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A study of rough $\mathcal{I}$-statistical convergence of order α

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Abstract. The purpose of the present article is to introduce the idea of rough ideal statistical convergence of order α of sequences in intuitionistic fuzzy normed spaces. In these spaces, we have defined the concepts of rough ideal statistical limit points, rough ideal cluster points, and rough ideal boundedness. The theory of convexity and closeness was next examined for the collection of approximative statistical limit points.

1. Introduction

The fascinating idea of rough convergence was first suggested by Phu [29] for sequences on finite-dimensional normed linear spaces in 2001. After that, numerous authors were inspired to work on various sequence-spaces, including those for double sequences [17, 25, 26], triple sequences [14], lacunary sequences [22], ideals [16, 27]. Despite this, it has been established in a variety of spaces, see [2, 3, 9]. It was later expanded to infinite-dimensional normed linear spaces [30]. In 2008, Aytar [8] also worked on same and introduced new generalized convergence named rough statistical convergence.

In 2000, Kostryko *et al.* [23] proposed the idea of ideal convergence($\mathcal{I}$-convergence) by generalizing the statistical convergence. With the help of ideals, in 2011, a new generalisation named rough ideal statistical convergence in normed spaces was defined by Das, Savaş and Ghosal [13]. They studied its fundamental properties. In 2010, Çolak [11] introduced the concept of statistical convergence of order α for sequences of real numbers. Later on, Savaş and Das [12] defined $\mathcal{I}$-statistical convergence of order $\alpha(\alpha \in (0,1])$. Maity [24] worked on rough statistical convergence of order α and studied some properties of rough statistical limit points in normed linear space. Futhermore, a lot of developments have been made in this area for sequences in 2-normed spaces [4–6, 15, 18, 19]. In [10], authors defined $\mathcal{I}$-statistical rough convergence of order α in normed linear spaces and established the necessary and sufficient condition for a sequence $\{x_k\}$ to be $\mathcal{I}$-statistically convergent of order α and $\mathcal{I}$-statistically bounded of order α.

Zadeh [32] proposed the idea of fuzzy sets as an extension of classical sets to study the vague qualitative or quantitative data. Fuzzy set theory is an efficient technique for describing uncertainty and ambiguity. As a generalisation of fuzzy sets that can deal with both the degree of non-membership and the degree of membership of the components for the given set, Atanassov [7] introduced intuitionistic fuzzy sets in 1986. Park [28] was the first to develop intuitionistic fuzzy metric spaces and the concept of Cauchy sequences in

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the same spaces using continuous t-norm and continuous t-conorms. Saadati and Park [31] expanded this idea to intuitionistic fuzzy topological spaces. Karakus [21] defined statistical convergence in intuitionistic fuzzy normed spaces and studied statistical convergence with properties in *IFNS*. Recently, Antal *et al.* [1] examined rough statistical convergence and some fundamental results in intuitionistic fuzzy normed spaces.

In present article, we introduce the notion of rough ideal statistical convergence of order $\alpha(\alpha \in (0,1])$ in *intuitionistic fuzzy normed spaces* (Briefly *IFNS*). We provided the results in their most generic manner compared to past studies, which exist in normed linear spaces. In addition, we have also defined rough ideal statistical cluster points in *IFNS* and worked on it. The theory of convexity and closeness was next examined for the collection of approximative statistical limit points. We shall provide some basic definitions and concepts relating to rough ideal statistical convergence of order α in the next section.

2. Preliminaries

This section reviews essential terminology and ideas associated with *rough ideal statistical convergence* and *intuitionistic fuzzy normed spaces (IFNS)*.

Definition 2.1. *[23] Let $\Upsilon \neq \emptyset$. A family $\mathcal{I} \subset P(\Upsilon)$ of subsets of Υ is called an ideal in Υ provided,*

(i) $\emptyset \in \mathcal{I}$,
(ii) $R, S \in \mathcal{I} \implies R \cup S \in \mathcal{I}$,
(iii) *For each* $R \in \mathcal{I}, S \subset R \implies S \in \mathcal{I}$.

where the collection of all subsets of Υ is denoted by $P(\Upsilon)$.

A non-trivial ideal ($\mathcal{I} \neq P(\Upsilon)$) which is proper subset of $P(\Upsilon)$ is called an admissible ideal in Υ iff it contains all the singletons.
Consider $\mathcal{I}$ as a non-trivial admissible ideal in set of natural numbers throughout the article.

Definition 2.2. *[23] Let $\Upsilon \neq \emptyset$. A non-empty class $\mathbb{F} \subset P(\Upsilon)$ is called filter on Υ provided,*

(i) $\emptyset \notin \mathbb{F}$,
(ii) $S, T \in \mathbb{F} \implies S \cap T \in \mathbb{F}$,
(iii) *For each* $S \in \mathbb{F}, S \subset T \implies T \in \mathbb{F}$.

Every ideal $\mathcal{I}$ is associated with a filter $\mathbb{F}(\mathcal{I})$ defined as follows:

$$\mathbb{F}(\mathcal{I}) = \{\mathbb{M} \subseteq \Upsilon : \mathbb{M}^c \in \mathcal{I}\} \text{ where } \mathbb{M}^c = \Upsilon - \mathbb{M}.$$

Definition 2.3. *[23] A sequence $y = \{y_m\}$ in Υ is called ideal convergent ($\mathcal{I}-$convergent) to ρ if for every $\varepsilon > 0$*

$$\mathbb{A}(\varepsilon) = \{n \in \mathbb{N} : |y_m - \rho| \geq \varepsilon\} \in \mathcal{I}.$$

Here, ρ is known as the $\mathcal{I}$-limit of the sequence $y = \{y_m\}$.

Now, we recall rough convergence as follows:

Definition 2.4. *[29] Let $(\Upsilon, \|.\|)$ be a normed linear space. Then, sequence $y = \{y_m\}$ in Υ is called rough convergent($r-$convergent) to $\xi \in \Upsilon$ for some non-negative real number r if there exists $m_0 \in \mathbb{N}$ for every $\varepsilon > 0$ such that $\|y_m - \xi\| < r + \varepsilon$ for all $m \geq m_0$.*
We write it as $y_m \xrightarrow{r} \xi$ or $r - \lim_{m\to\infty} y_m = \xi$, where r is known as roughness degree of rough convergence of the sequence $y = \{y_m\}$.

For any sequence $y = \{y_m\}$ in the normed linear space Υ the r-limit set is given as $\text{LIM}^r_{y_m} = \{\xi \in \mathbb{X} : y_m \xrightarrow{r} \xi\}$.
Also, $\text{LIM}^r_{y_m} = [\mathit{lim}\sup y - r,\ \mathit{lim}\inf y + r]$ is defined for any sequence $y = \{y_m\}$ of real numbers [29].

Definition 2.5. *[27] A sequence $y = \{y_m\}$ is called rough ideal convergent($r - \mathcal{I} -$ convergent) to ξ, where r is non-negative roughness degree, if for every $\varepsilon > 0$,*

$$\{m \in \mathbb{N} : \|y_m - \xi\| \geq r + \varepsilon\} \in \mathcal{I}.$$

Next, we mention about *IFNS* along with convergence of sequence in this space.

Definition 2.6. *[20] Let Υ be a vector space and ψ, η be two fuzzy sets on $\Upsilon \times \mathbb{R}$, then the triplet (Υ, ψ, η) is called an intuitionistic fuzzy normed space (IFNS) if for each $y, z \in \Upsilon$ and $p, q \in \mathbb{R}$, we have*

(i) $\psi(y, q) = 0$ *and* $\eta(y, q) = 1$ *for* $q \notin \mathbb{R}^+$,
(ii) $\psi(y, q) = 1$ *and* $\eta(y, q) = 0$ *for* $q \in \mathbb{R}^+$ *iff* $y = 0$,
(iii) $\psi(\alpha y; q) = \psi\left(y; \frac{q}{|\alpha|}\right)$ *and* $\eta(\alpha y; q) = \eta\left(y; \frac{q}{|\alpha|}\right)$, $\alpha \neq 0$ *is a real number,*
(iv) *min* $\{\psi(y, p), \psi(z, q)\} \leq \psi(y + z; p + q)$ *and max* $\{\eta(y, p), \eta(z, q)\} \geq \eta(y + z; p + q)$,
(v) $\lim_{q\to\infty} \psi(y, q) = 1, \lim_{q\to 0} \psi(y, q) = 0, \lim_{q\to\infty} \eta(y, q) = 0, \lim_{q\to 0} \eta(y, q) = 1.$

Example 2.7. *[31] Let $(\Upsilon, \|.\|)$ be any real normed space. For every $q > 0$ and all $y \in \Upsilon$, define $\psi(y, q) = \frac{q}{q+\|y\|}$, $\eta(y, q) = \frac{y}{q+\|y\|}$. Then (Υ, ψ, η) is an IFNS.*

Definition 2.8. *[31] Let (Υ, ψ, η) be an IFNS with intuitionistic fuzzy norms (ψ, η). A sequence $y = \{y_m\}$ in Υ is called convergent to $\xi \in \Upsilon$ with respect to the norm (ψ, η) if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $m_0 \in \mathbb{N}$ such that $\psi(y_m - \xi; \varepsilon) > 1 - \lambda$ and $\eta(y_m - \xi; \varepsilon) < \lambda$ for all $m \geq m_0$. It is denoted by $(\psi, \eta) - \lim_{m\to\infty} y_m = \xi$.*

Definition 2.9. *[1]Let (Υ, ψ, η) be an intuitionistic fuzzy normed space. A sequence $y = \{y_m\}$ in Υ is said to be rough convergent to $\xi \in \Upsilon$ with respect to norm (ψ, η) for some non-negative real number r if there exists $m_0 \in \mathbb{N}$ for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that*

$$\psi(y_m - \xi; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \xi; r + \varepsilon) < \lambda \text{ for all } m \geq m_0.$$

It is denoted by $y_m \xrightarrow{r(\psi,\eta)} \xi$ or $r(\psi, \eta) - \lim_{m\to\infty} y_m = \xi$.

Remark 2.10. *For $r = 0$, the rough convergence agrees with the usual convergence for the sequences in an IFNS.*

Definition 2.11. *[1] Let (Υ, ψ, η) be an intuitionistic fuzzy normed space. A sequence $y = \{y_m\}$ in Υ is said to be rough statistical convergent to $\xi \in \Upsilon$ with respect to norm (ψ, η) for some non-negative number r if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{m \leq n : \psi(y_m - \xi; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - \xi; r + \varepsilon) \geq \lambda\}\right| = 0.$$

It is denoted by $y_m \xrightarrow{r-st} \xi$ or $r - st - \lim_{m\to\infty} y_m = \xi$.
Let $st - LIM_y^r$ denotes the the set of all rough statistical limit points of the sequence $y = \{y_m\}$.

3. Main Results

In this section, we first define the concept of rough statistical convergence of order α and rough ideal statistical convergence of order $\alpha(\alpha \in (0, 1])$ in an *IFNS* and then proved some significant results. Throughout the article $\alpha \in (0, 1]$).

Definition 3.1. *Let (Υ, ψ, η) be an intuitionistic fuzzy normed space. A sequence $y = \{y_m\}$ in Υ is said to be rough statistically convergent of order α to $\xi \in \Upsilon$ with respect to norm (ψ, η) for some non-negative number r if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{m \leq n : \psi(y_m - \xi; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - \xi; r + \varepsilon) \geq \lambda\}\right| = 0.$$

It is denoted by $y_m \xrightarrow{r-st(\alpha)} \xi$ or $r - st(\alpha) - \lim_{m\to\infty} y_m = \xi$.
Let $st - LIM_r^\alpha y$ denotes the the set of all rough statistical limit points of order α of the sequence $y = \{y_m\}$.

Remark 3.2. *For $r = 0$, the notion rough statistical convergence of order α agrees with the statistical convergence of order α for the sequences on an IFNS and for $\alpha = 1$, this notion coincides with rough statistical convergence studied by Antal et al. [1] on IFNS.*

Definition 3.3. *Let (Υ, ψ, η) be an intuitionistic fuzzy normed space. A sequence $y = \{y_m\}$ in Υ is said to be rough ideal statistically convergent of order α to $\xi \in \Upsilon$ with respect to norm (ψ, η) for some non-negative number r if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$*

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \left|\{m \leq n : \psi(y_m - \xi; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - \xi; r + \varepsilon) \geq \lambda\}\right| \geq \delta\right\} \in \mathcal{I}$$

or

$$\delta_{\mathcal{I}}\left(\{m \in \mathbb{N} : \psi(y_m - \xi; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - \xi; r + \varepsilon) \geq \lambda\}\right) = 0,$$

where $\delta_{\mathcal{I}}(A) = \mathcal{I} - \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : m \in A\}|$ if exists.
It is denoted by $y_m \xrightarrow{r-\mathcal{I}-st(\alpha)} \xi$ or $r - st_{\mathcal{I}}^\alpha - \lim_{m\to\infty} y_m = \xi$.
Let $\mathcal{I} - st - LIM_r^\alpha y$ denotes the the set of all rough ideal statistical limit points of order α of the sequence $y = \{y_m\}$.

Remark 3.4. *For $r = 0$, the notion rough ideal statistical convergence of order α agrees with the ideal statistical convergence of order α in an IFNS.*

The following example shows that there is a sequence which is neither *rough ideal convergent* nor *ideal statistical convergent* but it is *rough ideal statistical convergent of order α*.

Example 3.5. *Let S be an infinite subset of $\mathbb{N}$ and $\mathcal{I}$ be an ideal such that $S \in \mathcal{I}$. Then $S^c = \{a_1 < a_2 < a_3 < ...\} \in \mathbb{F}(\mathcal{I})$. Consider a sequence $y = \{y_m\}_{m\in\mathbb{N}}$ such that*

$$y_m = \begin{cases} (-1)^m, & \text{if } m \notin S \text{ and } \alpha = 1, \\ m, & \text{if } m \in S. \end{cases}$$

Then

$$\mathcal{I} - st - LIM_r^\alpha y = \begin{cases} \phi & r < 1, \\ [1 - r, r - 1] & \text{otherwise.} \end{cases}$$

In general, a sequence's *rough limit* might not be unique. Therefore, we consider the *rough $\mathcal{I}$-st-limit* set of order α of sequences $y = \{y_m\}$ with respect to the norm (ψ, η) as

$$\mathcal{I} - st - \text{LIM}_r^\alpha y = \{\xi^* \in \Upsilon : y_m \xrightarrow{r-\mathcal{I}-st(\alpha)} \xi^*\}.$$

Moreover, sequence $y = \{y_m\}$ is *rough $\mathcal{I}$-st-convergent* provided $\mathcal{I} - st - \text{LIM}_r^\alpha y \neq \phi$ for some $r > 0$. In [29], it was observed that for a sequence $y = \{y_m\}$ of real numbers, the set of rough limit points,

$$\text{LIM}_y^r = [\limsup y - r, \liminf y + r].$$

In the similar way,

$$\mathcal{I} - st - \text{LIM}_r^\alpha y = [\mathcal{I} - st - \limsup y - r, \mathcal{I} - st - \liminf y + r].$$

Definition 3.6. *A sequence $y = \{y_m\}$ in intuitionistic fuzzy normed space (Υ, ψ, η) is $\mathcal{I}$-st-bounded of order $\alpha(\alpha \in (0,1])$ if for $\varepsilon > 0$, $\lambda \in (0,1)$ and some $r > 0$, there exists $H > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; H) \le 1-\lambda \text{ or } \eta(y_m; H) \ge \lambda\}\right| \ge \delta\right\} \in \mathcal{I}.$$

Theorem 3.7. *Let (Υ, ψ, η) be an IFNS with intuitionistic fuzzy norms (ψ, η). A sequence $y = \{y_m\}$ in Υ is $\mathcal{I}$-st-bounded of order $\alpha(\alpha \in (0,1])$ iff $\mathcal{I} - st - LIM_r^\alpha y \neq \phi$ for some $r > 0$.*

Proof. Necessary Part:- Consider the sequence $y = \{y_m\}$ which is $\mathcal{I}$-st-bounded of order α in an *IFNS* (Υ, ψ, η). Then, for every $\varepsilon > 0$, $\lambda \in (0,1)$ and some $r > 0$, $\exists H > 0$ such that

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; H) \le 1-\lambda \text{ or } \eta(y_m; H) \ge \lambda\}\right| \ge \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is admissible, therefore $M = \mathbb{N} \setminus G$ is a non-empty set, where

$$G = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; H) \le 1-\lambda \text{ or } \eta(y_m; H) \ge \lambda\}\right| \ge \delta\right\}.$$

Choose $m \in M$, then

$$\frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; H) \le 1-\lambda \text{ or } \eta(y_m; H) \ge \lambda\}\right| < \delta$$

$$\implies \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; H) > 1-\lambda \text{ and } \eta(y_m; H) < \lambda\}\right| \ge 1-\delta. \tag{1}$$

Let $\mathbb{K} = \{m \le n : \psi(y_m; H) > 1-\lambda \text{ and } \eta(y_m; H) < \lambda\}$.
Also,

$$\begin{aligned} \psi(y_m; r+H) &\ge \min\{\psi(0,r), \psi(y_m, H)\} \\ &= \min\{1, \psi(y_m; H)\} \\ &> 1-\lambda, \\ \text{and } \eta(y_m; r+H) &\le \max\{\eta(0,r), \eta(y_m, H)\} \\ &= \max\{0, \eta(y_m; H)\} \\ &< \lambda. \end{aligned}$$

Thus, $\mathbb{K} \subset \{m \le n : \psi(y_m; r+H) > 1-\lambda \text{ and } \eta(y_m; r+H) < \lambda\}$.
Using (1), it implies $1-\delta \le \frac{|\mathbb{K}|}{n^\alpha} \le \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; r+H) > 1-\lambda \text{ and } \eta(y_m; r+H) < \lambda\}\right|$.
Therefore,

$$\frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; r+H) \le 1-\lambda \text{ or } \eta(y_m; r+H) \ge \lambda\}\right| < 1-(1-\delta) < \delta.$$

Then,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m; r+H) \le 1-\lambda \text{ or } \eta(y_m; r+H) \ge \lambda\}\right| \ge \delta\right\} \subset G \in \mathcal{I}.$$

Hence, $0 \in \mathcal{I} - st - \text{LIM}_r^\alpha y$. Therefore, $\mathcal{I} - st - \text{LIM}_r^\alpha y \neq \emptyset$.
Sufficient Part:-
Let $\mathcal{I} - st - \text{LIM}_r^\alpha y \neq \emptyset$ for some $r > 0$, then there exists some $\xi \in \Upsilon$ such that $\xi \in \mathcal{I} - st - \text{LIM}_r^\alpha y$.
For every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \le n : \psi(y_m - \xi; r+\varepsilon) \le 1-\lambda \text{ or } \eta(y_m - \xi; r+\varepsilon) \ge \lambda\}\right| \ge \delta\right\} \in \mathcal{I}.$$

Therefore, almost all y_m's are enclosed in some ball with centre ξ in *IFNS*, which imply that $y = \{y_m\}$ is *$\mathcal{I}$-statistically bounded of order α* in an *IFNS*. □

Next, we will show that the algebraic characterization also hold for rough ideal statistical convergent sequences of order α in *intuitionistic fuzzy normed spaces*.

Theorem 3.8. *Let $\{x_m\}$ and $\{y_m\}$ be two sequences in an IFNS (Υ, ψ, η) with $\mathcal{I}$ as admissible ideal and $\alpha \in (0, 1])$, then the following results holds:*
(i) If $x_m \xrightarrow{r-\mathcal{I}-st(\alpha)} L$ and $c \in \mathbb{R}$, then $cx_m \xrightarrow{r-\mathcal{I}-st(\alpha)} cL$.
(ii) If $x_m \xrightarrow{r-\mathcal{I}-st(\alpha)} L_1$ and $y_m \xrightarrow{r-\mathcal{I}-st(\alpha)} L_2$ then $(x_m + y_m) \xrightarrow{r-\mathcal{I}-st(\alpha)} L_1 + L_2$.

Proof. (i) If $c = 0$ then we have nothing to prove. So assume $c \neq 0$. As $r - st_{\mathcal{I}}^{\alpha} - \lim_{m\to\infty} x_k = L$, then for given $\lambda > 0$ and $r \geq 0$,

$$G = \left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(x_m - L; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(x_m - L; r + \varepsilon) \geq \lambda\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is admissible, therefore $M = \mathbb{N} \setminus G$ is a non-empty set. Choose $m \in M$, then

$$\frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(x_m - L; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(x_m - L; r + \varepsilon) \geq \lambda\}\right| < \delta$$

$$\implies \frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(x_m - L; r + \varepsilon) > 1 - \lambda \text{ or } \eta(x_m - L; r + \varepsilon) < \lambda\}\right| \geq 1 - \delta.$$

$$\implies \frac{1}{n^{\alpha}} |\mathbb{K}| \geq 1 - \delta. \tag{2}$$

Where

$$\mathbb{K} = \{m \in \mathbb{N} : \psi(x_m - L; r + \varepsilon) > 1 - \lambda \text{ and } \eta(x_m - L; r + \varepsilon) < \lambda\}.$$

It is sufficient to prove that for each $\lambda > 0$ and $r \geq 0$;

$$\mathbb{K} \subset \{m \in \mathbb{N} : \psi(cx_m - cL; r + \varepsilon) > 1 - \lambda \text{ and } \eta(cx_m - cL; r + \varepsilon) < \lambda\}.$$

Let $m \in \mathbb{K}$, then $\psi(x_m - L; r + \varepsilon) > 1 - \lambda$ and $\eta(x_m - L; r + \varepsilon) < \lambda$.
So,

$$\begin{aligned}
\psi(cx_m - cL; r + \varepsilon) &= \psi\left(x_m - L, \frac{r + \varepsilon}{|c|}\right) \\
&\geq \min\left\{\psi(x_m - L, r + \varepsilon), \psi\left(0, \frac{r + \varepsilon}{|c|} - (r + \varepsilon)\right)\right\} \\
&= \min\{\psi(x_m - L, r + \varepsilon), 1\} \\
&= \psi(x_m - L, r + \varepsilon) > 1 - \lambda, \\
\text{and } \eta(cx_m - cL; r + \varepsilon) &= \eta\left(x_m - L, \frac{r + \varepsilon}{|c|}\right) \\
&\leq \max\left\{\eta(x_m - L, r + \varepsilon), \eta\left(0, \frac{r + \varepsilon}{|c|} - (r + \varepsilon)\right)\right\} \\
&= \max\{\eta(x_m - L, r + \varepsilon), 0\} \\
&= \eta(x_m - L, r + \varepsilon) < \lambda.
\end{aligned}$$

Which gives,

$$\mathbb{K} \subset \{m \in \mathbb{N} : \psi(cx_m - cL; r + \varepsilon) > 1 - \lambda \text{ and } \eta(cx_m - cL; r + \varepsilon) < \lambda\}.$$

Using (2),we have $1 - \delta \leq \frac{|\mathbb{K}|}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(cx_m - cL; r + \varepsilon) > 1 - \lambda \text{ and } \eta(cx_m - cL; r + \varepsilon) < \lambda\}\right|$.
Therefore,

$$\frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(cx_m - cL; r + \varepsilon) \leq 1 - \lambda \text{ and } \eta(cx_m - cL; r + \varepsilon) \geq \lambda\}\right| < 1 - (1 - \delta) < \delta.$$

Then,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(cx_m - cL; r+\varepsilon) \leq 1-\lambda \text{ or } \eta(cx_m - cL; r+\varepsilon) \geq \lambda\}\right| \geq \delta\right\} \subset \mathbb{G} \in \mathcal{I}.$$

which shows that $cL \in \mathcal{I} - st - \text{LIM}_r^{\alpha} cx_m$ in (Υ, ψ, η).
(ii) In the similar manner, we can prove (ii) part. So, we are omitting its proof. □

In next result, we will show the set $\mathcal{I} - st - \text{LIM}_r^{\alpha} y$ is closed.

Theorem 3.9. *Let $y = \{y_m\}$ be a sequence and r is some non-negative real number. Then, the set $\mathcal{I} - st - LIM_r^{\alpha} y$ of a sequence y in an IFNS (Υ, ψ, η) is a closed set.*

Proof. If $\mathcal{I} - st - \text{LIM}_r^{\alpha} y = \emptyset$ then the result is obvious as $\mathcal{I} - st - \text{LIM}_r^{\alpha} y$ is either empty set or singleton set. Let $\mathcal{I} - st - \text{LIM}_r^{\alpha} y \neq \emptyset$ for some $r > 0$.
Let $x = \{x_m\}$ be a convergent sequence in (Υ, ψ, η) with respect to (ψ, η), which converges to $x_0 \in \Upsilon$. For $\varepsilon > 0$ and $\lambda \in (0,1)$ then, there exists $m_0 \in \mathbb{N}$ such that

$$\psi\left(x_m - x_0; \frac{\varepsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(x_m - x_0; \frac{\varepsilon}{2}\right) < \lambda \text{ for all } m \geq m_0.$$

Let us take $x_{m_1} \in \mathcal{I} - st - \text{LIM}_r^{\alpha} y$ which gives the existence of a set $\mathbb{A}$ where

$$\mathbb{A} = \left\{m \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\left\{m \leq n : \psi\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) \leq 1-\lambda \text{ or } \eta\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) \geq \lambda\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is admissible so $G = \mathbb{N} \setminus \mathbb{A}$ is a non-empty set. Choose $n \in G$, then

$$\frac{1}{n^{\alpha}} \left|\left\{m \leq n : \psi\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) \leq 1-\lambda \text{ or } \eta\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) \geq \lambda\right\}\right| < \delta$$

$$\Rightarrow \frac{1}{n^{\alpha}} \left|\left\{m \leq n : \psi\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) < \lambda\right\}\right| \geq 1-\delta.$$

Put $\mathbb{B}_n = \left\{m \leq n : \psi\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) > 1-\lambda \text{ and } \eta\left(y_m - x_{m_1}; r+\frac{\varepsilon}{2}\right) < \lambda\right\}$.
Then, for $j \in \mathbb{B}_n$, $j \geq m_0$, we have

$$\begin{aligned} \psi(y_j - x_0; r+\varepsilon) &\geq \min\left\{\psi\left(y_j - x_{m_1}; r+\frac{\varepsilon}{2}\right), \psi\left(x_{m_1} - x_0; \frac{\varepsilon}{2}\right)\right\} \\ &> 1-\lambda, \\ \text{and } \eta(y_j - x_0; r+\varepsilon) &\leq \max\left\{\eta\left(y_j - x_{m_1}; r+\frac{\varepsilon}{2}\right), \eta\left(x_{m_1} - x_0; \frac{\varepsilon}{2}\right)\right\} \\ &< \lambda. \end{aligned}$$

Therefore;

$$j \in \{m \in \mathbb{N} : \psi(y_m - x_0; r+\varepsilon) > 1-\lambda \text{ and } \eta(y_m - x_0; r+\varepsilon) < \lambda\}.$$

Hence, $\mathbb{B}_n \subset \{m \in \mathbb{N} : \psi(y_m - x_0; r+\varepsilon) > 1-\lambda \text{ and } \eta(y_m - x_0; r+\varepsilon) < \lambda\}$, which implies that $1-\delta \leq \frac{|\mathbb{B}_n|}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(y_m - x_0; r+\varepsilon) > 1-\lambda \text{ and } \eta(y_m - x_0; r+\varepsilon) < \lambda\}\right|$.
Therefore, $\frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(y_m - x_0; r+\varepsilon) \leq 1-\lambda \text{ or } \eta(y_m - x_0; r+\varepsilon) \geq \lambda\}\right| < 1-(1-\delta) = \delta$.
Then,

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\{m \leq n : \psi(y_m - x_0; r+\varepsilon) \leq 1-\lambda \text{ or } \eta(y_m - x_0; r+\varepsilon) \geq \lambda\}\right| \geq \delta\right\} \subset \mathbb{A} \in \mathcal{I}.$$

which shows that $x_0 \in \mathcal{I} - st - \text{LIM}_r^{\alpha} y$ in (Υ, ψ, η). □

The convexity of the set $\mathcal{I} - st - \text{LIM}_r^{\alpha} y$ is demonstrated in the following result.

Theorem 3.10. *Let $y = \{y_m\}$ be any sequence in an IFNS (Υ, ψ, η) then the set $\mathcal{I} - st - LIM_r^\alpha y$ is a convex set for some non-negative number r.*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{I} - st - \mathrm{LIM}_r^\alpha y$. For convexity we have to show that $(1-\omega)\varphi_1 + \omega\varphi_2 \in \mathcal{I} - st - \mathrm{LIM}_r^\alpha y$ for any real number $\omega \in (0, 1)$.
Since $\varphi_1, \varphi_2 \in \mathcal{I} - st - \mathrm{LIM}_r^\alpha y_m$, then there exists $m \in \mathbb{N}$ for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that

$$\mathbb{A}_0 = \left\{m \in \mathbb{N} : \psi\left(y_m - \varphi_1; \frac{r+\varepsilon}{2(1-\omega)}\right) \leq 1 - \lambda \text{ or } \eta\left(y_m - \varphi_1; \frac{r+\varepsilon}{2(1-\omega)}\right) \geq \lambda\right\},$$

and

$$\mathbb{A}_1 = \left\{m \in \mathbb{N} : \psi\left(y_m - \varphi_2; \frac{r+\varepsilon}{2\omega}\right) \leq 1 - \lambda \text{ or } \eta\left(y_m - \varphi_2; \frac{r+\varepsilon}{2\omega}\right) \geq \lambda\right\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_0 \cup \mathbb{A}_1\}| \geq \delta\right\} \in \mathcal{I}.$$

Now choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Let

$$\mathbb{A} = \left\{n \in \mathbb{N} : \frac{1}{n} |\{m \leq n : m \in \mathbb{A}_0 \cup \mathbb{A}_1\}| \geq \delta_1\right\} \in \mathcal{I}.$$

Now for $n \notin \mathbb{A}$

$$\frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_0 \cup \mathbb{A}_1\}| < 1 - \delta_1$$

$$\frac{1}{n^\alpha} |\{m \leq n : m \notin \mathbb{A}_0 \cup \mathbb{A}_1\}| \geq 1 - (1 - \delta_1) = \delta_1.$$

This implies $\{m \leq n : m \notin \mathbb{A}_0 \cup \mathbb{A}_1\} \neq \emptyset$.
Let $m_0 \in (\mathbb{A}_0 \cup \mathbb{A}_1)^c = \mathbb{A}_0^c \cap \mathbb{A}_1^c$.
Then,

$$\begin{aligned}\psi\left(y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\varepsilon\right) &= \psi[(1-\omega)(y_{m_0} - \varphi_1) + \omega(y_{m_0} - \varphi_2); r+\varepsilon] \\ &\geq \min\left\{\psi\left((1-\omega)(y_{m_0} - \varphi_1); \frac{r+\varepsilon}{2}\right), \psi\left(\omega(y_{m_0} - \varphi_2); \frac{r+\varepsilon}{2}\right)\right\} \\ &= \min\left\{\psi\left(y_{m_0} - \varphi_1; \frac{r+\varepsilon}{2(1-\omega)}\right), \psi\left(y_{m_0} - \varphi_2; \frac{r+\varepsilon}{2\omega}\right)\right\} > 1 - \lambda,\end{aligned}$$

and

$$\begin{aligned}\eta\left(y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\varepsilon\right) &= \eta[(1-\omega)(y_{m_0} - \varphi_1) + \omega(y_{m_0} - \varphi_2); r+\varepsilon] \\ &\leq \max\left\{\eta\left((1-\omega)(y_{m_0} - \varphi_1); \frac{r+\varepsilon}{2}\right), \eta\left(\omega(y_{m_0} - \varphi_2); \frac{r+\varepsilon}{2}\right)\right\} \\ &= \max\left\{\eta\left(y_{m_0} - \varphi_1; \frac{r+\varepsilon}{2(1-\omega)}\right), \eta\left(y_{m_0} - \varphi_2; \frac{r+\varepsilon}{2\omega}\right)\right\} < \lambda.\end{aligned}$$

This implies $\mathbb{A}_0^c \cap \mathbb{A}_1^c \subset \mathbb{B}^c$.
Where
$\mathbb{B} = \{m \in \mathbb{N} : \psi\left(y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\varepsilon\right) \leq 1 - \lambda \text{ or } \eta\left(y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\varepsilon\right) \geq \lambda\}$.
So for $n \notin \mathbb{A}$,

$$\delta_1 \leq \frac{1}{n^\alpha} |\{m \leq n : m \notin \mathbb{A}_0 \cup \mathbb{A}_1\}| \leq \frac{1}{n^\alpha} |\{m \leq n : m \notin \mathbb{B}\}|$$

or $\frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{B}\}| < 1 - \delta_1 < \delta$.
Thus, $\mathbb{A}^c \subset \{n \in \mathbb{N} : \frac{1}{n^\alpha}|m \leq n : m \in \mathbb{B}| < \delta\}$. Since $\mathbb{A}^c \in \mathbb{F}(\mathcal{I})$, then $\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} |m \leq n : m \in \mathbb{B}| < \delta\right\} \in \mathbb{F}(\mathcal{I})$, which implies that $\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} |m \leq n : m \in \mathbb{B}| \geq \delta\right\} \in \mathcal{I}$. Therefore, $\mathcal{I} - st - \mathrm{LIM}_r^\alpha y_m$ is a convex set. □

Theorem 3.11. *A sequence $y = \{y_m\}$ in an IFNS (Υ, ψ, η) is rough ideal statistically convergent of order α to $\rho \in \Upsilon$ with respect to the norm (ψ, η) for some $r > 0$ if there exists a sequence $z = \{z_m\}$ in Υ such that $\mathcal{I} - st - LIM^{\alpha} z = \rho$ in Υ and for every $\lambda \in (0, 1)$ have $\psi(y_m - z_m; r + \varepsilon) > 1 - \lambda$ and $\eta(y_m - z_m; r + \varepsilon) < \lambda$ for all $m \in \mathbb{N}$.*

Proof. Consider $z = \{z_m\}$ be a sequence in Υ, which is $\mathcal{I}$-statistically convergent of order α to ρ and $\psi(y_m - z_m; r + \varepsilon) < \lambda$ and $\eta(y_m - z_m; r + \varepsilon) > 1 - \lambda$.
Then by definition, for any $\varepsilon, \delta > 0$ and $\lambda \in (0, 1)$ the set

$$M = \left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left| \{ m \leq n : \psi(z_m - \rho; \varepsilon) \leq 1 - \lambda \text{ or } \eta(z_m - \rho; \varepsilon) \geq \lambda \} \right| \geq \delta \right\} \in \mathcal{I}.$$

Define

$$\mathbb{A}_1 = \{ m \in \mathbb{N} : \psi(z_m - \rho; \varepsilon) \leq 1 - \lambda \text{ or } \eta(z_m - \rho; \varepsilon) \geq \lambda \}$$
$$\mathbb{A}_2 = \{ m \in \mathbb{N} : \psi(y_m - z_m; r) \leq 1 - \lambda \text{ or } \eta(y_m - z_m; r) \geq \lambda \}.$$

For $\delta > 0$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} |\{ m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2 \}| \geq \delta \right\} \in \mathcal{I}.$$

Now choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$ and let

$$\mathbb{A} = \left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} |\{ m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2 \}| \geq \delta_1 \right\} \in \mathcal{I}.$$

Now for $n \notin \mathbb{A}$

$$\frac{1}{n^{\alpha}} |\{ m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2 \}| < 1 - \delta_1$$
$$\frac{1}{n^{\alpha}} |\{ m \leq n : m \notin \mathbb{A}_1 \cup \mathbb{A}_2 \}| \geq 1 - (1 - \delta_1) = \delta_1.$$

This implies $\{ m \leq n : m \notin \mathbb{A}_1 \cup \mathbb{A}_2 \} \neq \emptyset$.
Let $m \in (\mathbb{A}_1 \cup \mathbb{A}_2)^c = \mathbb{A}_1^c \cap \mathbb{A}_1^c$.
Then,

$$\begin{aligned} \psi\left(y_m - \rho; r + \varepsilon\right) &\geq \min \left\{ \psi(y_m - z_m; r), \psi(z_m - \rho; \varepsilon) \right\} \\ &> 1 - \lambda, \\ \text{and } \eta\left(y_m - \rho; r + \varepsilon\right) &\leq \max \left\{ \eta(y_m - z_m; r), \eta(z_m - \rho; \varepsilon) \right\} \\ &< \lambda. \end{aligned}$$

Which gives $\mathbb{A}_1^c \cap \mathbb{A}_2^c \subset \mathbb{B}^c$, where

$$\mathbb{B} = \{ m \in \mathbb{N} : \psi\left(y_m - \rho; r + \varepsilon\right) \leq 1 - \lambda \text{ or } \eta\left(y_m - \rho; r + \varepsilon\right) \geq \lambda \}.$$

So for $n \notin \mathbb{A}$,

$$\delta_1 \leq \frac{1}{n^{\alpha}} |\{ m \leq n : m \notin \mathbb{A}_1 \cup \mathbb{A}_2 \}| \leq \frac{1}{n^{\alpha}} |\{ m \leq n : m \notin \mathbb{B} \}|$$

or

$$\frac{1}{n^{\alpha}} |\{ m \leq n : m \in \mathbb{B} \}| < 1 - \delta_1 < \delta.$$

Thus, $\mathbb{A}^c \subset \left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} |m \leq n : m \in \mathbb{B}| < \delta \right\}$. Since $\mathbb{A}^c \in \mathbb{F}(\mathcal{I})$ then $\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} |m \leq n : m \in \mathbb{B}| < \delta \right\} \in \mathbb{F}(\mathcal{I})$, which implies $\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} |m \leq n : m \in \mathbb{B}| \geq \delta \right\} \in \mathcal{I}$. Hence, $y_m \xrightarrow{r-\mathcal{I}-st(\alpha)} \rho$ with respect to the norm (ψ, η). □

Theorem 3.12. *Let $y = \{y_m\}$ be a sequence in an IFNS (Υ, ψ, η) then there does not exist two elements $\alpha_1, \alpha_2 \in \mathcal{I} - st - LIM_r^{\alpha} y$ for $r > 0$ and $\lambda \in (0, 1)$ such that $\psi(\alpha_1 - \alpha_2; cr) \leq 1 - \lambda$ and $\eta(\alpha_1 - \alpha_2; cr) \geq \lambda$ for $c > 2$.*

Proof. We shall use contradiction to support our conclusion. Assume there exists two elements $\alpha_1, \alpha_2 \in \mathcal{I} - st - \text{LIM}_r^\alpha y$ such that

$$\psi(\alpha_1 - \alpha_2; cr) \leq 1 - \lambda \text{ and } \eta(\alpha_1 - \alpha_2; cr) \geq \lambda \text{ for } c > 2. \tag{3}$$

As $\alpha_1, \alpha_2 \in \mathcal{I} - st - \text{LIM}_r^\alpha y$ then for every $\varepsilon > 0$ and $\lambda \in (0, 1)$. Define,

$$\mathbb{A}_1 = \left\{ m \in \mathbb{N} : \psi\left(y_m - \alpha_1; r + \frac{\varepsilon}{2}\right) \leq 1 - \lambda \text{ or } \eta\left(y_m - \alpha_1; r + \frac{\varepsilon}{2}\right) \geq \lambda \right\}$$

$$\mathbb{A}_2 = \left\{ m \in \mathbb{N} : \psi\left(y_m - \alpha_2; r + \frac{\varepsilon}{2}\right) \leq 1 - \lambda \text{ or } \eta\left(y_m - \alpha_2; r + \frac{\varepsilon}{2}\right) \geq \lambda \right\}.$$

Then,

$$\frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_1\}| + \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_2\}|$$

So, by the property of $\mathcal{I}$-*convergence*, we have

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{n^\alpha} |m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2| \leq \mathcal{I} - \lim_{n\to\infty} \frac{1}{n^\alpha} |m \leq n : m \in \mathbb{A}_1| + \mathcal{I} - \lim_{n\to\infty} \frac{1}{n^\alpha} |m \leq n : m \in \mathbb{A}_2|$$

Thus,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \geq \delta \right\} \in \mathcal{I}, \text{for all } \delta > 0.$$

Now choose $0 < \delta_1 = 1/2 < 1$ such that $0 < 1 - \delta_1 < \delta$.
Let

$$\mathbb{K} = \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \geq \delta_1 \right\} \in \mathcal{I}.$$

Now for $n \notin \mathbb{K}$

$$\frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{A}_1 \cup \mathbb{A}_2\}| < 1 - \delta_1 = 1/2$$

$$\frac{1}{n^\alpha} |\{m \leq n : m \notin \mathbb{A}_1 \cup \mathbb{A}_2\}| \geq 1 - (1 - \delta_1) = 1/2.$$

This implies $\{m \leq n : m \notin \mathbb{A}_1 \cup \mathbb{A}_2\} \neq \emptyset$. Then for $m \in \mathbb{A}_1^c \cap \mathbb{A}_2^c$, we have

$$\begin{aligned} \psi(\alpha_1 - \alpha_2; 2r + \varepsilon) &\geq \min \left\{ \psi\left(y_m - \alpha_2 : r + \frac{\varepsilon}{2}\right), \psi\left(y_m - \alpha_1; r + \frac{\varepsilon}{2}\right) \right\} \\ &> 1 - \lambda, \\ \text{and } \eta(\alpha_1 - \alpha_2; 2r + \varepsilon) &\leq \max \left\{ \eta\left(y_m - \alpha_2 : r + \frac{\varepsilon}{2}\right), \eta\left(y_m - \alpha_1; r + \frac{\varepsilon}{2}\right) \right\} \\ &< \lambda. \end{aligned}$$

Hence,

$$\psi(\alpha_1 - \alpha_2; 2r + \varepsilon) > 1 - \lambda \text{ and } \eta(\alpha_1 - \alpha_2; 2r + \varepsilon) < \lambda. \tag{4}$$

Then from (4) we have $\psi(\alpha_1 - \alpha_2; cr) > 1 - \lambda$ and $\eta(\alpha_1 - \alpha_2; cr) < \lambda$ for $c > 2$. which is contradiction to (3). Therefore, there does not exist two elements such that $\psi(\alpha_1 - \alpha_2; cr) \leq 1 - \lambda$ and $\eta(\alpha_1 - \alpha_2; cr) \geq \lambda$ for $c > 2$. □

Definition 3.13. *Let (Υ, ψ, η) be an intuitionistic fuzzy normed space. Then $\gamma \in \Upsilon$ is called rough $\mathcal{I}$-statistical cluster point of order α of the sequence $y = \{y_m\}$ in Υ with respect to norm (ψ, η) for some $r > 0$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$*

$$\delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi(y_m - \gamma; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \gamma; r + \varepsilon) < \lambda\}) \neq 0,$$

where $\delta_{\mathcal{I}}(A) = \mathcal{I} - \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \le n : m \in A\}|$ *if exists. In this case* γ *is known as r-*$\mathcal{I}$*-statistical cluster point of order* α *of a sequence* $y = \{y_m\}$*.*
Let $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$ *denotes the set of all r-*$\mathcal{I}$*-statistical cluster points of order* α *with respect to the norm* (ψ, η) *of a sequence* $y = \{y_m\}$ *in an IFNS* (Υ, ψ, η)*. If* $r = 0$ *and* $\alpha = 1$ *then we get ideal statistical cluster point with respect to the norm* (ψ, η) *in an IFNS* (Υ, ψ, η) *i.e.* $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) = \Gamma_{st(\psi,\eta)}(\mathcal{I})$*.*

Theorem 3.14. *Let* (Υ, ψ, η) *be an intuitionistic fuzzy normed space. Then, the set* $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$ *of any sequence* $y = \{y_m\}$ *is closed for some* $r > 0$*.*

Proof. If $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) = \emptyset$, then we have nothing to prove.
So Assume, $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) \neq \emptyset$. Take a sequence $x = \{x_m\} \subseteq \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$ such that $x_m \xrightarrow{(\psi,\eta)} x_0$. It is sufficient to show that $x_0 \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$.
As $x_m \xrightarrow{(\psi,\eta)} x_0$, then for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $m_\varepsilon \in \mathbb{N}$ such that $\psi\left(x_m - x_0; \frac{\varepsilon}{2}\right) > 1 - \lambda$ and $\eta\left(x_m - x_0; \frac{\varepsilon}{2}\right) < \lambda$ for $m \ge m_\varepsilon$.
Choose some $m_0 \in \mathbb{N}$ such that $m_0 \ge m_\varepsilon$. Then we have $\psi\left(x_{m_0} - x_0; \frac{\varepsilon}{2}\right) > 1 - \lambda$ and $\eta\left(x_{m_0} - x_0; \frac{\varepsilon}{2}\right) < \lambda$.
Again as $x = \{x_m\} \subseteq \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$, we have $x_{m_0} \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$.

$$\implies \delta_I\left(\left\{m \in \mathbb{N} : \psi(y_m - x_{m_0}; r + \frac{\varepsilon}{2}) > 1 - \lambda \text{ and } \eta(y_m - x_{m_0}); r + \frac{\varepsilon}{2}) < \lambda\right\}\right) \neq 0. \tag{5}$$

Consider $\mathbb{G} = \left\{m \in \mathbb{N} : \psi(y_m - x_{m_0}; r + \frac{\varepsilon}{2}) > 1 - \lambda \text{ and } \eta(y_m - x_{m_0}; r + \frac{\varepsilon}{2}) < \lambda\right\}$. Choose $j \in \mathbb{G}$, then we have $\psi(y_j - x_{m_0}; r + \frac{\varepsilon}{2}) > 1 - \lambda$ and $\eta(y_j - x_{m_0}; r + \frac{\varepsilon}{2}) < \lambda$.
Now,

$$\begin{aligned}\psi\left(y_j - x_0; r + \varepsilon\right) &\ge \min\left\{\psi\left(y_j - x_{m_0}; r + \frac{\varepsilon}{2}\right), \psi\left(x_{m_0} - x_0; r + \frac{\varepsilon}{2}\right)\right\} \\ &> 1 - \lambda, \\ \text{and } \eta\left(y_j - x_0; r + \varepsilon\right) &\le \max\left\{\eta\left(y_j - x_{m_0}; r + \frac{\varepsilon}{2}\right), \eta\left(x_{m_0} - x_0; r + \frac{\varepsilon}{2}\right)\right\} \\ &< \lambda.\end{aligned}$$

Thus,

$$j \in \{m \in \mathbb{N} : \psi(y_m - x_0; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - x_0; r + \varepsilon) < \lambda\}.$$

Hence,

$$\begin{aligned}&\{m \in \mathbb{N} : \psi\left(y_m - x_{m_0}; r + \frac{\varepsilon}{2}\right) > 1 - \lambda \text{ and } \eta\left(y_m - x_{m_0}; r + \frac{\varepsilon}{2}\right) < \lambda\} \\ &\qquad \subseteq \{m \in \mathbb{N} : \psi(y_m - x_0; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - x_0; r + \varepsilon) < \lambda\}.\end{aligned}$$

$$\begin{aligned}&\delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi\left(y_m - x_{m_0}; r + \frac{\varepsilon}{2}\right) > 1 - \lambda \text{ and } \eta\left(y_m - x_{m_0}; r + \frac{\varepsilon}{2}\right) < \lambda\}) \\ &\qquad \le \delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi(y_m - x_0; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - x_0; r + \varepsilon) < \lambda\}).\end{aligned} \tag{6}$$

From (5), the set on left side hand of (6) has natural density more than zero, which implies that

$$\delta_{\mathcal{I}}\left(\{m \in \mathbb{N} : \psi(y_m - x_0; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - x_0; r + \varepsilon) < \lambda\}\right) \neq 0.$$

Therefore, $x_0 \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$. Hence the result proved. □

Theorem 3.15. *Let $\Gamma^{(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$ be the set of all $\mathcal{I}$-statistical cluster points of order α of the sequence $y = \{y_m\}$ in an intuitionistic fuzzy normed space(Υ, ψ, η). Then for any arbitrary $\nu \in \Gamma^{(\alpha)}_{st(\psi,\eta)}(\mathcal{I}), r > 0$ and $\lambda \in (0,1)$, we have $\psi(\zeta - \nu; r) > 1 - \lambda$ and $\eta(\zeta - \nu; r) < \lambda$ for all $\zeta \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$.*

Proof. Since $\nu \in \Gamma^{(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$ then for $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\delta_{\mathcal{I}}\left(\{m \in \mathbb{N} : \psi(y_m - \nu; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \nu; \varepsilon) < \lambda\}\right) \neq 0. \tag{7}$$

Now, it is sufficient to show that if any $\zeta \in \Upsilon$ satisfy $\psi(\zeta - \nu; \varepsilon) > 1 - \lambda$ and $\eta(\zeta - \nu; \varepsilon) < \lambda$, then $\zeta \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$. Suppose $j \in \{m \in \mathbb{N} : \psi(y_m - \nu; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \nu; \varepsilon) < \lambda\}$ then $\psi(y_j - \nu; \varepsilon) > 1 - \lambda$ and $\eta(y_j - \nu; \varepsilon) < \lambda$. Now,

$$\begin{aligned} \psi\left(y_j - \zeta; r + \varepsilon\right) &\geq \min\left\{\psi\left(y_j - \nu; \varepsilon\right), \psi(\zeta - \nu; r)\right\} \\ &> 1 - \lambda, \\ \text{and } \eta\left(y_j - \zeta; r + \varepsilon\right) &\leq \max\left\{\eta\left(y_j - \nu; \varepsilon\right), \eta(\zeta - \nu; r)\right\} \\ &< \lambda. \end{aligned}$$

So, we have $\psi\left(y_j - \zeta; r + \varepsilon\right) > 1 - \lambda$ and $\eta\left(y_j - \zeta; r + \varepsilon\right) < \lambda$.
Thus, $j \in \{m \in \mathbb{N} : \psi(y_m - \zeta; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \zeta; r + \varepsilon) < \lambda\}$ which gives the inclusion

$$\{m \in \mathbb{N} : \psi(y_m - \nu; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \nu; \varepsilon) < \lambda\} \subseteq \{m \in \mathbb{N} : \psi(y_m - \zeta; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \zeta; r + \varepsilon) < \lambda\}.$$

Then

$$\begin{aligned} &\delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi(y_m - \nu; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \nu; \varepsilon) < \lambda\}) \\ &\qquad \leq \delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi(y_m - \zeta; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \zeta; r + \varepsilon) < \lambda\}). \end{aligned}$$

Therefore, from (7),

$$\delta_I\left(\{m \in \mathbb{N} : \psi(y_m - \zeta; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \zeta; r + \varepsilon) < \lambda\}\right) \neq 0.$$

Hence, $\zeta \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$. □

Theorem 3.16. *Let $y = \{y_m\}$ be a sequence in intuitionistic fuzzy normed space(Υ, ψ, η) and $\overline{B(\rho, \lambda, r)} = \{y \in \Upsilon : \psi(y - \rho; r) \geq 1 - \lambda, \eta(y - \rho; r) \leq \lambda\}$, is a closed ball for some $r > 0$, $\alpha \in (0,1]$ and $\lambda \in (0,1)$ and fixed $\rho \in \Upsilon$ then*

$$\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) = \bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho, \lambda, r)}.$$

Proof. Let $\zeta \in \bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho, \lambda, r)}$ then there exists some $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$ for $r > 0$, $\lambda \in (0,1)$ and $\alpha \in (0,1]$ such that $\psi(\rho - \zeta; r) > 1 - \lambda$ and $\eta(\rho - \zeta; r) < \lambda$.
As $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})(y)$ then there exists a set $\mathbb{M} = \{m \in \mathbb{N} : \psi(y_m - \rho; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \rho; \varepsilon) < \lambda\}$, with $\delta_{\mathcal{I}}(\mathbb{M}) \neq 0$. For $m \in \mathbb{M}$,

$$\begin{aligned} \psi(y_m - \zeta; r + \varepsilon) &\geq \min\{\psi(y_m - \rho; \varepsilon), \psi(\rho - \zeta; r)\} \\ &> 1 - \lambda, \\ \text{and } \eta(y_m - \zeta; r + \varepsilon) &\leq \max\{\eta(y_m - \rho; \varepsilon), \eta(\rho - \zeta; r)\} \\ &< \lambda. \end{aligned}$$

This implies that $\delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi(y_m - \zeta; r+\varepsilon) > 1-\lambda \text{ and } \eta(y_m - \zeta; r+\varepsilon) < \lambda\}) \neq 0$.
Hence, $\zeta \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$. So, $\bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)} \subseteq \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$.
Conversely, Take $\zeta \in \Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})$ and we want to show that $\zeta \in \bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)}$. We will prove the result by contadiction, if possible let $\zeta \notin \bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)}$ *i.e* $\zeta \notin \overline{B(\rho,\lambda,r)}$ for all $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$. Then $\psi(\zeta-\rho;r) \leq 1-\lambda$ or $\eta(\zeta-\rho;r) \geq \lambda$ for every $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$. But by Theorem (3.15), we have $\psi(\zeta-\rho;r) > 1-\lambda$ and $\eta(\zeta-\rho;r) < \lambda$ for every $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$. Which is contadiction to our supposition. Hence, $\zeta \in \bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)}$ and $\zeta \in \bigcup_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)}$. This completes the proof. □

Theorem 3.17. *Let $y = \{y_m\}$ be a sequence in intuitionistic fuzzy normed space(Υ,ψ,η), Then for $\lambda \in (0,1)$ and $\alpha \in (0,1]$,*
(i) If $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$ then $\mathcal{I} - st - LIM^{\alpha}_r \subseteq \overline{B(\rho,\lambda,r)}$.
(ii) $\mathcal{I} - st - LIM^{\alpha}_r = \bigcap_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)} = \left\{\xi \in \Upsilon : \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I}) \subseteq \overline{B(\xi,\lambda,r)}\right\}$.

Proof. Let $\xi \in \mathcal{I} - st - \mathrm{LIM}^{\alpha}_r$ and $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$
For $\varepsilon > 0, \lambda \in (0,1)$ and $\alpha \in (0,1]$ Consider

$$\mathbb{G} = \{m \in \mathbb{N} : \psi(y_m - \xi; r+\varepsilon) > 1-\lambda \text{ and } \eta(y_m - \xi; r+\varepsilon) < \lambda\}$$

and

$$\mathbb{H} = \{m \in \mathbb{N} : \psi(y_m - \rho; \varepsilon) > 1-\lambda \text{ and } \eta(y_m - \rho; \varepsilon) < \lambda\}$$

with $\delta_{\mathcal{I}}(\mathbb{G}^c) = 0$ and $\delta_{\mathcal{I}}(\mathbb{H}) \neq 0$ respectively. Now for $m \in \mathbb{G} \bigcap \mathbb{H}$,

$$\begin{aligned} \psi(\xi-\rho;r) &\geq \min\{\psi(y_m-\rho;\varepsilon), \psi(y_m-\xi;r+\varepsilon)\} \\ &> 1-\lambda, \\ \text{and } \eta(\xi-\rho;r) &\leq \max\{\eta(y_m-\rho;\varepsilon), \eta(y_m-\xi;r+\varepsilon)\} \\ &< \lambda. \end{aligned}$$

Which gives $\xi \in \overline{B(\rho,\lambda,r)}$.
(ii) From (i) part, one can easily see that $\mathcal{I} - st - \mathrm{LIM}^{\alpha}_r \subseteq \bigcap_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)}$.
Take $y \in \bigcap_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)}$ then $\psi(y-\rho;r) \geq 1-\lambda$ and $\eta(y-\rho;r) \leq \lambda$ for all $\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})$. This implies that $\Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})(y) \subseteq \overline{B(y,\lambda,r)}$ *i.e.* $\bigcap_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)} \subseteq \left\{\xi \in \Upsilon : \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I}) \subseteq \overline{B(\xi,\lambda,r)}\right\}$.
Now assume $y \notin \mathcal{I} - st - \mathrm{LIM}^{\alpha}_r$, then

$$\delta_{\mathcal{I}}(\{m \in \mathbb{N} : \psi(y_m - y; r+\varepsilon) \leq 1-\lambda \text{ or } \eta(y_m - y; r+\varepsilon) \geq \lambda\}) \neq 0, \text{ for all } \varepsilon > 0, \lambda \in (0,1].$$

Then there exists some $\mathcal{I}$-statistical cluster point ρ for the sequence $y = \{y_m\}$ such that $\psi(y-\rho;r+\varepsilon) \leq 1-\lambda$ or $\eta(y-\rho;r+\varepsilon) \geq \lambda$.
Thus, $\Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I}) \nsubseteq \overline{B(y,\lambda,r)}$ and $y \notin \left\{\xi \in \Upsilon : \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I}) \subseteq \overline{B(\xi,\lambda,r)}\right\}$.
Hence, $\left\{\xi \in \Upsilon : \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I}) \subseteq \overline{B(\xi,\lambda,r)}\right\} \subseteq \mathcal{I} - st - \mathrm{LIM}^{\alpha}_r$.
And $\bigcap_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)} \subseteq \mathcal{I} - st - \mathrm{LIM}^{\alpha}_r$.
So, $\mathcal{I} - st - \mathrm{LIM}^{\alpha}_r = \bigcap_{\rho \in \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho,\lambda,r)} = \left\{\xi \in \Upsilon : \Gamma^{\alpha}_{st(\psi,\eta)}(\mathcal{I}) \subseteq \overline{B(\xi,\lambda,r)}\right\}$. □

Theorem 3.18. *Let $y = \{y_m\}$ be a sequence in intuitionistic fuzzy normed space(Υ,ψ,η), which is ideal statistically convergent of order α to ρ then $\overline{B(\rho,\lambda,r)} = \mathcal{I} - st - LIM^{\alpha}_r$.*

Proof. Since sequence $y = \{y_m\}$ is ideal statistically convergent of order α to ρ with respect to the norm (ψ, η) *i.e* $y_m \xrightarrow{\mathcal{I}-st(\psi,\eta)} \rho$, then there is a set $\mathbb{A}$ such that

$$\mathbb{A} = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \leq n : \psi(y_m - \rho; \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - \rho; \varepsilon) \geq \lambda\}\right| > \delta\right\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is admissible so $G = \mathbb{N} \setminus \mathbb{A}$ is a non-empty set, then for $m \in G^c$,

$$\frac{1}{n^\alpha}\left|\{m \leq n : \psi(y_m - \rho; \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - \rho; \varepsilon) \geq \lambda\}\right| < \delta$$

$$\Rightarrow \frac{1}{n^\alpha}\left|\{m \leq n : \psi(y_m - \rho; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \rho; \varepsilon) < \lambda\}\right| \geq 1 - \delta.$$

Put $\mathbb{B}_n = \{m \leq n : \psi(y_m - \rho; \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - \rho; \varepsilon) < \lambda\}$ for $j \geq m$.
Now for $j \in \mathbb{B}_n$, we have $\psi\left(y_j - \rho; \varepsilon\right) > 1 - \lambda$ and $\eta\left(y_j - \rho; \varepsilon\right) < \lambda$.
Let $y \in \overline{B(\rho, \lambda, r)}$. We will prove $y \in \mathcal{I} - st - \text{LIM}_r^\alpha y$

$$\begin{aligned}\psi(y_j - y; r + \varepsilon) &\geq \min\left\{\psi(y_j - \rho, \varepsilon), \psi(y - \rho, r)\right\}\\ &> 1 - \lambda,\\ \text{and } \eta(y_j - y; r + \varepsilon) &\leq \max\left\{\psi(y_j - \rho, \varepsilon), \eta(y - \rho, r)\right\}\\ &< \lambda.\end{aligned}$$

Hence, $\mathbb{B}_n \subset \{m \in \mathbb{N} : \psi(y_m - y; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - y; r + \varepsilon) < \lambda\}$, which implies $1 - \delta \leq \frac{|\mathbb{B}_n|}{n^\alpha} \leq \frac{1}{n^\alpha}\left|\{m \leq n : \psi(y_m - y; r + \varepsilon) > 1 - \lambda \text{ and } \eta(y_m - y; r + \varepsilon) < \lambda\}\right|$.
Therefore, $\frac{1}{n^\alpha}\left|\{m \leq n : \psi(y_m - y; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - y; r + \varepsilon) \geq \lambda\}\right| < 1 - (1 - \delta) = \delta$.
Then,

$$\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{m \leq n : \psi(y_m - y; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - y; r + \varepsilon) \geq \lambda\}\right| \geq \delta\} \subset \mathbb{A} \in \mathcal{I}.$$

Which shows that $y \in \mathcal{I} - st - \text{LIM}_r^\alpha y$ in (Υ, ψ, η).
Hence $\overline{B(\rho, \lambda, r)} \subseteq \mathcal{I} - st - \text{LIM}_r^\alpha$. Also $\mathcal{I} - st - \text{LIM}_r^\alpha \subseteq \overline{B(\rho, \lambda, r)}$ Therefore; $\mathcal{I} - st - \text{LIM}_r^\alpha = \overline{B(\rho, \lambda, r)}$. □

Theorem 3.19. *Let $y = \{y_m\}$ be a sequence in intuitionistic fuzzy normed space(Υ, ψ, η), which is ideal statistically convergent of order α to ξ then $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})y = \mathcal{I} - st - LIM_r^\alpha y$.*

Proof. Firstly, Assume $y_m \xrightarrow{\mathcal{I}-st(\psi,\eta)} \xi$, which gives $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I})y = \{\xi\}$. Then for $r > 0, \lambda \in (0, 1)$ and $\alpha \in (0, 1]$ by Theorem (3.16) , we have $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) = \overline{B(\xi, \lambda, r)}$. Also from Theorem (3.18), $\overline{B(\xi, \lambda, r)} = \mathcal{I} - st - \text{LIM}_r^\alpha y$.
Hence, $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) = \mathcal{I} - st - \text{LIM}_r^\alpha y$.
Conversely, assume $\Gamma^{r(\alpha)}_{st(\psi,\eta)}(\mathcal{I}) = \mathcal{I} - st - \text{LIM}_r^\alpha y$, then by Theorem (3.16) and (3.17)(ii),

$$\bigcap_{\rho \in \Gamma^\alpha_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho, \lambda, r)} = \bigcup_{\rho \in \Gamma^\alpha_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho, \lambda, r)}.$$

This is possible only if either $\Gamma^\alpha_{st(\psi,\eta)}(\mathcal{I}) = \emptyset$ or $\Gamma^\alpha_{st(\psi,\eta)}(\mathcal{I})$ is a singlton set. Then $\mathcal{I} - st - \text{LIM}_r^\alpha y = \bigcap_{\rho \in \Gamma^\alpha_{st(\psi,\eta)}(\mathcal{I})} \overline{B(\rho, \lambda, r)} = \overline{B(\xi, \lambda, r)}$ for some $\xi \in \Gamma^\alpha_{st(\psi,\eta)}(\mathcal{I})$. Also, by Theorem (3.16), $\mathcal{I} - st - \text{LIM}_r^\alpha y = \{\xi\}$. □

Theorem 3.20. *Let $0 < \alpha \leq \beta \leq 1$. Then, $rS^\alpha_\mathcal{I} \subseteq rS^\beta_\mathcal{I}$, where $rS^\alpha_\mathcal{I}$ and $rS^\beta_\mathcal{I}$ denote the collection of rough ideal statistical convergent sequences of order α and β respectively.*

Proof. For every $\varepsilon > 0$ and $r > 0$ with limit y, define

$$\mathbb{G} = \{m \leq n : \psi(y_m - y; r + \varepsilon) \leq 1 - \lambda \text{ or } \eta(y_m - y; r + \varepsilon) \geq \lambda\}$$

such that $\left\{n \in \mathbb{N} : \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{G}\}| \geq \delta\right\} \in \mathcal{I}$.
As $0 < \alpha \leq \beta \leq 1$, then

$$\frac{1}{n^\beta} |\{m \leq n : m \in \mathbb{G}\}| \leq \frac{1}{n^\alpha} |\{m \leq n : m \in \mathbb{G}\}$$

Which clearly shows that $rS^\alpha_\mathcal{I} \subseteq rS^\beta_\mathcal{I}$. □

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