

$$X = X_0 + F_G H F_G,$$

where $X_0 = G^\dagger D + F_G(G^\dagger D)^* + F_G D^*(DG^*)^\dagger D F_G$, and $H \in \mathbb{C}_{\mathrm{H}}^{p\times p} \geq 0$ is arbitrary.
(b). The ME $GX = D$ has a $X \in \mathbb{C}_{\mathrm{H}}^{p\times p} > 0$ ($\mathbb{C}_{\mathrm{H}}^{p\times p} > 0$ is the Hermitian positive definite set) if and only if $GG^\dagger D = D$, $P_1^* D G^* P_1 > 0$, in this case, the GSs are

$$X = X_0 + F_G H F_G,$$

where $X_0 = G^\dagger D + F_G(G^\dagger D)^* + F_G D^*(DG^*)^\dagger D F_G$, and $H \in \mathbb{C}_{\mathrm{H}}^{p\times p} > 0$ is arbitrary.

3. Main results

In this section, we will put forward a general theory elaborating how to solve Problem P.
Firstly, inserting (6) into (3) yields

$$Y_2^*(A_1 + E_{X_1} A_{22} E_{X_1}) X_2 = D. \tag{7}$$

For notation simplicity, we set $Y_{21} = Y_2^* E_{X_1}$, $X_{12} = E_{X_1} X_2$. By Lemma 2.2, (7) has a solution A_{22} if and only if

$$Y_{21} Y_{21}^\dagger (D - Y_2^* A_1 X_2) X_{12}^\dagger X_{12} = D - Y_2^* A_1 X_2. \tag{8}$$

In this case, the GS is

$$A_{22} = Y_{21}^\dagger (D - Y_2^* A_1 X_2) X_{12}^\dagger + F_{Y_{21}} S_1 + S_2 E_{X_{12}}, \tag{9}$$

where S_1, S_2 are arbitrary. Substituting (9) into (6) leads to

$$A = A_1 + E_{X_1}(Y_{21}^\dagger (D - Y_2^* A_1 X_2) X_{12}^\dagger + F_{Y_{21}} S_1 + S_2 E_{X_{12}}) E_{X_1}. \tag{10}$$

Thus, solving Problem P can be deduced equivalently to finding suitable matrices S_1, S_2 such that

$$A + A^* = A_0 + E_{X_1} F_{Y_{21}} S_1 E_{X_1} + E_{X_1} S_2 E_{X_{12}} E_{X_1} + E_{X_1} S_1^* F_{Y_{21}} E_{X_1} + E_{X_1} E_{X_{12}} S_2^* E_{X_1} \geq 0\ (> 0), \tag{11}$$

where

$$A_0 = A_1 + E_{X_1} Y_{21}^\dagger (D - Y_2^* A_1 X_2) X_{12}^\dagger E_{X_1} + A_1^* + E_{X_1}(Y_{21}^\dagger (D - Y_2^* A_1 X_2) X_{12}^\dagger)^* E_{X_1}.$$

According to Lemmas 2.3 and 2.4, we can obtain that

$$F_{Y_{21}} E_{X_1} = E_{X_1} - (Y_2^* E_{X_1})^\dagger Y_2^* E_{X_1} = E_{X_1} F_{Y_{21}} = P_{\mathcal{N}(X_1^*)\cap\mathcal{N}(Y_{21})} \triangleq E_{21}, \tag{12}$$

$$E_{X_1} E_{X_{12}} = E_{X_1} - E_{X_1} X_2 (E_{X_1} X_2)^\dagger = E_{X_{12}} E_{X_1} = P_{\mathcal{N}(X_{12}^*)\cap\mathcal{N}(X_1^*)} \triangleq E_{12}. \tag{13}$$

Then, (11) can be written as

$$A + A^* = A_0 + E_{21} S_1 E_{X_1} + E_{X_1} S_2 E_{12} + E_{X_1} S_1^* E_{21} + E_{12} S_2^* E_{X_1} \geq 0\ (> 0). \tag{14}$$

To solve the Problem P, we introduce a variable R to rewrite (14) as follows:

$$A_0 + E_{21} S_1 E_{X_1} + E_{X_1} S_2 E_{12} + E_{X_1} S_1^* E_{21} + E_{12} S_2^* E_{X_1} - R = 0,\ R \geq 0\ (> 0), \tag{15}$$

which can be equivalently written as

$$A_0 + \begin{bmatrix} E_{21} & E_{12} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2^* \end{bmatrix} E_{X_1} + E_{X_1} \begin{bmatrix} S_1^* & S_2 \end{bmatrix} \begin{bmatrix} E_{21} \\ E_{12} \end{bmatrix} - R = 0, \ R \geq 0 \ (> 0). \tag{16}$$

Let $G = \begin{bmatrix} E_{21} & E_{12} \end{bmatrix}, S^* = \begin{bmatrix} S_1^* & S_2 \end{bmatrix}$, then we have

$$GSE_{X_1} + (GSE_{X_1})^* = R - A_0, \ R \geq 0 \ (> 0), \tag{17}$$

thus, we will elucidate how to solve (17) in the following.

Combined with Lemma 2.5, Eq. (17) has solutions S if and only if

$$\begin{aligned} &E_G(R - A_0)E_G = 0, \\ &X_1X_1^\dagger(R - A_0)X_1X_1^\dagger = 0, \end{aligned} \tag{18}$$

$$[G, E_{X_1}][G, E_{X_1}]^\dagger(R - A_0) = R - A_0. \tag{19}$$

In this case, the solutions S are

$$S = G^\dagger \left(\tilde{\Theta} + F_L S_{12} F_L GG^\dagger \right) E_{X_1} + M_S - G^\dagger G M_S E_{X_1}, \tag{20}$$

where $S_{12} \in \mathbb{C}^{n\times n}$ is skew-Hermitian, and $M_S \in \mathbb{C}^{2n\times n}$ is arbitrary (If $S_{12} = -S_{12}^*$, then S_{12} is skew-Hermitian), and $\tilde{\Theta}$ is given by $\tilde{\Theta} = \frac{1}{2}(R - A_0)(2I_n - GG^\dagger) + \frac{1}{2}(\Psi - \Psi^*)GG^\dagger$ with $\Psi = 2L^\dagger X_1X_1^\dagger(R - A_0) + \left(I_n - L^\dagger X_1X_1^\dagger\right)(R - A_0)L^\dagger L$, $L = X_1X_1^\dagger GG^\dagger$.

Note that

$$I - [G, E_{X_1}][G, E_{X_1}]^\dagger = P_{\mathcal{N}(G^*)\cap\mathcal{R}(X_1)} \triangleq P_{\mathcal{T}_1}, \tag{21}$$

combined with Lemma 2.4, we can find that

$$E_G = P_{\mathcal{N}(G^*)} \triangleq P_{\mathcal{T}_2}, \quad X_1X_1^\dagger = P_{\mathcal{R}(X_1)} \triangleq P_{\mathcal{T}_3}, \tag{22}$$

thus we has the property that

$$P_{\mathcal{T}_2}P_{\mathcal{T}_1} = P_{\mathcal{T}_1} = P_{\mathcal{T}_1}P_{\mathcal{T}_2}, \ \ P_{\mathcal{T}_3}P_{\mathcal{T}_1} = P_{\mathcal{T}_1} = P_{\mathcal{T}_1}P_{\mathcal{T}_3}, \tag{23}$$

at the same time, (18), (19) can be written as

$$\begin{aligned} &P_{\mathcal{T}_2}RP_{\mathcal{T}_2} = P_{\mathcal{T}_2}A_0P_{\mathcal{T}_2}, \\ &P_{\mathcal{T}_3}RP_{\mathcal{T}_3} = P_{\mathcal{T}_3}A_0P_{\mathcal{T}_3}, \end{aligned} \tag{24}$$

$$P_{\mathcal{T}_1}R = P_{\mathcal{T}_1}A_0. \tag{25}$$

By Lemma 2.6, Eq. (25) has solutions $R \in \mathbb{C}_{\mathrm{H}}^{n\times n} \geq 0$ if and only if

$$P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1} \geq 0, \quad \mathcal{R}\left(P_{\mathcal{T}_1}A_0\right) = \mathcal{R}\left(P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1}\right), \tag{26}$$

in this case, we obtain the GS of (25) as

$$R = R_0 + (I_n - P_{\mathcal{T}_1})K(I_n - P_{\mathcal{T}_1}), \tag{27}$$

where

$$R_0 = P_{\mathcal{T}_1}A_0 + A_0P_{\mathcal{T}_1} - P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1} + (I_n - P_{\mathcal{T}_1})A_0P_{\mathcal{T}_1}(P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1})^\dagger P_{\mathcal{T}_1}A_0(I_n - P_{\mathcal{T}_1}), \tag{28}$$

and $K \in \mathbb{C}_{\mathrm{H}}^{n\times n} \geq 0$ is arbitrary.

Assume that the spectral decomposition (SD) of $P_{\mathcal{T}_1}$ is

$$P_{\mathcal{T}_1} = Q\begin{bmatrix} I_g & 0 \\ 0 & 0 \end{bmatrix}Q^* = Q_1Q_1^*, \tag{29}$$

where $g = \mathrm{rank}(P_{\mathcal{T}_1})$ and $Q = [Q_1, Q_2] \in \mathbb{C}^{n\times n}$ is a UM with $Q_1 \in \mathbb{C}^{n\times g}$. By Lemma 2.6, (25) has solutions $R \in \mathbb{C}_{\mathrm{H}}^{n\times n} > 0$ if and only if

$$Q_1^*P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1}Q_1 > 0, \tag{30}$$

in this case, the GS of (25) is

$$R = R_0 + (I_n - P_{\mathcal{T}_1})K(I_n - P_{\mathcal{T}_1}), \tag{31}$$

where R_0 is given by (28), and $K \in \mathbb{C}_{\mathrm{H}}^{n\times n} > 0$ is arbitrary.

Substituting (27) ((31)) into (24), then combined with (23), we have

$$(P_{\mathcal{T}_i} - P_{\mathcal{T}_1})K(P_{\mathcal{T}_i} - P_{\mathcal{T}_1}) = (P_{\mathcal{T}_i} - P_{\mathcal{T}_1})N(P_{\mathcal{T}_i} - P_{\mathcal{T}_1}), i = 2, 3, \tag{32}$$

where

$$N = A_0 - A_0P_{\mathcal{T}_1}(P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1})^{\dagger}P_{\mathcal{T}_1}A_0. \tag{33}$$

Notice that

$$P_{\mathcal{T}_i} - P_{\mathcal{T}_1} = (P_{\mathcal{T}_i} - P_{\mathcal{T}_1})^* = (P_{\mathcal{T}_i} - P_{\mathcal{T}_1})^2, i = 2, 3,$$

which shows that $P_{\mathcal{T}_2} - P_{\mathcal{T}_1}$ and $P_{\mathcal{T}_3} - P_{\mathcal{T}_1}$ are OPs (see [1, p.81, Ex.68]). Hence, there exist UMs $B, C \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} P_{\mathcal{T}_2} - P_{\mathcal{T}_1} &= B\begin{bmatrix} I_c & 0 \\ 0 & 0 \end{bmatrix}B^* = B_1B_1^*, \\ P_{\mathcal{T}_3} - P_{\mathcal{T}_1} &= C\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}C^* = C_1C_1^*, \end{aligned} \tag{34}$$

where $c = \mathrm{rank}(P_{\mathcal{T}_2} - P_{\mathcal{T}_1}), d = \mathrm{rank}(P_{\mathcal{T}_3} - P_{\mathcal{T}_1})$, and $B_1 \in \mathbb{C}^{n\times c}, C_1 \in \mathbb{C}^{n\times d}$ are column UMs. Substituting (34) into (32), we can arrive at

$$B_1^*KB_1 = B_1^*NB_1, \;\; C_1^*KC_1 = C_1^*NC_1. \tag{35}$$

According to [19] that the generalized singular value decomposition (GSVD) of $[B_1, C_1]$ is of the following form:

$$B_1 = \Upsilon\Sigma_1E^*, \quad C_1 = \Upsilon\Sigma_2F^*, \tag{36}$$

where $\Upsilon \in \mathbb{C}^{n\times n}$ is a nonsingular matrix and $E \in \mathbb{C}^{c\times c}, F \in \mathbb{C}^{d\times d}$ are UMs, and

$$\Sigma_1 = \begin{bmatrix} I & 0 \\ 0 & \Xi \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{matrix} c-r \\ r \\ k-c \\ n-k \end{matrix}, \qquad \Sigma_2 = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}\begin{matrix} c-r \\ r \\ k-c \\ n-k \end{matrix},$$
$$\qquad\quad c-r \quad r \qquad\qquad\qquad\qquad s \quad d-s$$

$k = \mathrm{rank}\,([B_1, C_1]) = c + d - r$, and

$$\Xi = \mathrm{diag}(\xi_1, \cdots, \xi_r), \;\; \Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_r)$$

with

$$1>\xi_1\geq\xi_2\geq\cdots\geq\xi_r>0,\ \ 0<\gamma_1\leq\gamma_2\leq\cdots\leq\gamma_r<1,\\ \xi_i^2+\gamma_i^2=1,\ i=1,\cdots,r.$$

Plugging (36) into (35) gives

$$\Sigma_1^*\Upsilon^*K\Upsilon\Sigma_1=\Sigma_1^*\Upsilon^*N\Upsilon\Sigma_1,\quad \Sigma_2^*\Upsilon^*K\Upsilon\Sigma_2=\Sigma_2^*\Upsilon^*N\Upsilon\Sigma_2, \tag{37}$$

partition $\Upsilon^*K\Upsilon,\Upsilon^*N\Upsilon$ into the following forms:

$$\begin{aligned}\Upsilon^*K\Upsilon&=\begin{array}{c}\left[\begin{array}{cccc}K_{11}&K_{12}&K_{13}&K_{14}\\K_{12}^*&K_{22}&K_{23}&K_{24}\\K_{13}^*&K_{23}^*&K_{33}&K_{34}\\K_{14}^*&K_{24}^*&K_{34}^*&K_{44}\end{array}\right]\begin{array}{c}c-r\\r\\k-c\\n-k\end{array}\\ \begin{array}{cccc}c-r&r&k-c&n-k\end{array}\end{array},\\ \Upsilon^*N\Upsilon&=\begin{array}{c}\left[\begin{array}{cccc}N_{11}&N_{12}&N_{13}&N_{14}\\N_{12}^*&N_{22}&N_{23}&N_{24}\\N_{13}^*&N_{23}^*&N_{33}&N_{34}\\N_{14}^*&N_{24}^*&N_{34}^*&N_{44}\end{array}\right]\begin{array}{c}c-r\\r\\k-c\\n-k\end{array}\\ \begin{array}{cccc}c-r&r&k-c&n-k\end{array}\end{array}\end{aligned}, \tag{38}$$

inserting (38) into (37), then combined with an obtained result in [24] leads to
(a). Eq. (35) has a common $K\in\mathbb{C}_{\mathrm{H}}^{n\times n}\geq0$ if and only if

$$\left[\begin{array}{cc}N_{11}&N_{12}\\N_{12}^*&N_{22}\end{array}\right]\geq0,\quad\left[\begin{array}{cc}N_{22}&N_{23}\\N_{23}^*&N_{33}\end{array}\right]\geq0. \tag{39}$$

In this case, the GS of (35) is

$$K=(\Upsilon^*)^{-1}\left[\begin{array}{cc}\mathcal{M}(K_{13})&\mathcal{M}(K_{13})H_1\\H_1^*\mathcal{M}(K_{13})&H_2+H_1^*\mathcal{M}(K_{13})H_1\end{array}\right]\Upsilon^{-1}, \tag{40}$$

where

$$\mathcal{M}(K_{13})\triangleq\left[\begin{array}{ccc}N_{11}&N_{12}&K_{13}\\N_{12}^*&N_{22}&N_{23}\\K_{13}^*&N_{23}^*&N_{33}\end{array}\right] \tag{41}$$

with

$$K_{13}=N_{12}N_{22}^{\dagger}N_{23}+\left(N_{11}-N_{12}N_{22}^{\dagger}N_{12}^*\right)^{\frac{1}{2}}\mathcal{J}\left(N_{33}-N_{23}^*N_{22}^{\dagger}N_{23}\right)^{\frac{1}{2}}, \tag{42}$$

and $H_1\in\mathbb{C}^{k\times(n-k)}$ is arbitrary, $H_2\in\mathbb{C}_{\mathrm{H}}^{(n-k)\times(n-k)}\geq0$ is arbitrary and $\mathcal{J}\in\mathbb{C}^{(c-r)\times(k-c)}$ is an arbitrary CM.

(b). Eq. (35) has a common $K\in\mathbb{C}_{\mathrm{H}}^{n\times n}>0$ if and only if

$$\left[\begin{array}{cc}N_{11}&N_{12}\\N_{12}^*&N_{22}\end{array}\right]>0,\quad\left[\begin{array}{cc}N_{22}&N_{23}\\N_{23}^*&N_{33}\end{array}\right]>0.$$

In this case, the GS of (35) is

$$K=(\Upsilon^*)^{-1}\left[\begin{array}{cc}\mathcal{M}(K_{13})&H_3\\H_3^*&H_4+H_3^*\left(\mathcal{M}(K_{13})\right)^{-1}H_3\end{array}\right]\Upsilon^{-1}, \tag{43}$$

where $\mathcal{M}(K_{13})$ is defined by (41) with

$$K_{13}=N_{12}N_{22}^{-1}N_{23}+\left(N_{11}-N_{12}N_{22}^{-1}N_{12}^*\right)^{\frac{1}{2}}\mathcal{J}\left(N_{33}-N_{23}^*N_{22}^{-1}N_{23}\right)^{\frac{1}{2}}, \tag{44}$$

and $H_3 \in \mathbb{C}^{k\times(n-k)}$ is arbitrary, $H_4 \in \mathbb{C}_{\mathrm{H}}^{(n-k)\times(n-k)} > 0$ is arbitrary and $\mathcal{J} \in \mathbb{C}^{(c-r)\times(k-c)}$ is an arbitrary SCM.

Once we achieve the $K \in \mathbb{C}_{\mathrm{H}}^{n\times n} \geq 0$ ($K \in \mathbb{C}_{\mathrm{H}}^{n\times n} > 0$) of Eq. (35), then R in (27) ((31)) is completely specified. Substituting (27) and (31) into (20), (10), we can obtain the solutions to Problem P. On the basis of the preceding discussion, we can derive the following result.

Theorem 3.1. *For given matrices $X_1, Z, W \in \mathbb{C}^{n\times k}$, $Y_2 \in \mathbb{C}^{n\times m}$, $X_2 \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{m\times p}$, let $Y_{21} = Y_2^* E_{X_1}, X_{12} = E_{X_1}X_2$, $E_{21} = F_{Y_{21}}E_{X_1}, E_{12} = E_{X_1}E_{X_{12}}$, $G = \begin{bmatrix} E_{21} & E_{12} \end{bmatrix}, S^* = \begin{bmatrix} S_1^* & S_2 \end{bmatrix}$, and $P_{\mathcal{T}_1}, P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ be given by* (21), (22)*, respectively, and suppose that SDs of $P_{\mathcal{T}_1}, P_{\mathcal{T}_2} - P_{\mathcal{T}_1}$ and $P_{\mathcal{T}_3} - P_{\mathcal{T}_1}$ are respectively given by* (29) *and* (34)*, N is given by* (33)*. Furthermore, assume that the GSVD of $[B_1, C_1]$ is given by* (36)*, and the partition of $\Upsilon^* K\Upsilon, \Upsilon^* N\Upsilon$ is given by* (38)*. Thus:*

(*a*)*. Problem P has a Re-nnd solution A on the linear manifold if and only if*

$$P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1} \geq 0, \quad \mathcal{R}(P_{\mathcal{T}_1}A_0) = \mathcal{R}(P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1}), \tag{45}$$

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^* & N_{22} \end{bmatrix} \geq 0, \begin{bmatrix} N_{22} & N_{23} \\ N_{23}^* & N_{33} \end{bmatrix} \geq 0. \tag{46}$$

In this case, the Re-nnd solutions of Problem P are

$$A = A_1 + E_{X_1}Y_{21}^{\dagger}(D - Y_2^* A_1 X_2)X_{12}^{\dagger}E_{X_1} + E_{X_1}F_{Y_{21}}S_1E_{X_1} + E_{X_1}S_2E_{X_{12}}E_{X_1},$$

by choosing the suitable parameter matrices S_1, S_2 in (20)*, where R, K are respectively given by* (27), (40)*, A_1 is given by* (5).

(*b*)*. Problem P has a Re-pd solution A on the linear manifold if and only if*

$$Q_1^* P_{\mathcal{T}_1}A_0P_{\mathcal{T}_1}Q_1 > 0, \tag{47}$$

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^* & N_{22} \end{bmatrix} > 0, \begin{bmatrix} N_{22} & N_{23} \\ N_{23}^* & N_{33} \end{bmatrix} > 0. \tag{48}$$

In this case, the Re-pd solutions of Problem P are

$$A = A_1 + E_{X_1}Y_{21}^{\dagger}(D - Y_2^* A_1 X_2)X_{12}^{\dagger}E_{X_1} + E_{X_1}F_{Y_{21}}S_1E_{X_1} + E_{X_1}S_2E_{X_{12}}E_{X_1},$$

by choosing the suitable parameter matrices S_1, S_2 in (20)*, where R, K are respectively given by* (31), (43)*, A_1 is given by* (5).

4. Conclusions

In 2015, Yuan and Zuo [25] discussed the Re-nnd and Re-pd solutions of $AXB = D$. However, the Re-nnd and Re-pd solutions to the matrix IP $AXB = D$ on a linear manifold have not been given. In Theorem 3.1, we provide the solutions to the Problem P. Actually, in special cases, Problem P can degenerate to a IEP of Re-nnd and Re-pd matrices on a linear manifold. It is worth noting that the IEPs, especially considered on linear manifolds, have wide applications in solid mechanics, structural vibration design, automatic control, and system identification of physical parameters and etc. In this paper, the solvability conditions and the GS of Problem P are discussed. Based on Lemma 2.1, an improved result (6) is provided, which shows a compact form of A. Subsequently, by introducing a variable R, an inequality problem is transformed into an equality problem. In this way, Problem P is solved in terms of some matrix decompositions, MPIs and OPs. The solvability conditions and the explicit expression of the GS for Problem P are given.

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