Filomat 38:30 (2024), 10529–10537
https://doi.org/10.2298/FIL2430529D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Impact of concircular curvature tensor in $f(\mathcal{R}^*)$-gravity

**Uday Chand De[a], Fatemah Mofarreh[b]**

[a]*Department of Pure Mathematics, University of Calcutta, West Bengal, India*
[b]*Department of Mathematical Science, Faculty of Science, Princess Nourah Bint Abdulrahaman University, Riyadh 11546, Saudi Arabia*

**Abstract.** This paper concerns with the characterization of a spacetime and $f(\mathcal{R}^*)$-gravity endowed with concircular curvature tensor. We prove that a concircularly flat perfect fluid spacetime is either a de-Sitter spacetime or locally isometric to Minkowski spacetime . Moreover, it is established that a perfect fluid spacetime admitting harmonic concircular curvature tensor represents a Robertson-Walker spacetime. Finally, we examine the impact of concircularly flat perfect fluid spacetime solutions in two forms of $f(\mathcal{R}^*)$-gravity.

## 1. Introduction

A spacetime is a Lorentzian manifold $M^4$ that allows for a globally time-oriented vector and has a Lorentzian metric $g$ with signature $(-,+,+,+)$. Numerous scholars have examined spacetimes in various contexts (see; [14], [15], [20], [30]).

A $n$ dimensional $(n > 2)$ Lorentzian manifold having the local structure

$$ds^2 = -(d\zeta)^2 + \phi^2(\zeta)\, g^*_{v_1v_2} dx^{v_1} dx^{v_2} \tag{1}$$

is called generalized Robertson-Walker (GRW) spacetime ([3], [12]), $\phi$ indicates a function dependent on $\zeta$ and $g^*_{v_1v_2} = g^*_{v_1v_2}(x^{v_3})$ are only functions of $x^{v_3}$ $(v_1, v_2, v_3 = 2, 3, \ldots, n)$ . The last equation (1) can also be presented as $-\mathcal{I} \times \phi^2\tilde{M}$, where $\mathcal{I} \subset \mathbb{R}$ and $\tilde{M}$ denotes $(n-1)$-dimensional Riemannian manifold. If $\tilde{M}$ is 3-dimensional and of constant sectional curvature, then the GRW spacetime represents a Robertson-Walker (RW) spacetime.

$M^4$ is described as a perfect fluid spacetime (PFS) if the Ricci tensor $\mathcal{R}_{jk}$ having the form

$$\mathcal{R}_{jk} = c g_{jk} + d u_j u_k, \tag{2}$$

in which $c$, $d$ denote scalars and $u_k$ indicates a unit timelike vector. The energy-momentum tensor (EMT) $\mathcal{T}$, which is used to describe the matter content of spacetimes in general relativity (GR) theory. In a PFS, the EMT [26] is given by

$$\mathcal{T}_{jk} = p g_{jk} + \left(p + \mu\right) u_j u_k, \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 53Z05; Secondary 83C05, 83C10.
*Keywords*. Robertson-Walker spacetime, Concircular curvature tensor, perfect fluid spacetime, $f(\mathcal{R}^*)$-gravity, energy condition.
Received: 08 June 2024; Accepted: 05 July 2024
Communicated by Ljubica Velimirović
ORCID iD: 0000-0002-8990-4609 (Uday Chand De)
*Email addresses:* uc_de@yahoo.com (Uday Chand De), fyalmofarrah@pnu.edu.sa (Fatemah Mofarreh)

where $p$ and $\mu$ denote isotropic pressure and energy density, respectively. Also, the equation of state (EoS) $p = p(\mu)$ interconnects $p$ and $\mu$, and the PFS is named isentropic. Moreover, if $p = \mu$, $p + \mu = 0$, $p = 0$ and $p = \frac{\mu}{3}$, the PFS is called stiff matter, the dark matter era, the dust matter fluid and the radiation era [11], respectively. From Einstein's field equations (EFE), we infer

$$\mathcal{R}_{jk} - \frac{1}{2} g_{jk}\mathcal{R} = \kappa\mathcal{T}_{jk}, \tag{4}$$

in which $\mathcal{R}$ is the Ricci scalar and $\kappa$ indicates the gravitational constant.

Concircular transformation was introduced by Yano [28]. It preserves geodesic circles (A curve called a geodesic circle has a first curvature that is constant and a second curvature of zero). The conformal transformation

$$\bar{g}_{lk} = \varphi^2 g_{lk}$$

of the Riemannian tensor $g_{lk}$ fulfilling the partial differential equation

$$\nabla_k \varphi_l = \psi g_{lk},$$

$\psi$ being a scalar changes a geodesic circle to a geodesic circle. The circular transformation is a term used to describe such a transition [28].

In a Riemannian or a semi-Riemannian space, tensor $\mathcal{H}^l_{ijk}$ of type $(1,3)$ defined by Yano and Kon [29]

$$\mathcal{H}^l_{ijk} = \mathcal{R}^l_{ijk} - \frac{\mathcal{R}}{n(n-1)}\left\{\delta^l_k g_{ij} - \delta^l_j g_{ik}\right\} \tag{5}$$

is called a concircular curvature tensor, $\mathcal{R}^l_{ijk}$ stands for the Riemannian curvature tensor. Under such a transformation the expression

$$\mathcal{R}^l_{ijk} - \frac{\mathcal{R}}{n(n-1)}\left\{\delta^l_k g_{ij} - \delta^l_j g_{ik}\right\}$$

remains invariant.

It is widely circulated in GR theory, the energy conditions (ECs) are important tools for examining wormholes and black holes in many modified gravity, like $f(\mathcal{R})$, $f(\mathcal{T})$, $f(G)$, $f(\mathcal{R},\mathcal{T})$, $f(\mathcal{R},G)$ and $f(\mathcal{R},L_m)$-gravity ([4], [5], [6], [7], [8], [16], [13], [21], [22], [23]). The Raychaudhuri equations [27] provides the fascinating feature of gravity through the criterion $\mathcal{R}_{jk}v^j v^k \geq 0$ (positivity condition), where $v^j$ is the null vector. In GR theory, the last criterion on geometry is identical to the null EC (NEC) $\mathcal{T}_{jk}v^j v^k \geq 0$ on matter. Particularly, the weak EC (WEC) states that $\mathcal{T}_{jk}u^j u^k \geq 0$, for every time-like vector $u^j$ and allows a positive local energy density. Various changes to EFE have been established and extensively studied [9]. In ([10], [18]), the authors assert a specific model in which the adjustment to GR is a polynomial function of $\mathcal{R}^2$, $\mathcal{R}_{jk}\mathcal{R}^{jk}$ and $\mathcal{R}_{lijk}\mathcal{R}^{lijk}$ quadratic curvature invariants ($\mathcal{R}_{lijk}$ stands for the Riemannian curvature tensor).In this article, we study "$f(\mathcal{R}^*)$-gravity theory", where $\mathcal{R}^* = \mathcal{R}_{lk}\mathcal{R}^{lk}$, which was developed by Li et al. [24]. A few $f(\mathcal{R}^*)$-models, for example, $f(\mathcal{R}^*) = \beta(\mathcal{R}^*)^\gamma$ ($\beta$ and $\gamma$ are constants) introduced by Li et al. [24] and we choose a novel model $f(\mathcal{R}^*) = ln(\delta\mathcal{R}^*) - \lambda e^{-\frac{\mathcal{R}^*}{\lambda}}$ ($\lambda$ and $\delta$ are constants), to discuss different ECs.

After preliminaries the properties of PFS allowing concircular curvature tensor are provided in Section 3. Lastly, we explore concircularly flat PFS solutions in $f(\mathcal{R}^*)$-gravity.

## 2. Preliminaries

We choose a 4-dimensional spacetime throughout the paper. A spacetime is named concircularly flat if the concircular curvature tensor vanishes at each point of the spacetime. Let us choose a concircularly flat spacetime. Hence, equation (5) yields

$$\mathcal{R}^l_{ijk} = \frac{\mathcal{R}}{12}\,[\delta^l_k g_{ij} - \delta^l_j g_{ik}], \tag{6}$$

which entails that the spacetime is of constant sectional curvature.
Contraction of (6) provides

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{4}\, g_{ij}. \tag{7}$$

Thus we state:

**Proposition 2.1.** *A concircularly flat spacetime is an Einstein spacetime.*

Covariant derivative of equation (5) gives (for $n = 4$)

$$\nabla_l \mathcal{H}^l_{ijk} = \nabla_l \mathcal{R}^l_{ijk} - \frac{\nabla_l \mathcal{R}}{12}\left\{\delta^l_k g_{ij} - \delta^l_j g_{ik}\right\}. \tag{8}$$

we know that

$$\nabla_l \mathcal{R}^l_{ijk} = \nabla_k \mathcal{R}_{ij} - \nabla_j \mathcal{R}_{ik}. \tag{9}$$

Making use of equation (9) in (8), we infer

$$\nabla_l \mathcal{H}^l_{ijk} = \nabla_k \mathcal{R}_{ij} - \nabla_j \mathcal{R}_{ik} - \frac{1}{12}\left\{g_{ij}\nabla_k \mathcal{R} - g_{ik}\nabla_j \mathcal{R}\right\}. \tag{10}$$

Suppose $\mathcal{H}^l_{ijk}$ is harmonic, that is, $\nabla_l \mathcal{H}^l_{ijk} = 0$, then equation (10) provides

$$\nabla_k \mathcal{R}_{ij} - \nabla_j \mathcal{R}_{ik} = \frac{1}{12}\left\{g_{ij}\nabla_k \mathcal{R} - g_{ik}\nabla_j \mathcal{R}\right\}. \tag{11}$$

Multiplying (11) with $g^{ij}$ infers

$$\nabla_k \mathcal{R} - \nabla_j \mathcal{R}^j_k = \frac{1}{4}\nabla_k \mathcal{R}. \tag{12}$$

Since $\nabla_j \mathcal{R}^j_k = \frac{1}{2}\nabla_k \mathcal{R}$, thus from the equation (12), we acquire

$$\mathcal{R} = \text{constant}. \tag{13}$$

Therefore, equation (11) yields

$$\nabla_k \mathcal{R}_{ij} - \nabla_j \mathcal{R}_{ik} = 0, \tag{14}$$

which entails $\mathcal{R}_{jk}$ is of Codazzi type.

Conversely, for a Codazzi type of Ricci tensor $\mathcal{R}_{lk}$, we get

$$\nabla_j \mathcal{R}_{ik} - \nabla_k \mathcal{R}_{ij} = 0. \tag{15}$$

Multiplying (15) with $g^{ij}$, we find

$$\nabla_k \mathcal{R} = 0. \tag{16}$$

The equations (10), (15) and (16) turns into

$$\nabla_l \mathcal{H}^l_{ijk} = 0. \tag{17}$$

Thus, we write

**Proposition 2.2.** *In a Riemannian or a semi-Riemannian space the concircular curvature tensor is harmonic iff the Ricci tensor is of Codazzi type.*

**Remark 2.3.** *Theorem 6 and Theorem 7 of the paper [2] are wrongly stated.*

## 3. PFS admitting concircular curvature tensor

Here, we choose a concircularly flat PFS obeying EFE.
From equations (3), (4) and (7), we get

$$\kappa(\boldsymbol{p}+\boldsymbol{\mu})u_j u_k + (\kappa\boldsymbol{p} + \frac{\mathcal{R}}{4})g_{jk} = 0. \tag{18}$$

Now multiplying equation (18) with $g^{jk}$ yields

$$\mathcal{R} + \kappa(3\boldsymbol{p} - \boldsymbol{\mu}) = 0. \tag{19}$$

Again, multiplying (18) with $u^j$ provides

$$\mathcal{R} = 4\kappa\boldsymbol{\mu}. \tag{20}$$

From the last two equations, we obtain

$$\boldsymbol{p} = -\boldsymbol{\mu}, \tag{21}$$

which means that a dark matter era [11]. Therefore, we provide:

**Theorem 3.1.** *A concircularly flat* PFS *satisfying* EFE *represents a dark matter era.*

Equations (3) and (4) jointly yield

$$\mathcal{R}_{jk} = \kappa(\boldsymbol{p}+\boldsymbol{\mu})u_j u_k + (\kappa\boldsymbol{p} + \frac{\mathcal{R}}{2})g_{jk}. \tag{22}$$

Now, multiplying equation (22) by $u^j u^k$, we acquire

$$\mathcal{R}_{jk}u^j u^k = \kappa\boldsymbol{\mu} - \frac{\mathcal{R}}{2}. \tag{23}$$

Therefore, from (20) and (23), we find

$$\mathcal{R}_{jk}u^j u^k = -\kappa\boldsymbol{\mu}. \tag{24}$$

A spacetime satisfies the strong EC (SEC) [17] if for every time-like vector $v$, $\mathcal{R}_{hj}v^h v^j \geq 0$ holds. Let the spacetime under consideration obeys the SEC. Thus

$$\kappa\boldsymbol{\mu} \leq 0. \tag{25}$$

As $\boldsymbol{\mu}$ is non-negative and $\kappa > 0$, then (20) and (25) infer

$$\mathcal{R} = 0. \tag{26}$$

Then equation (6) provides $\mathcal{R}^l_{ijk} = 0$, which entails that the spacetime has vanishing sectional curvature. Hence, a concircularly flat PFS and Minkowski spacetime are locally isometric ([17], p. 67).
Thus, we write:

**Theorem 3.2.** *A concircularly flat* PFS *satisfying the* SEC *is locally isometric to Minkowski spacetime.*

Since $\boldsymbol{\mu}$ is non-negative, equation (20) states that

$$\mathcal{R} \geq 0, \tag{27}$$

which entails $\mathcal{R} > 0$, or $\mathcal{R} = 0$.

**Case 1.** If $\mathcal{R} > 0$, then equation (6) implies that the space is of positive constant curvature. Thus, it is a de-Sitter spacetime [17].

**Case 2.** For $\mathcal{R} = 0$, equation (6) reveals $\mathcal{R}^l_{ijk} = 0$. Therefore, the spacetime is locally isometric to Minkowski spacetime.
Hence, we state:

**Theorem 3.3.** *A concircularly flat* PFS *is either locally isometric to Minkowski spacetime or a de-Sitter spacetime.*

Since the de-Sitter spacetime is conformally flat and therefore belongs to Petrov classification *O*. Hence, we provide:

**Corollary 3.4.** *A concircularly flat* PFS *belongs to Petrov classification O or locally isometric to Minkowski spacetime.*

Let the concircular curvature tensor be harmonic, that is, $\nabla_l \mathcal{H}^l_{ijk} = 0$.

We know that a Yang pure space [19] is a Lorentzian manifold obeying the Yang's equation:

$$\nabla_h \mathcal{R}_{lk} - \nabla_k \mathcal{R}_{lh} = 0. \tag{28}$$

Hence, by Proposition 2, we infer that a spacetime allowing harmonic concircular curvature tensor is a Yang pure space.
From [19], we write the subsequent:

**Theorem A.** A 4-dimensional PFS obeying $\mu + p \neq 0$ represents a Yang pure space iff it is a RW spacetime.

Therefore, using Theorem A, we provide:

**Theorem 3.5.** *A* PFS *obeying $p + \mu \neq 0$ and harmonic concircular curvature tensor is a* RW *spacetime.*

## 4. Concircularly flat PFS solutions fulfilling $f(\mathcal{R}^*)$-gravity

The modified Einstein-Hilbert action term is given by

$$S = \int \left\{ L_m + \frac{\mathcal{R} + f(\mathcal{R}^*)}{2\kappa} \right\} d^4x \sqrt{-g}, \tag{29}$$

where $L_m$ indicates the matter Lagrangian density depends on the metric $g_{lk}$ and Ricci-tensor-squared gravity $\mathcal{R}^*$ is presented as

$$\mathcal{R}^* = \mathcal{R}_{lk}\mathcal{R}^{lk}. \tag{30}$$

The variation of action term (29) provides the modified EFE of $f(\mathcal{R}^*)$-gravity as [24]

$$\mathcal{R}_{lk} + 2 f_{\mathcal{R}^*} \mathcal{R}^h_l \mathcal{R}_{kh} - \frac{1}{2}\{\mathcal{R} + f(\mathcal{R}^*)\} g_{lk} = \kappa \mathcal{T}^f_{lk} \tag{31}$$

in which $f_{\mathcal{R}^*} = \dfrac{\partial f}{\partial \mathcal{R}^*}$ and $\mathcal{T}^f_{lk}$ is the EMT of the fluid.
In $f(\mathcal{R}^*)$ modified gravity, the ECs are given by

$$\begin{aligned}
\text{NEC} &\Longleftrightarrow \mu + p \geq 0,\\
\text{WEC} &\Longleftrightarrow \mu \geq 0 \quad \text{and} \quad \mu + p \geq 0,\\
\text{DEC} &\Longleftrightarrow \mu \geq 0 \quad \text{and} \quad \mu \pm p \geq 0,\\
\text{SEC} &\Longleftrightarrow \mu + 3p \geq 0 \quad \text{and} \quad \mu + p \geq 0,
\end{aligned}$$

in which DEC indicates the dominant EC.

Here, we choose PFS solutions in $f(\mathcal{R}^*)$-gravity equation allowing the EMT equation (3). Thus, (3), (7) and (31) together imply

$$\frac{\mathcal{R}^2}{8} g_{lk} f_{\mathcal{R}^*} - \frac{\mathcal{R}}{4} g_{lk} - \frac{1}{2} g_{lk} f(\mathcal{R}^*) = \kappa p g_{lk} + \kappa \left(p + \mu\right) u_l u_k. \tag{32}$$

Multiplying (32) with $u^l$, we obtain

$$\mu = \frac{\mathcal{R}}{4\kappa} + \frac{1}{2\kappa} f(\mathcal{R}^*) - \frac{\mathcal{R}^2}{8\kappa} f_{\mathcal{R}^*}. \tag{33}$$

Again, multiplying equations (32) with $g^{lk}$ infers

$$3\kappa p - \kappa\mu = \frac{\mathcal{R}^2}{2} f_{\mathcal{R}^*} - \mathcal{R} - 2f(\mathcal{R}^*). \tag{34}$$

From (33) and (34), it follows that

$$p = -\frac{\mathcal{R}}{4\kappa} - \frac{1}{2\kappa} f(\mathcal{R}^*) + \frac{\mathcal{R}^2}{8\kappa} f_{\mathcal{R}^*}. \tag{35}$$

Hence, we provide:

**Theorem 4.1.** *In a concircularly flat* PFS *solutions in $f(\mathcal{R}^*)$-gravity the energy density $\mu$ and the isotropic pressure $p$ are described by* (33) *and* (35), *respectively.*

Equations (33) and (35) together imply

$$p + \mu = 0,$$

which tells that NEC is verified.

From (7), it follows that

$$\mathcal{R}^{lk} = \frac{\mathcal{R}}{4} g^{lk}. \tag{36}$$

Equations (7) and (36) together imply

$$\mathcal{R}_{lk}\mathcal{R}^{lk} = \frac{\mathcal{R}^2}{4}. \tag{37}$$

Therefore the Ricci-tensor-squared gravity is

$$\mathcal{R}^* = \frac{\mathcal{R}^2}{4}. \tag{38}$$

We now investigate the ECs for two distinct $f(\mathcal{R}^*)$-gravity models in the subsequent subsections.

### A. $f(\mathcal{R}^*) = \beta(\mathcal{R}^*)^\gamma$

Using equations (33), (35) and (38), the energy density and pressure are described as

$$\mu = \frac{\mathcal{R}}{4\kappa} + \frac{\beta(1-\gamma)\mathcal{R}^{2\gamma}}{2\kappa \times 4^\gamma}, \tag{39}$$

$$p = -\frac{\mathcal{R}}{4\kappa} - \frac{\beta(1-\gamma)\mathcal{R}^{2\gamma}}{2\kappa \times 4^\gamma}. \tag{40}$$

The ECs are now discussed using equations (39) and (40).

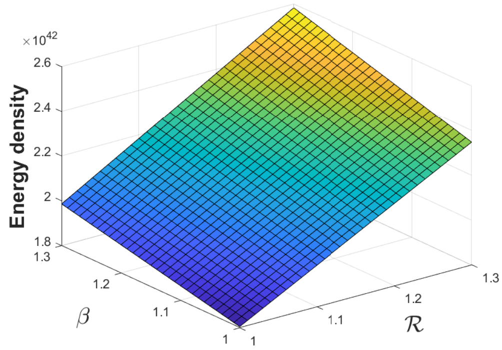


**Fig. 1:** Advancement of $\mu$

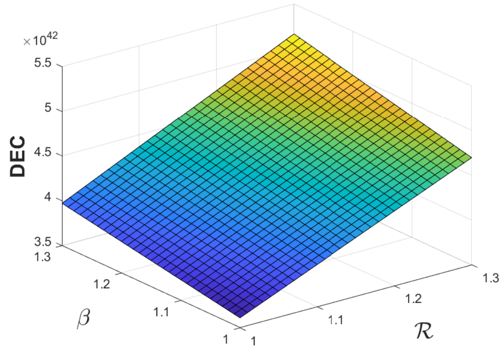


**Fig. 2:** Advancement of DEC

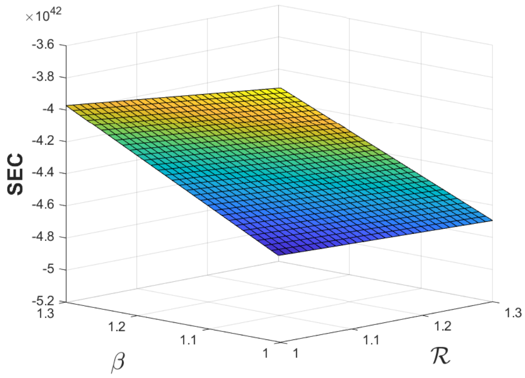


**Fig. 3:** Advancement of SEC

Figs. 1, 2 and 3, gave the profiles of $\mu$, DEC and SEC. In this set up, $\mu + p$ is zero. As NEC belongs to WEC, consequently NEC and WEC are verified. Fig. 2 gives the DEC profile.SEC is violated, and this result provides the Universe's late-time acceleration [25]. Moreover, for this set up, the EoS is $\frac{p}{\mu} = -1$.

**B.** $f(\mathcal{R}^*) = ln(\delta\mathcal{R}^*) - \lambda e^{-\frac{\mathcal{R}^*}{\lambda}}$

Here, using (33), (35) and (38), the energy density and pressure are represented as

$$\mu = \frac{1}{\kappa}\left\{\left(\frac{\mathcal{R}-2}{4}\right) - \left(\frac{\mathcal{R}^2+4\lambda}{8}\right)e^{-\frac{\mathcal{R}^2}{4\lambda}} + \frac{1}{2}ln\left(\frac{\delta\mathcal{R}^2}{4}\right)\right\}, \tag{41}$$

$$p = \frac{1}{\kappa}\left\{-\left(\frac{\mathcal{R}-2}{4}\right) + \left(\frac{\mathcal{R}^2+4\lambda}{8}\right)e^{-\frac{\mathcal{R}^2}{4\lambda}} - \frac{1}{2}ln\left(\frac{\delta\mathcal{R}^2}{4}\right)\right\}. \tag{42}$$

Using (41) and (42), we can discuss about the ECs for this model.

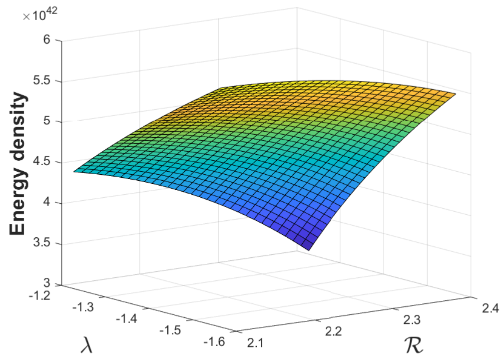


**Fig. 4:** Advancement of $\boldsymbol{\mu}$

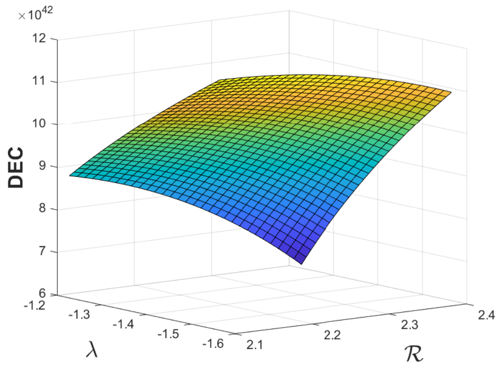


**Fig. 5:** Advancement of DEC

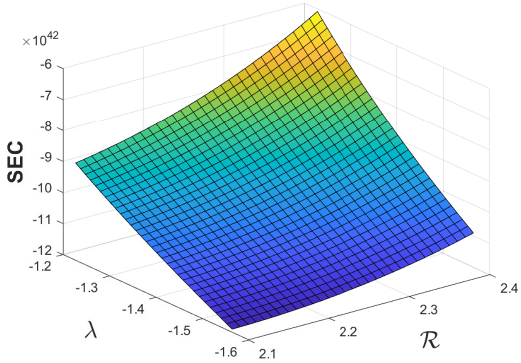


**Fig. 6:** Advancement of SEC

Figs. 4, 5, and 6 shows the profiles of $\boldsymbol{\mu}$, DEC, and SEC. Also, we see that $\boldsymbol{\mu}$ and DEC are obeyed for $\mathcal{R} > 2$ and $\mathcal{R}^2 + 4\lambda < 0$ with $\delta = 4$ but the SEC is not satisfied. Moreover, as $\boldsymbol{\mu} + \boldsymbol{p} = 0$ for this set up, NEC and WEC are also verified.

## 5. Conclusion

The prime focus of this paper is to explore the concircularly flat PFS solutions in $f(\mathcal{R}^*)$-gravity. Our outcomes have been examined analytically and graphically. To create our formulation, we used here the analytic technique and discuss the stability of two cosmological models, like $f(\mathcal{R}^*) = \beta(\mathcal{R}^*)^{\gamma}$ and $f(\mathcal{R}^*) = ln(\delta\mathcal{R}^*) - \lambda e^{-\frac{\mathcal{R}^*}{\lambda}}$. For the 1st model, Figs. 1, 2 and 3 gave the profiles of ECs. For parameters $\mathcal{R} > 0$ and $\beta > 0$, the evolution of $\boldsymbol{\mu}$ is seen to be positive. Although WEC, NEC and DEC are verified, SEC violated the agreement. Moreover, the EoS is $\frac{\boldsymbol{p}}{\boldsymbol{\mu}} = -1$, which indicates the dark matter era. Besides, all conclusion is consistent with the ΛCDM model [1]. Likewise to the 1st model, Figs. 4, 5 and 6 depict all ECs for the 2nd model. The findings we discovered for the 2nd model are identical with those of the 1st model.

## 6. Acknowledgement

The author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R27), Princess Noura bint Abdulrahman

University, Riyadh, Saudi Arabia.

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