Filomat 38:30 (2024), 10555–10566
https://doi.org/10.2298/FIL2430555G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Asymptotic estimations of eigenvalues and eigenfunctions for a multi-point nonlocal boundary value problem

Yu Gan[a], Zhaowen Zheng[b,a,*], Kun Li[a]

[a] *School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China*
[b] *College of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, China*

Abstract. In this paper, the multi-point nonlocal second order linear Sturm-Liouville problem is considered consisting of the equation $-u''(t) + q(t)u(t) = \lambda u(t)$ on $[0, 1]$ and multi-point boundary value conditions. The geometric multiplicity of eigenvalues, the asymptotic formulas for eigenvalues and eigenfunctions are expressed explicitly under certain mild conditions, these results can be used to investigate the inverse spectral problems of certain Sturm-Liouville operators.

1. Introduction

Sturm-Liouville theory is well known in qualitative analysis of eigenvalues and eigenfunctions of second-order linear boundary value problems ([6, 19]), and many researches are involved such as self-adjointness of operator, distribution of eigenvalues, asymptotic properties of eigenvalues, oscillation of eigenfunctions, completeness of system of eigenfunctions. Among these researches, nonlocal boundary value problems have been of growing interest in recent years since they can describe states and properties that related to past moments ([8, 12, 15, 24, 34]), such processes arise in mathematical physics, biology and biotechnology and other fields. The multi-point nonlocal boundary value problem for the second-order ordinary differential equation was first proposed by Ilyin and Moisseev in [16], later, some scholars devoted themselves to the study of such problems (see [1–3, 14]). The authors in [7, 11, 29] studied eigenvalues for such problems with nonlocal boundary conditions of Ionkin-Samarskii or integral type. More complicated cases of Sturm-Liouville problem with one classical boundary condition and another Bitsadze-Samarskii type or integral type nonlocal boundary condition were considered in [25, 26]. In particular, Sturm-Liouville problem in three cases of Bitsadze-Samarskii type nonlocal two-point boundary conditions were investigated by Peciulyte and Stikonas in [25]. They obtained the general properties of eigenvalues and eigenfunctions, the qualitative behavior of eigenvalues dependent on boundary condition parameters was also described.

2020 *Mathematics Subject Classification*. Primary 47E05; Secondary 34B10.
Keywords. nonlocal boundary value problem, multi-point boundary value problem, asymptotic formulas of eigenvalues and eigenfunctions.
Received: 06 June 2024; Accepted: 04 September 2024
Communicated by Snežana Č. Živković-Zlatanović
Research is partly supported by the NSF of Shandong Province (Nos. ZR2023MA023, ZR2024MA020), NNSF of China (No. 12401160) and Guangdong Provincial Featured Innovation Projects of High School (No. 2023KTSCX067).
* Corresponding author: Zhaowen Zheng
Email addresses: ygan@qfnu.edu.cn (Yu Gan), zhwzheng@126.com (Zhaowen Zheng), qslikun@163.com (Kun Li)

In 2007, the following Sturm-Liouville problem with one classical boundary condition and another Bitsadze-Samarskii type nonlocal boundary condition

$$-u''(t) = \lambda u(t), t \in (0, 1)$$

$$u(0) = 0$$

$$u(1) = \gamma u(\xi)$$

was considered in [31], where $\gamma \in \overline{\mathbb{C}}$ and $\xi \in [0, 1]$. The author proved general properties of the eigenfunction and spectrum for such a problem in the complex case and analyzed the case of real eigenvalues. Then the characteristic function method has been used in [32] for the purpose of presenting some new results of the spectrum of the above problem.

Recently, Sen and Stikonas in [33] obtained the asymptotic formulas of eigenvalues and eigenfunctions for the following second order nonlocal boundary value problem with potential function $q(t)$ in the differential equation

$$-u''(t) + q(t)u(t) = \lambda u(t), t \in [0, 1]$$

$$u(0) = 0,$$

$$u(1) = \gamma u(\xi),$$

where $\gamma \in \mathbb{R}$ and $\xi \in (0, 1)$.

Meanwhile, the multi-point boundary value problems of ordinary differential equations have become an important area of research in recent years, the authors in [17] considered the second order linear Sturm-Liouville problem which involve one or two multi-point boundary conditions. They got certain interlacing relations between the eigenvalues of Sturm-Liouville problems with multi-point boundary conditions and those with two-point separated boundary conditions. Algebraic multiplicity of eigenvalues was also involved. Furthermore, the studies of multi-point Sturm-Liouville problems will set up a foundation for the further investigations of nonlinear boundary value problems with multi-point boundary conditions. There were some literatures on such problems (see [4, 5, 9, 10, 17, 20–23, 27, 28] for more details).

In [9], the authors investigated the structure of eigenvalues for the multi-point boundary value problem in the following form:

$$-y''(x) + q(x)y = \lambda y, x \in [0, 1]$$

$$y(0) = 0,$$

$$y(1) - \sum_{k=1}^{m} \alpha_k y(\eta_k) = 0,$$

where $q \in L^1([0, 1], \mathbb{R})$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, and $0 < \eta_1 < \cdots < \eta_m < 1$. They gave a sufficient condition for real eigenvalues, but did not give the asymptotic formulas of eigenvalues and eigenfunctions.

Inspired by the above literatures, in this paper, we discuss a class of boundary value problems consisting the equation

$$-u''(t) + q(t)u(t) = \lambda u(t), t \in [0, 1] \tag{1}$$

together with the $(m + 2)$-point boundary conditions of the form

$$u(0) = 0, \tag{2}$$

$$u(1) = \sum_{j=1}^{m} \alpha_j u(\eta_j), \tag{3}$$

where the potential $q(t)$ is a real-valued function which is continuous in $[0,1]$, $\lambda = s^2$ is a complex spectral parameter and $s = x+iy, x, y \in \mathbb{R}$. $\alpha_j \in \mathbb{R}$ and $\alpha_j > 0$ (or $\alpha_j < 0$), $j = 1, 2, \cdots, m$, simultaneously. $0 < \eta_1 < \cdots < \eta_j < 1$.

The rest of the paper is organized as follows. In Section 2, we will give the asymptotic formulas of solutions satisfying the initial conditions, and prove that the geometric multiplicity of the eigenvalue is 1. In Section 3, by comparing the characteristic equation with the case $q = 0$, we analyze the characteristic equation of problem (1)-(3). Finally, the asymptotic formulas of eigenvalues and eigenfunctions for the multi-point nonlocal boundary value problem are expressed explicitly under certain mild conditions.

In the sequel, we will use the following symbols:

$$\mathbb{R}_s^- = \{s = x+iy \in \mathbb{C} : x = 0, y > 0\}, \quad \mathbb{R}_s^+ = \{s = x+iy \in \mathbb{C} : x > 0, y = 0\},$$
$$\mathbb{C}_s^+ = \{s = x+iy \in \mathbb{C} : x > 0, y > 0\}, \quad \mathbb{C}_s^- = \{s = x+iy \in \mathbb{C} : x > 0, y < 0\},$$
$$\mathbb{R}_s^0 = \{s = 0\},$$

where $\mathbb{R}_s = \mathbb{R}_s^- \cup \mathbb{R}_s^+ \cup \mathbb{R}_s^0$, $\mathbb{C}_s = \mathbb{R}_s \cup \mathbb{C}_s^+ \cup \mathbb{C}_s^-$.

2. Preliminaries

In this section, we will give estimations of solutions satisfying the initial conditions and analyze the geometric multiplicity of eigenvalues.

Let $\varphi_\lambda(t) = \varphi(t, \lambda)$ be the solution of equation (1) satisfying the initial conditions

$$\varphi_\lambda(0) = 0, \quad \varphi'_\lambda(0) = -1. \tag{4}$$

Then we know that $\varphi_\lambda(t)$ is uniquely determined by the existence and uniqueness theorem of the solution, and $\varphi_\lambda(t)$ is an analytic function with respect to λ on the complex plane by the differentiability theorem of the solution to the parameters.

Lemma 2.1. *[19] The solution $\varphi_\lambda(t)$ of equation* (1) *satisfying initial condition* (4) *has the following expression for $\lambda \neq 0$:*

$$\varphi_\lambda(t) = -\frac{1}{s}\sin(st) + \frac{1}{s}\int_0^t \sin(s(t-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau. \tag{5}$$

Lemma 2.2. *[33] Let $s \in \mathbb{C}_s$, then there exists $q_0 > 0$ such that for $|s| > q_0$ one has the estimate*

$$\varphi_\lambda(t) = O(s^{-1}e^{|y|t}), \quad |s| \to \infty, \tag{6}$$

and more precisely,

$$\varphi_\lambda(t) = -s^{-1}\sin(st) + O(s^{-2}e^{|y|t}), \quad |s| \to \infty. \tag{7}$$

These estimates hold uniformly for $0 \le t \le 1$.

Lemma 2.3. *[33] Let $s \in \mathbb{C}_s$, then there exists $q_0 > 0$ such that for $|s| > 2q_0$ one has the estimate*

$$\varphi'_\lambda(t) = O(e^{|y|t}), \quad |s| \to \infty, \tag{8}$$

and more precisely,

$$\varphi'_\lambda(t) = -\cos(st) + O(s^{-1}e^{|y|t}), \quad |s| \to \infty. \tag{9}$$

These estimates hold uniformly for $0 \le t \le 1$.

Theorem 2.4. *The geometric multiplicity of eigenvalues of problem* (1)-(3) *is* 1.

Proof. We suppose that λ is the eigenvalue of (1)-(3) and $\psi_\lambda(t)$ is the corresponding eigenfunction. The statement is obvious since all solutions with $\psi_\lambda(0) = 0$ are multiples of φ_λ by the uniqueness property of the initial value problem. □

3. Main results

In this section, we will get the main results of the paper.

Substituting $\varphi_\lambda(t)$ into $u(1)=\sum_{j=1}^{m}\alpha_j u(\eta_j)$, we get the characteristic equation of problem (1)-(3)

$$\varphi_\lambda(1)-\sum_{j=1}^{m}\alpha_j\varphi_\lambda(\eta_j)=0. \tag{10}$$

For $s=\sqrt{\lambda}$, we define a function

$$F(s):=-s\left(\varphi_\lambda(1)-\sum_{j=1}^{m}\alpha_j\varphi_\lambda(\eta_j)\right). \tag{11}$$

We see that $F(s)$ is an analytic function of s. The set of eigenvalues for problem (1)-(3) is equal to the set $\left\{\lambda:\lambda=s^2,-F(s)/s=\varphi_\lambda(1)-\sum_{j=1}^{m}\alpha_j\varphi_\lambda(\eta_j)=0\right\}$.

Substituting (5) into (11), we obtain

$$\begin{aligned}F(s)&=\sin s-\int_0^1\sin(s(1-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau\\&\quad-\sum_{j=1}^{m}\alpha_j\left(\sin(\eta_j s)-\int_0^{\eta_j}\sin(s(\eta_j-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau\right),\end{aligned}$$

$$\begin{aligned}F'(s)&=\cos s-\sum_{j=1}^{m}\alpha_j\eta_j\cos(\eta_j s)\\&\quad-\int_0^1(1-\tau)\cos(s(1-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau+\sum_{j=1}^{m}\alpha_j\int_0^{\eta_j}(\eta_j-\tau)\cos(s(\eta_j-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau.\end{aligned}$$

Next, we estimate $F(s)$ and $F'(s)$. Since

$$\begin{aligned}\left|-\int_0^1\sin(s(1-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau\right|&\le\int_0^1|\sin(s(1-\tau)||q(\tau)||\varphi_\lambda(\tau)|\mathrm{d}\tau\\&\le\int_0^1 e^{|y|(1-\tau)}|q(\tau)|C(s^{-1}e^{|y|\tau})\mathrm{d}\tau\\&=C\int_0^1|q(\tau)|\mathrm{d}\tau\cdot s^{-1}e^{|y|}=O(s^{-1}e^{|y|}),\end{aligned}$$

and for $0<\eta_j<1$ ($j=1,\cdots,m$),

$$\begin{aligned}\left|\alpha_j\int_0^{\eta_j}\sin(s(\eta_j-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau\right|&\le|\alpha_j|\int_0^{\eta_j}e^{|y|(\eta_j-\tau)}|q(\tau)|C(s^{-1}e^{|y|\tau})\mathrm{d}\tau\\&\le C|\alpha_j|\int_0^1|q(\tau)|\mathrm{d}\tau\cdot s^{-1}e^{|y|}\\&=O(s^{-1}e^{|y|}),\end{aligned}$$

where $C>0$ is a constant determined by the solution. We can write $F(s)$ and $F'(s)$ as

$$F(s)=f(s)+f_0(s)=\sin s-\sum_{j=1}^{m}\alpha_j\sin(\eta_j s)+O(s^{-1}e^{|y|}), \tag{12}$$

$$F'(s) = \cos s - \sum_{j=1}^{m} \alpha_j \eta_j \cos(\eta_j s) + O(s^{-1}e^{|y|}). \tag{13}$$

Theorem 3.1. *If $\alpha_j > 0$ ($j = 1, 2, \cdots, m$) and $\sum_{j=1}^{m} \alpha_j < 1$, then there exists a countable number of positive eigenvalues for problem* (1)-(3) *and the equation $F(x) = 0$ has at least one positive root in each interval $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ for k large enough.*

Proof. Suppose $s = x,\ 0 < x \in \mathbb{R}$, according to (12), we get

$$F(x) = \sin x - \sum_{j=1}^{m} \alpha_j \sin(\eta_j x) + O(x^{-1}).$$

For x large enough,

$$\begin{aligned} &\left| \sum_{j=1}^{m} \alpha_j \sin(\eta_j x) + O(x^{-1}) \right| \\ &\leq \alpha_1 |\sin(\eta_1 x)| + \alpha_2 |\sin(\eta_2 x)| + \cdots + \alpha_m |\sin(\eta_m x)| + |O(x^{-1})| \\ &< \sum_{j=1}^{m} \alpha_j < 1. \end{aligned}$$

Since $\sin x$ takes the local maximum value 1 at $-\frac{3\pi}{2} + 2k\pi,\ k \in \mathbb{N}$, and the local minimum value -1 at $-\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{N}$. Using the intermediate value theorem, the equation $F(x) = 0$ has at least one root in each interval $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi),\ K < k \in \mathbb{N}$ for K large enough. So the equation $F(x) = 0$ has countable numbers of roots. □

Remark 3.2. *Similarly, equation $f(x) = \sin x - \sum_{j=1}^{m} \alpha_j \sin(\eta_j x) = 0$ has at least one positive root in the interval $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ for k large enough. And it is obtained that the root of equation $f(x) = 0$ is unique in each interval $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ by using the implicit function theorem as in [27], so we represent the root as $x_k = k\pi + g_k(\alpha_1, \cdots, \alpha_m)$, where $|g_k(\alpha_1, \cdots, \alpha_m)| \leq \frac{\pi}{2}$.*

If $q(t) = 0$, we note that the characteristic equation of problem (1)-(3) is exact the equation $f(s) = \sin s - \sum_{j=1}^{m} \alpha_j \sin(\eta_j s) = 0$. Remark 3.2 shows that for $\sum_{j=1}^{m} \alpha_j < 1$, equation $f(s) = 0$ has countable numbers of positive simple roots x_k, which can be obtained by solving equation $f(x) = 0$. According to Rouche's theorem, there exists $\widetilde{x_k}$ between two roots x_k and x_{k+1} of the above equation such that $f'(\widetilde{x_k}) = 0$, that is, $\widetilde{x_k}$ is the root of $\cos x - \sum_{j=1}^{m} \alpha_j \eta_j \cos(\eta_j x) = 0$.

Remark 3.3. *Suppose $\alpha_j > 0$ ($j = 1, 2, \cdots, m$) with $\sum_{j=1}^{m} \alpha_j < 1$ and $0 < \eta_j$. If x_k is a root of $\sin x - \sum_{j=1}^{m} \alpha_j \sin(\eta_j x) = 0$, due to the simplicity of the roots of the equation ([27]), there exists $\kappa > 0$ such that $\left| \cos x_k - \sum_{j=1}^{m} \alpha_j \eta_j \cos(\eta_j x_k) \right| \geq \kappa > 0$, where κ depends on x.*

Remark 3.4. *Suppose* $\alpha_j > 0$ $(j = 1, 2, \cdots, m)$ *with* $\sum_{j=1}^{m} \alpha_j < 1$ *and* $0 < \eta_j < 1$. *If* $\cos x - \sum_{j=1}^{m} \alpha_j \cos(\eta_j x) = 0$, *let* $x = a_k := (k + \frac{1}{2})\pi$, $k \in \mathbb{N}$. *At this time, by* $\cos a_k = 0$, *we have* $\alpha_1 \cos(\eta_1 a_k) + \alpha_2 \cos(\eta_2 a_k) + \cdots + \alpha_m \cos(\eta_m a_k) = 0$, *then there exists* $\widetilde{\kappa} > 0$ *such that* $|\sin a_k| - \sum_{j=1}^{m} \alpha_j |\sin(\eta_j a_k)| \geq \widetilde{\kappa} =: 1 - \sum_{j=1}^{m} \alpha_j > 0$.

Let $D_k = \{s : |x| \leq a_k = (k + \frac{1}{2})\pi, |y| \leq a_k\}$, $D_k^s = D_k \cap \mathbb{C}_s$. Define a contour $\Gamma_k^s = \partial D_k \cap \mathbb{C}_s$, it can be seen from [32] that the corresponding contour Γ_k^λ is the boundary of the domain D_k^λ in the plane $\mathbb{C}_\lambda$, where $\lambda = s^2$ is the bijection from $\mathbb{C}_s$ to $\mathbb{C}_\lambda$.

Lemma 3.5. *Suppose* $\sum_{j=1}^{m} \alpha_j < 1$, *then there exists* $l > 0$ *such that all eigenvalues of problem* (1)-(3) *are positive in the domain* $\{s \in \mathbb{C}_s : |s| > l\}$.

Proof. On the vertical part of the contour Γ_k^s, i.e. $s = a_k + iy$, $y \in [-a_k, a_k]$,

$$f(s) = \sin s - \sum_{j=1}^{m} \alpha_j \sin(\eta_j s).$$

Taking the real part of $f(s)$, we get

$$\mathrm{Re} f(s) = \sin a_k \cosh y - \sum_{j=1}^{m} \alpha_j \sin(\eta_j a_k) \cosh(\eta_j y),$$

$$\begin{aligned} |f(s)| \geq |\mathrm{Re} f(s)| &\geq |\sin a_k| \cosh y - \sum_{j=1}^{m} \alpha_j |\sin(\eta_j a_k)| \cosh(\eta_j y) \\ &\geq \left(|\sin a_k| - \sum_{j=1}^{m} \alpha_j |\sin(\eta_j a_k)| \right) \cosh y. \end{aligned}$$

According to Remark 3.4, we get $|f(s)| \geq \widetilde{k} \cosh y \geq M_1 e^{|y|}$, where $M_1 > 0$.

On the rest part of the contour Γ_k^s, i.e. $y = \pm a_k$, $0 \leq x \leq a_k$. Since

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

we get

$$\begin{aligned} |\sin s| &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \\ &= \sqrt{\cosh^2 y - \cos^2 x}. \end{aligned}$$

Then

$$|\sin s| \geq |\sinh y|, \quad |\sin(\eta_j s)| \leq \cosh(\eta_j y).$$

So we have

$$\begin{aligned} |f(s)| &= \left| \sin s - \sum_{j=1}^{m} \alpha_j \sin(\eta_j s) \right| \\ &\geq |\sin s| - \sum_{j=1}^{m} \alpha_j |\sin(\eta_j s)| \\ &\geq |\sinh y| - \sum_{j=1}^{m} \alpha_j \cosh(\eta_j y) = h(y) e^{|y|}, \end{aligned}$$

where $h(y)$ is defined by

$$h(y) = \left(|\sinh y| - \sum_{j=1}^{m} \alpha_j \cosh(\eta_j y)\right) e^{-|y|}$$
$$= \frac{1}{2} - \frac{e^{-2|y|}}{2} - \sum_{j=1}^{m} \alpha_j \frac{e^{(\eta_j-1)|y|}}{2} - \sum_{j=1}^{m} \alpha_j \frac{e^{-(\eta_j+1)|y|}}{2}.$$

We know that there exists $\widetilde{y} > 0$ such that $h(y) \geq \frac{1}{5}$ for $|y| > \widetilde{y}$ by analyzing the function $h(y)$, so $|f(s)| \geq \frac{e^{|y|}}{5}$. Taking $M = \min\{M_1, \frac{1}{5}\}$, Thus we get $|f(s)| \geq Me^{|y|}$ on Γ_k^s for k large enough. By (12) and $|s| > l$, we obtain

$$|f_0(s)| \leq c_1|s|^{-1}e^{|y|} < Me^{|y|} \leq |f(s)|,$$

where c_1 is a constant determined by the solution. According to Rouche's theorem, it is obvious that $f(s) = 0$ and $F(s) = f(s) + f_0(s) = 0$ have the same number of zeros inside Γ_k^s. Since $f(s) = 0$ has only one root in the area between contours Γ_{k-1}^s and Γ_k^s, equation $F(s) = 0$ also has one root in this area. Combined with Theorem 3.1, we obtain that the roots of $F(s) = 0$ in the domain $\{s \in \mathbb{C}_s : |s| > l\}$ are positive. □

Let s_k be a root of $F(s) = 0$, we know that s_k is positive by Lemma 3.5 for k large enough. Next we will study the distribution of positive eigenvalues of problem (1)-(3), and we only consider the case of $s = x > 0$, since eigenvalues are real, as $|s| \to \infty$, we have

$$F(s) = f(s) + O(s^{-1}), \quad \text{where} \quad f(s) = \sin s - \sum_{j=1}^{m} \alpha_j \sin(\eta_j s).$$

$$F'(s) = f'(s) + O(s^{-1}), \quad \text{and} \quad f'(s) = \cos s - \sum_{j=1}^{m} \alpha_j \eta_j \cos(\eta_j s).$$

It follows from $x_k, s_k \in ((k - 1/2)\pi, (k + 1/2)\pi)$ that

$$s_k \sim x_k \sim k\pi \text{ as } k \to \infty.$$

Let $\omega_k = s_k - x_k$, we get

$$\lim_{k\to\infty} w_k = 0 \quad i.e., \quad w_k = o(1), \ k \to \infty.$$

By (7), we obtain

$$\varphi_s(t) = -\frac{\sin(st)}{s} + O(s^{-2}), \ |s| \to \infty. \tag{14}$$

Theorem 3.6. *Let $q \in C[0, 1]$, suppose $\alpha_j > 0$ $(j = 1, 2, \cdots, m)$ and $\sum_{j=1}^{m} \alpha_j < 1$. Then as $k \to \infty$, the asymptotic formulas of eigenvalues λ_k and eigenfunctions u_k of problem* (1)-(3) *have the forms*

$$s_k = x_k + O(k^{-1}), \quad u_k(t) = -\frac{\sin(x_k t)}{x_k} + O(k^{-2}). \tag{15}$$

Proof. Substituting $s_k = x_k + \omega_k$ into $F(s) = \sin s - \sum_{j=1}^{m} \alpha_j \sin(\eta_j s) + O(s^{-1}) = 0$, as $k \to \infty$, we get

$$\sin x_k - \sum_{j=1}^{m} \alpha_j \sin(\eta_j x_k) + \omega_k \left(\cos x_k - \sum_{j=1}^{m} \alpha_j \eta_j \cos(\eta_j x_k)\right) + O(\omega_k^2) = O(k^{-1}).$$

Since x_k is the root of $f(s) = 0$, i.e. $\sin x_k - \sum_{j=1}^{m} \alpha_j \sin(\eta_j x_k) = 0$, we have

$$\left(\cos x_k - \sum_{j=1}^{m} \alpha_j \eta_j \cos(\eta_j x_k) + O(\omega_k)\right)\omega_k = O(k^{-1}),\ k \to \infty.$$

Then it is obtained from Remark 3.3 that $\omega_k = O(k^{-1})$. Similarly, substituting $s_k = x_k + \omega_k$ into (14), we get

$$\begin{aligned} u_k(t) = \varphi_{\lambda_k}(t) &= -\frac{\sin((x_k+\omega_k)t)}{x_k+\omega_k} + O(k^{-2}) \\ &= -\frac{\sin(x_k t)}{x_k} - \left[\frac{\sin((x_k+\omega_k)t)}{x_k+\omega_k} - \frac{\sin(x_k t)}{x_k}\right] + O(k^{-2}) \\ &= -\frac{\sin(x_k t)}{x_k} - \frac{x_k \sin(x_k t) O(\omega_k^2) + x_k \omega_k t \cos(x_k t) - \sin(x_k t)\omega_k}{x_k^2} + O(k^{-2}) \\ &= -\frac{\sin(x_k t)}{x_k} - \frac{x_k t \cos(x_k t) - \sin(x_k t)}{x_k^2}\omega_k - \frac{\sin(x_k t)}{x_k} O(\omega_k^2) + O(k^{-2}) \\ &= -\frac{\sin(x_k t)}{x_k} + O(k^{-2}),\ k \to \infty. \end{aligned}$$

□

As $k \to \infty$, we normalize $u_k(t)$ as

$$\begin{aligned} \alpha_k^2 = \int_0^1 u_k^2 \mathrm{d}t = \int_0^1 \left(\frac{\sin^2(x_k t)}{x_k^2} + O(k^{-3})\right)\mathrm{d}t \\ &= \frac{1}{x_k^2}\left[\frac{1}{2} - \frac{1}{4x_k}\sin(2x_k)\right] + O(k^{-3}) \\ &= \frac{1}{2x_k^2}(1 + O(k^{-1})), \end{aligned}$$

$$-\frac{1}{\alpha_k} = -\sqrt{2}x_k + O(1).$$

Therefore, as $k \to \infty$, the normalized eigenfunctions have asymptotic formulas

$$v_k(t) = (-\sqrt{2}x_k + O(1))\left[-\frac{\sin(x_k t)}{x_k} + O(k^{-2})\right] = \sqrt{2}\sin(x_k t) + O(k^{-1}).$$

In order to obtain more exact asymptotic formulas of eigenvalues and eigenfunctions, we assume that $q \in C^1[0,1]$, then as $|s| \to \infty$, the following formulas hold:

$$\int_0^t q(\tau)\cos(2s\tau)\mathrm{d}\tau = O(s^{-1}), \quad \int_0^t q(\tau)\sin(2s\tau)d\tau = O(s^{-1}).$$

Let $Q(t) = \frac{1}{2}\int_0^t q(\tau)d\tau$, it is obvious that $Q(t)$ is bounded. Substituting (14) into (5), we obtain

$$\varphi_\lambda(t) = -\frac{1}{s}\sin(st) + \frac{1}{s}\int_0^t \sin(st - s\tau)q(\tau)\left[-\frac{\sin(s\tau)}{s} + O(s^{-2})\right]\mathrm{d}\tau,\ |s| \to \infty.$$

Since

$$-\frac{1}{s^2}\int_0^t \sin(s\tau)\sin(st-s\tau)q(\tau)\mathrm{d}\tau$$
$$=-\frac{\sin(st)}{2s^2}\int_0^t \sin(2s\tau)q(\tau)\mathrm{d}\tau$$
$$+\frac{Q(t)\cos(st)}{s^2}-\frac{\cos(st)}{2s^2}\int_0^t \cos(2s\tau)q(\tau)\mathrm{d}\tau,$$

as $|s|\to\infty$, we have

$$\left|-\frac{\sin(st)}{2s^2}\int_0^t \sin(2s\tau)q(\tau)\mathrm{d}\tau\right|=\frac{|\sin(st)|}{2s^2}O(s^{-1})=O(s^{-3}),$$

$$\left|-\frac{\cos(st)}{2s^2}\int_0^t \cos(2s\tau)q(\tau)\mathrm{d}\tau\right|=O(s^{-3}),$$

$$\frac{1}{s}\int_0^t \sin(st-s\tau)q(\tau)O(s^{-2})\mathrm{d}\tau=O(s^{-3}).$$

So we get

$$\varphi_s(t)=-\frac{1}{s}\sin(st)+\frac{Q(t)\cos(st)}{s^2}+O(s^{-3}),\ |s|\to\infty. \tag{16}$$

Then by (11), we obtain

$$F(s)=\sin s-\sum_{j=1}^m \alpha_j\sin(\eta_j s)-\frac{Q(1)\cos s-\sum_{j=1}^m \alpha_j Q(\eta_j)\cos(\eta_j s)}{s}+O(s^{-2}),\ |s|\to\infty. \tag{17}$$

Define

$$W(\alpha_1,\cdots,\alpha_m;\eta_1,\cdots,\eta_m;s):=\frac{Q(1)\cos s-\sum_{j=1}^m \alpha_j Q(\eta_j)\cos(\eta_j s)}{\cos s-\sum_{j=1}^m \alpha_j\eta_j\cos(\eta_j s)}.$$

For convenience, we abbreviate $W(\alpha_1,\cdots,\alpha_m;\eta_1,\cdots,\eta_m;s)$ to $W(\alpha_j,\eta_j,s)$ in the following assertions.

Theorem 3.7. *If $q\in C^1[0,1]$, $\alpha_j>0$ $(j=1,2,\cdots,m)$ and $\sum_{j=1}^m \alpha_j<1$. Then as $k\to\infty$, the asymptotic formulas of eigenvalues and eigenfunctions of problem* (1)-(3) *have forms*

$$s_k=x_k+W(\alpha_j,\eta_j,x_k)x_k^{-1}+O(k^{-2}), \tag{18}$$

$$u_k(t)=-\frac{\sin(x_k t)}{x_k}+(Q(t)-tW(\alpha_j,\eta_j,x_k))\frac{\cos(x_k t)}{x_k^2}+O(k^{-3}). \tag{19}$$

Proof. Substituting $s_k = x_k + \omega_k$ into (17), as $k \to \infty$, we get

$$\sin(x_k + \omega_k) - \sum_{j=1}^{m} \alpha_j \sin((x_k + \omega_k)\eta_j) - \frac{Q(1)\cos(x_k + \omega_k)}{x_k + \omega_k}$$
$$+ \frac{\sum_{j=1}^{m} \alpha_j Q(\eta_j)\cos((x_k + \omega_k)\eta_j)}{x_k + \omega_k} = O(k^{-2}).$$

The four parts on the left side of the above formula will be discussed below. As $k \to \infty$, we have

$$\sin(x_k + \omega_k) = \sin x_k + \omega_k \cos x_k + O(\omega_k^2),$$

$$-\sum_{j=1}^{m} \alpha_j \sin(x_k\eta_j + \omega_k\eta_j) = -\sum_{j=1}^{m} \alpha_j \sin(x_k\eta_j) - \sum_{j=1}^{m} \alpha_j\omega_k\eta_j \cos(x_k\eta_j) + O(\omega_k^2),$$

$$-\frac{Q(1)\cos(x_k + \omega_k)}{x_k + \omega_k} = -\frac{Q(1)\cos x_k}{x_k} + \frac{Q(1)\sin x_k}{x_k}\omega_k + O(\omega_k^2),$$

$$\frac{\sum_{j=1}^{m} \alpha_j Q(\eta_j)\cos(\eta_j(x_k + \omega_k))}{x_k + \omega_k} = \frac{\sum_{j=1}^{m} \alpha_j Q(\eta_j)}{x_k}\cos(x_k\eta_j) \quad -\frac{\sum_{j=1}^{m} \alpha_j\eta_j Q(\eta_j)\omega_k \sin(x_k\eta_j)}{x_k} + O(\omega_k^2).$$

Then we can obtain that

$$\sin x_k - \sum_{j=1}^{m} \alpha_j \sin(x_k\eta_j) - \frac{Q(1)\cos x_k - \sum_{j=1}^{m} \alpha_j Q(\eta_j)\cos(x_k\eta_j)}{x_k}$$
$$+ \left[\cos x_k - \sum_{j=1}^{m} \alpha_j\eta_j \cos(x_k\eta_j) + (Q(1)\sin x_k - \omega_k \sum_{j=1}^{m} \alpha_j\eta_j Q(\eta_j)\sin(x_k\eta_j))x_k^{-1}\right]\omega_k$$
$$+ O(\omega_k^2) = O(k^{-2}),\ k \to \infty.$$

Since $\sin x_k - \sum_{j=1}^{m} \alpha_j \sin(x_k\eta_j) = 0$, the above formula can be written as

$$\left(\cos x_k - \sum_{j=1}^{m} \alpha_j\eta_j \cos(x_k\eta_j) + O(k^{-1})\right)\omega_k = \frac{Q(1)\cos x_k - \sum_{j=1}^{m} \alpha_j Q(\eta_j)\cos(x_k\eta_j)}{x_k} + O(k^{-2}),\ k \to \infty,$$

or

$$\omega_k = \frac{Q(1)\cos x_k - \sum_{j=1}^{m} \alpha_j Q(\eta_j)\cos(x_k\eta_j)}{x_k\left[\cos x_k - \sum_{j=1}^{m} \alpha_j\eta_j \cos(x_k\eta_j)\right]} + O(k^{-2})$$
$$= \frac{W(\alpha_j, \eta_j, x_k)}{x_k} + O(k^{-2}),\ k \to \infty.$$

Substituting $s_k = x_k + \omega_k$ into (16), we get

$$u_k(t) = -\frac{\sin(x_k t)}{x_k} + \frac{Q(t)\cos(x_k t)}{x_k^2} - \frac{t\cos(x_k t)}{x_k}\omega_k + O(k^{-3}),\ k \to \infty.$$

Finally, it follows by $\omega_k = W(\alpha_j, \eta_j, x_k)x_k^{-1} + O(k^{-2})$ that

$$u_k(t) = -\frac{\sin(x_k t)}{x_k} + (Q(t) - tW(\alpha_j, \eta_j, x_k))\frac{\cos(x_k t)}{x_k^2} + O(k^{-3}),\ k \to \infty.$$

□

Next, we give the normalized eigenfunctions as follows. By (19), as $k \to \infty$, we conclude

$$\begin{aligned}
u_k^2 &= \frac{\sin^2(x_k t)}{x_k^2} - 2\frac{(Q(t) - tW(\alpha_j, \eta_j, x_k))\sin(x_k t)\cos(x_k t)}{x_k^3} + (Q(t) - tW(\alpha_j, \eta_j, x_k))^2\frac{\cos^2(x_k t)}{x_k^4} \\
&\quad + 2\left[-\frac{\sin(x_k t)}{x_k} + (Q(t) - tW(\alpha_j, \eta_j, x_k))\frac{\cos(x_k t)}{x_k^2}\right]O(k^{-3}) + O(k^{-6}) \\
&= \frac{\sin^2(x_k t)}{x_k^2} + \frac{tW(\alpha_j, \eta_j, x_k)\sin(2x_k t)}{x_k^3} + \frac{W^2(\alpha_j, \eta_j, x_k)t^2\cos^2(x_k t)}{x_k^4} - \frac{Q(t)\sin(2x_k t)}{x_k^3} \\
&\quad + \frac{Q^2(t)\cos^2(x_k t)}{x_k^4} - 2\frac{Q(t)tW(\alpha_j, \eta_j, x_k)\cos^2(x_k t)}{x_k^4} + O(k^{-4}) \\
&= \frac{\sin^2(x_k t)}{x_k^2} + \frac{tW(\alpha_j, \eta_j, x_k)\sin(2x_k t)}{x_k^3} - \frac{Q(t)\sin(2x_k t)}{x_k^3} + O(k^{-4}),
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha_k^2 &= \int_0^1 u_k^2(t)dt \\
&= \frac{1}{x_k^2}\int_0^1 \sin^2(x_k t)dt - \frac{1}{x_k^3}\int_0^1 Q(t)\sin(2x_k t)dt + \frac{1}{x_k^3}\int_0^1 tW(\alpha_j, \eta_j, x_k)\sin(2x_k t)dt + O(k^{-4}) \\
&= \frac{1}{x_k^2}\left(\frac{1}{2} - \frac{1}{4x_k}\sin(2x_k) + O(k^{-4})\right) \\
&= \frac{1}{2x_k^2}\left(1 - \frac{1}{2x_k}\sin(2x_k) + O(k^{-2})\right),\ k \to \infty.
\end{aligned}$$

Further, we calculate that

$$\begin{aligned}
\frac{1}{\alpha_k} &= \sqrt{2}x_k\frac{1}{\sqrt{1 - \frac{1}{2x_k}\sin(2x_k) + O(k^{-2})}} \\
&= \sqrt{2}x_k\left(\frac{1}{1 + \frac{1}{4x_k}\sin(2x_k) + O(k^{-2})}\right) \\
&= \sqrt{2}x_k\left(1 + \frac{1}{4x_k}\sin(2x_k) + O(k^{-2})\right),\ k \to \infty.
\end{aligned}$$

Thus we have

$$\begin{aligned}
v_k(t) &= -\sqrt{2}x_k\left(1 + \frac{1}{4x_k}\sin(2x_k) + O(k^{-2})\right)\left(-\frac{\sin(x_k t)}{x_k} + (Q(t) - tW(\alpha_j, \eta_j, x_k))\frac{\cos(x_k t)}{x_k^2} + O(k^{-3})\right) \\
&= \sqrt{2}\sin(x_k t) + \sqrt{2}\frac{0.25\sin(2x_k)\sin(x_k t) - (Q(t) - tW(\alpha_j, \eta_j, x_k))\cos(x_k t)}{x_k} + O(k^{-2}),\ k \to \infty.
\end{aligned}$$

Conflict of Interest Statement

The authors declare that the paper has no conflict of interest.

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