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Weak* integration of functions with values in the set of Hilbert space operators

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Abstract. In this work, we consider weakly* measurable operator-valued functions $\Omega \ni t \mapsto A_t \in \mathcal{B}(\mathcal{H})$ and $\Omega \ni t \mapsto \bigoplus_{n=1}^{+\infty} A_t^{(n)} \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$. Also, we will consider such functions as the elements of the normed space $L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and we show that this space is not complete in general with respect to the norm $\|(A_t)_{t\in\Omega}\|_G = \sup_{\|f\|=\|g\|=1} \int_\Omega |\langle Af, g\rangle|\, d\mu(t)$. Extensions of previous results are given and new problems related to this topic are discussed. For the families $(f_n(t)A_t)_{t\in\Omega}$ and $(f(t)A_t)_{t\in\Omega}$ in $\mathcal{B}(\mathcal{H})$, we have proved

$$\lim_{n\to\infty}\left\| \int_\Omega f_n(t)A_t\, d\mu(t) - \int_\Omega f(t)A_t\, d\mu(t)\right\| = 0,$$

under some additional conditions. Furthermore, necessary and sufficient conditions are discussed for the mapping $\Omega \ni t \mapsto \bigoplus_{n=1}^{+\infty} A_t^{(n)} \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ to belongs to the space $L^1_G\left(\Omega, \mu, \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)\right)$. If these conditions are satisfied we proved that the operator $\bigoplus_{n=1}^{+\infty} \int_\Omega A_t^{(n)} d\mu(t)$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ and we have

$$\int_\Omega \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) = \bigoplus_{n=1}^{+\infty} \int_\Omega A_t^{(n)} d\mu(t).$$

We also consider the vector measures $v : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ and $v : \mathfrak{M} \to \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ associated to the families from the spaces $L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and $L^1_G\left(\Omega, \mu, \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)\right)$ respectively. New results related to operator-valued measures are obtained. We also give a formula that connects the weak* integral and the spectral integral, when the family of operators $(A_t)_{t\in\Omega}$ arises from the same spectral measure. Throughout the paper, specific examples of operator-valued functions are given to illustrate the general results, and in these examples we can see how weak* integrals $\int_\Omega A_t\, d\mu(t)$ can be computed in concrete cases.

1. Introduction and the normed space $L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$

Let $(\Omega, \mathfrak{M}, \mu)$ be a measurable space. To avoid trivialities, we assume that there exists $A \in \mathfrak{M}$ such that $0 < \mu(A) < +\infty$. When we use that the measure μ is σ-finite or finite, we emphasize this. The Lebesgue measure on $\mathbb{R}$ is denoted by m. Let $\mathcal{H}$ be a Hilbert space and we denote the algebra of all linear and bounded

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operators on $\mathcal{H}$ with $\mathcal{B}(\mathcal{H})$. If $\mathcal{K}$ is another Hilbert space, then the space of all linear and bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. The ideal of operators generated by the symmetric norming function (s.n.) Φ is denoted with $\mathfrak{C}_\Phi(\mathcal{H})$, for separable $\mathcal{H}$. Specially, we have the ideal of nuclear operators $\mathfrak{C}_1(\mathcal{H})$ and the ideal of compact operators $\mathfrak{C}_\infty(\mathcal{H})$. Schatten-von Neumann ideals generated by s.n. functions ℓ^p, where $p \in [1,+\infty)$, are denoted by $\mathfrak{C}_p(\mathcal{H})$. The norm on $\mathfrak{C}_p(\mathcal{H})$ is denoted by $\|\cdot\|_p$ or ℓ_p. If $A \in \mathfrak{C}_p(\mathcal{H})$ is a positive operator and $p \in [1,+\infty)$, we have $\|A^p\|_1 = \|A\|_p^p$. If $A \in \mathcal{B}(\mathcal{H})$, then $|A| = \sqrt{A^*A}$ expresses the absolute value of the operator A and we have the equality $\|A\| = \||A|\|$. Furthermore, $A \in \mathfrak{C}_\Phi(\mathcal{H})$ iff $A^* \in \mathfrak{C}_\Phi(\mathcal{H})$ iff $|A| \in \mathfrak{C}_\Phi(\mathcal{H})$ iff $|A^*| \in \mathfrak{C}_\Phi(\mathcal{H})$ and in this case equalities $\|A\|_\Phi = \|A^*\|_\Phi = \||A|\|_\Phi = \||A^*|\|_\Phi$ hold. If $A, B \in \mathcal{B}(\mathcal{H})$, $0 \leqslant A \leqslant B$, $B \in \mathfrak{C}_\Phi(\mathcal{H})$ then we have $A \in \mathfrak{C}_\Phi(\mathcal{H})$ and $\|A\|_\Phi \leqslant \|B\|_\Phi$. For more information on the theory of ideals of compact operators, we refer the reader to [6] and [17].

If $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of Hilbert spaces, then the set

$$\bigoplus_{n=1}^{+\infty}\mathcal{H}_n = \left\{ f = (f_n)_{n\in\mathbb{N}} : \mathbb{N} \to \prod_{n=1}^{+\infty}\mathcal{H}_n \;\middle|\; f_n \in \mathcal{H}_n \text{ for all } n \in \mathbb{N}, \sum_{n=1}^{+\infty}\|f_n\|^2 < +\infty \right\}$$

with the inner product $\langle\cdot,\cdot\rangle : \bigoplus_{n=1}^{+\infty}\mathcal{H}_n \times \bigoplus_{n=1}^{+\infty}\mathcal{H}_n \to \mathbb{C}$ defined by the expression

$$\langle (f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}\rangle = \sum_{n=1}^{+\infty}\langle f_n, g_n\rangle \quad \text{for all} \quad (f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty}\mathcal{H}_n \tag{1}$$

is a Hilbert space. If we have $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{N}$, then we get the Hilbert space $\bigoplus_{n=1}^{+\infty}\mathcal{H}_n = \ell^2(\mathcal{H})$.

Let $A_n \in \mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$. If we have $\sup_{n\in\mathbb{N}}\|A_n\| < +\infty$, then the operator $\bigoplus_{n=1}^{+\infty}A_n : \bigoplus_{n=1}^{+\infty}\mathcal{H}_n \to \bigoplus_{n=1}^{+\infty}\mathcal{H}_n$ defined by $\left(\bigoplus_{n=1}^{+\infty}A_n\right)((f_n)_{n\in\mathbb{N}}) = (A_nf_n)_{n\in\mathbb{N}}$ for all $(f_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty}\mathcal{H}_n$ is linear and bounded and we have the equality $\left\|\bigoplus_{n=1}^{+\infty}A_n\right\| = \sup_{n\in\mathbb{N}}\|A_n\|$. If $\sup_{n\in\mathbb{N}}\|A_n\| = +\infty$, then the operator $A \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$ such that $A|_{\mathcal{H}_n} = A_n$ for all $n \in \mathbb{N}$ does not exist. Properties of direct sums of operators on a Hilbert space were considered in [1], [3] and [9]. We will deal with, among other things, weakly* measurable mappings from measurable space Ω to the Hilbert space $\bigoplus_{n=1}^{+\infty}\mathcal{H}_n$. With I_n we denote the identity operator on $\mathcal{H}_n$.

With $\mathrm{tr} : \mathfrak{C}_1(\mathcal{H}) \to \mathbb{C}$ we denote the trace functional. This is a linear and continuous functional on $\mathfrak{C}_1(\mathcal{H})$ and we have the equality $\|\mathrm{tr}\| = 1$. The mapping $\Omega \ni t \mapsto A_t \in \mathcal{B}(\mathcal{H})$ is weakly* measurable if the scalar function $\Omega \ni t \mapsto \mathrm{tr}(A_tY) \in \mathbb{C}$ is $\mathfrak{M}$-measurable for all $Y \in \mathfrak{C}_1(\mathcal{H})$. Dual space of $\mathfrak{C}_1(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$ (see [6, Theorem 12.1]). If the function $\Omega \ni t \mapsto \mathrm{tr}(A_tY) \in \mathbb{C}$ belongs to $L^1(\Omega,\mathfrak{M},\mu)$ for all $Y \in \mathfrak{C}_1(\mathcal{H})$, by [2, Theorem 11.52], there exists a unique element in $\mathcal{B}(\mathcal{H})$ which we denote by $\int_\Omega A_t\,d\mu(t)$, such that

$$\mathrm{tr}\left(\int_\Omega A_t\,d\mu(t)Y\right) = \int_\Omega \mathrm{tr}(A_tY)\,d\mu(t) \quad \text{for all} \quad Y \in \mathfrak{C}_1(\mathcal{H}). \tag{2}$$

In this case, we say that the family $(A_t)_{t\in\Omega}$ is *weakly* integrable* and the operator $\int_\Omega A_t\,d\mu(t) \in \mathcal{B}(\mathcal{H})$ is the *weak* integral* or *Gelfand integral* of the family $(A_t)_{t\in\Omega}$. We will also say that the function $A : \Omega \to \mathcal{B}(\mathcal{H})$ is weakly* integrable, where the letter A is assigned to the weakly* integrable family $(A_t)_{t\in\Omega}$. If we choose a one-dimensional operator Y in (2), we obtain

$$\left\langle \int_\Omega A_t\,d\mu(t)f, g\right\rangle = \int_\Omega \langle A_tf, g\rangle\,d\mu(t) \quad \text{for all vectors} \quad f, g \in \mathcal{H}. \tag{3}$$

The relation (3) is important because it gives the sesquilinear form of the operator $\int_\Omega A_t\,d\mu(t) \in \mathcal{B}(\mathcal{H})$ and will be used frequently in the following text. If the function $\Omega \ni t \mapsto \langle A_tf, g\rangle \in \mathbb{C}$ is $\mathfrak{M}$-measurable for all $f, g \in \mathcal{H}$, then the function $A : \Omega \to \mathcal{B}(\mathcal{H})$ is weakly* measurable. Also, if the function $\Omega \ni t \mapsto \langle A_tf, g\rangle \in \mathbb{C}$ belongs to the space $L^1(\Omega,\mathfrak{M},\mu)$ for all $f, g \in \mathcal{H}$ then the function $A : \Omega \to \mathcal{B}(\mathcal{H})$ is weakly* integrable and we have (2) (see [13, Lemma 1.1]). For weakly* integrable family $(A_t)_{t\in\Omega}$ and all $\mathfrak{M} \ni E \subseteq \Omega$ we have the equality

$$\left\langle \int_E A_t\, d\mu(t) f, g \right\rangle = \int_E \langle A_t f, g\rangle\, d\mu(t) \quad \text{for all} \quad f, g \in \mathcal{H}. \tag{4}$$

Thus, norm of the operator $\int_E A_t\, d\mu(t)$ is given by $\left\| \int_E A_t\, d\mu(t) \right\| = \sup_{\|f\|=\|g\|=1} \left| \int_E \langle A_t f, g\rangle\, d\mu(t) \right|$. Using Closed Graph argument, from the proof of [13, Lemma 1.1], we have

$$\|(A_t)_{t\in\Omega}\|_G = \sup_{\|f\|=\|g\|=1} \int_\Omega |\langle A_t f, g\rangle|\, d\mu(t) < +\infty. \tag{5}$$

Note that $\sup_{\|f\|=\|g\|=1} \left| \int_\Omega \langle A_t f, g\rangle\, d\mu(t) \right| = \sup_{\|f\|=\|g\|=1} \int_\Omega |\langle A_t f, g\rangle|\, d\mu(t)$ does not have to hold in a general case. A simple example is given by $\Omega = \{t_1, t_2\}$, $t_1 \neq t_2$ and $0 \neq A_{t_1} = -A_{t_2} \in \mathcal{B}(\mathcal{H})$.

If $\mathcal{K}$ is a Hilbert space, $A : \Omega \to \mathcal{B}(\mathcal{H})$ is a weakly* integrable function and $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then the families $(A_t^*)_{t\in\Omega}$, $(BA_t)_{t\in\Omega}$ and $(A_t C)_{t\in\Omega}$ are weakly* integrable and we have

$$\left(\int_E A_t\, d\mu(t) \right)^* = \int_E A_t^*\, d\mu(t),\ B \int_E A_t\, d\mu(t) = \int_E BA_t\, d\mu(t),\ \int_E A_t\, d\mu(t) C = \int_E A_t C\, d\mu(t) \tag{6}$$

for all sets $E \in \mathfrak{M}$ and this can be easily verified by considering their quadratic forms and applying (4).

If a family of the operators $(A_t)_{t\in\Omega}$ is weakly* integrable, then the function $v_A : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ defined by $v_A(E) = \int_E A_t\, d\mu(t)$, $E \in \mathfrak{M}$, is a *weakly countable additive vector measure* on the σ-algebra $\mathfrak{M}$. In fact, the function $\mathfrak{M} \ni E \mapsto \int_E \langle A_t f, g\rangle\, d\mu(t) \in \mathbb{C}$ is a complex measure on the σ-algebra $\mathfrak{M}$. So, if we have the sequence $(E_n)_{n\in\mathbb{N}}$ of mutually disjoint sets in $\mathfrak{M}$, then

$$\left\langle v_A\left(\bigsqcup_{n=1}^{+\infty} E_n \right) f, g \right\rangle = \int_{\bigsqcup_{n=1}^{+\infty} E_n} \langle A_t f, g\rangle\, d\mu(t) = \sum_{n=1}^{+\infty} \int_{E_n} \langle A_t f, g\rangle\, d\mu(t) = \left\langle \left(\sum_{n=1}^{+\infty} v_A(E_n) \right) f, g \right\rangle \quad \text{for all} \quad f, g \in \mathcal{H}.$$

If we have the equality $v_A\left(\bigsqcup_{n=1}^{+\infty} E_n \right) = \sum_{n=1}^{+\infty} v_A(E_n)$ in the norm of $\mathcal{B}(\mathcal{H})$ (or in the norm of an ideal $\mathfrak{C}_\Phi(\mathcal{H})$) for all sequences $(E_n)_{n\in\mathbb{N}}$ with mutually disjoint members in $\mathfrak{M}$, then we say that the function v_A is a $\mathcal{B}(\mathcal{H})$-valued (or $\mathfrak{C}_\Phi(\mathcal{H})$-valued) measure. For related questions, see [13, Theorem 1.3]. In this paper we give a generalization of the mentioned theorem.

To illustrate the introduced concepts, the simplest weakly* integrable family is $(f(t)A)_{t\in\Omega}$ for a given scalar function $f \in L^1(\Omega, \mathfrak{M}, \mu)$ and $A \in \mathcal{B}(\mathcal{H})$. It is easy to see that

$$\int_E f(t) A\, d\mu(t) = \left(\int_E f(t)\, d\mu(t) \right) A \quad \text{for all} \quad E \in \mathfrak{M}. \tag{7}$$

We will often use the equality (7) in the following text.

If the family $(A_t)_{t\in\Omega}$ of operators is weakly* measurable, then the function $\Omega \ni t \mapsto \|A_t f\| \in \mathbb{R}$ is $\mathfrak{M}$-measurable. This is a direct consequence of the Bessel equality $\|A_t f\| = \sqrt{\sum_{n=1}^{+\infty} |\langle A_t f, e_n\rangle|^2}$, which is satisfied for all vectors $f \in \mathcal{H}$. Moreover, the function $\Omega \ni t \mapsto \|A_t\| \in \mathbb{R}$ is $\mathfrak{M}$-measurable because we have the equality $\|A_t\| = \sup_{0\neq f\in\mathcal{D}} \frac{\|A_t f\|}{\|f\|}$, where $\mathcal{D} \subseteq \mathcal{H}$ is a countable and dense set in the separable Hilbert space $\mathcal{H}$.

For weakly* integrable family $(A_t)_{t\in\Omega}$ we have the inequality $\left\| \int_E A_t\, d\mu(t) \right\| \leqslant \int_E \|A_t\|\, d\mu(t)$ for all $E \in \mathfrak{M}$. Note that every unitary invariant norm on an ideal of compact operators is a symmetrical norm and vice versa. If the ideal $\mathfrak{C}_\Phi(\mathcal{H})$ is separable and $A_t \in \mathfrak{C}_\Phi(\mathcal{H})$ for μ-almost all $t \in \Omega$, then the map $\Omega \ni t \mapsto \|A_t\|_\Phi \in \mathbb{R}$ is $\mathfrak{M}$-measurable by [7, Theorem A.4]. If we also have $\int_\Omega \|A_t\|_\Phi\, d\mu(t) < +\infty$, then $\int_\Omega A_t\, d\mu(t) \in \mathfrak{C}_\Phi(\mathcal{H})$ and the inequality $\left\| \int_\Omega A_t\, d\mu(t) \right\|_\Phi \leqslant \int_\Omega \|A_t\|_\Phi\, d\mu(t)$ follows by [7, Theorem A.5].

The family $(A_t)_{t\in\Omega}$ is said to be *square integrable* if $\int_\Omega \|A_t f\|^2\, d\mu(t) < +\infty$ for all vectors $f \in \mathcal{H}$. In this case, the Gelfand integral $\int_\Omega A_t^* A_t\, d\mu(t) \in \mathfrak{B}(\mathcal{H})$ exists and using (3) and $\int_\Omega A_t^* A_t\, d\mu(t) \geqslant 0$ we have

$$\sup_{\|f\|=1} \int_\Omega \|A_t f\|^2\, d\mu(t) = \left\| \int_\Omega A_t^* A_t\, d\mu(t) \right\| < +\infty. \tag{8}$$

For more details on the integration of weakly* measurable function with values in $\mathfrak{B}(\mathcal{H})$ see [11], [13], [14].

Let X be a Banach space, where the dual space of X is denoted by X^* and let $P_1(\Omega, \mathfrak{M}, \mu, X)$ denote the set of functions (equivalence classes) $f : \Omega \to X$ such that: the scalar function $\Omega \ni t \mapsto x^*(f(t)) \in \mathbb{C}$ belongs to $L^1(\Omega, \mathfrak{M}, \mu)$ for all $x^* \in X^*$ and for all $E \in \mathfrak{M}$ there is the vector $x_E = \int_E f(t)\, d\mu(t) \in X$ such that $x^*\left(\int_E f(t)\, d\mu(t)\right) = \int_E x^*(f(t))\, d\mu(t)$ for all $x^* \in X^*$. The element $x_E \in X$ is called *Pettis integral* of the function $f : \Omega \to X$ on $E \in \mathfrak{M}$. The introduced vector space $P_1(\Omega, \mathfrak{M}, \mu, X)$ of *Pettis-integrable functions* is normed by

$$\|f\|_P = \sup_{\substack{x^*\in X^* \\ \|x^*\|=1}} \int_\Omega |x^*(f(t))|\, d\mu(t) \quad \text{for all} \quad f \in P_1(\Omega, \mathfrak{M}, \mu, X). \tag{9}$$

From the Closed Graph argument it follows that the expression (9) is finite for all $f \in P_1(\Omega, \mathfrak{M}, \mu, X)$. If X is reflexive and $\|f\|_P < +\infty$ then there exists $x_E \in X$ such that $x^*(x_E) = \int_E x^*(f(t))\, d\mu(t)$ for all $x^* \in X^*$.

The normed space $P_1(\Omega, \mathfrak{M}, \mu, X)$ is not Banach in the general case (see the Theorem 1.2). For more information on the theory of Pettis integration see [4], [15].

For the general definition of the Gelfand integral of an arbitrary weakly* measurable function $f : \Omega \mapsto X^*$, see [2, Section 11.9] and [4, Chapter 2]. The monograph [4] is a fundamental literature for the general theory of integration in Banach space. For a quick introduction to the theory of integration in Banach spaces, see [2, Sections 11.8, 11.9, 11.10], where the relationship between Bochner, Pettis, Gelfand and Danford integration types is explained in a concise form.

In the following we will introduce the normed space $L^1_G(\Omega, \mathfrak{M}, \mu, \mathfrak{B}(\mathcal{H}))$ which consists of (classes of) weakly* integrable functions (families) $A : \Omega \to \mathfrak{B}(\mathcal{H})$.

Definition 1.1. *Two weakly* measurable families $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ in $\mathfrak{B}(\mathcal{H})$ are μ-equivalent if for every $f \in \mathcal{H}$ the functions $\Omega \ni t \mapsto \langle A_t f, f\rangle \in \mathbb{C}$ and $\Omega \ni t \mapsto \langle B_t f, f\rangle \in \mathbb{C}$ are equal for μ-almost all $t \in \Omega$. Vector space of all weakly* integrable families $(A_t)_{t\in\Omega}$ in $\mathfrak{B}(\mathcal{H})$ (or functions) is denoted by $L^1_G(\Omega, \mathfrak{M}, \mu, \mathfrak{B}(\mathcal{H}))$, where two such families (functions) are identified if they are μ-equivalent.*

Closely related to the above definition is [13, Lemma 1.1]. For μ-equivalent families $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ we have $A_t = B_t$ for μ-almost all $t \in \Omega$ if $\mathcal{H}$ is separable. In this case, let X be the subspace in $\mathfrak{B}(\mathcal{H})$. With $L^1_G(\Omega, \mu, X)$ we will denote the vector subspace in $L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ consisting of elements $A = (A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ such that $A_t \in X$ for μ-almost all $t \in \Omega$

The introduced vector space $L^1_G(\Omega, \mathfrak{M}, \mu, \mathfrak{B}(\mathcal{H}))$ is infinite-dimensional if $\dim(\mathcal{H}) = +\infty$ and this space can be naturaly normed by $\|\cdot\|_G$ from (5). If it is understood what the σ-algebra is, we use the notation $P_1(\Omega, \mu, X)$ and $L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$.

Theorem 1.2. *([10, Theorem 3]) Let X be a Banach space and $\dim(X) = +\infty$. $P_1([0,1], \mathcal{B}_{[0,1]}, m, X)$ is not Banach.*

Lemma 1.3. *Let $\mathcal{H}$ be Hilbert space, $\dim(\mathcal{H}) = +\infty$, $a \in \mathcal{H}$, $\|a\| = 1$ and*

$$X_a = \{A \in \mathfrak{B}(\mathcal{H}) : Ax = 0 \text{ for all } x \in \{a\}^\perp\}.$$

- **(a)** *The normed space X_a is closed in $\mathfrak{B}(\mathcal{H})$ and X_a is isometrically-isomorphic to $\mathcal{H}$.*
- **(b)** *If $F \in \mathfrak{B}(X_a, \mathbb{C})$ then there is a unique $x_F \in \mathcal{H}$ such that $F(A) = \langle Aa, x_F\rangle$ for all $A \in X_a$ and we have the equality $\|F\| = \|x_F\|$. Also, for every $x \in \mathcal{H}$ the function $F_x : X_a \to \mathbb{C}$ given by $F_x(A) = \langle Aa, x\rangle$ defines the bounded linear functional on X_a and we have $\|F_x\| = \|x\|$.*
- **(c)** *The normed space $L^1_G(\Omega, \mu, X_a)$ is closed in $L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ if $\mathcal{H}$ is separable.*

Proof. **(a)** The set X_a is a vector subspace of $\mathfrak{B}(\mathcal{H})$. If the sequence $(A_n)_{n\in\mathbb{N}}$ in X_a converges to some $A \in \mathfrak{B}(\mathcal{H})$ then for all $x \in \{a\}^\perp$ we have $0 = \lim\limits_{n\to\infty} A_n x = Ax$. Thus, $A = 0$ on $\{a\}^\perp$. Define the mapping $\varphi : X_a \to \mathcal{H}$ by $\varphi_a(A) = Aa$ for all $A \in X_a$. The mapping φ_a is linear, isometric and bijective between X_a and $\mathcal{H}$. Indeed, we have $\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} |\langle x, a\rangle| \cdot \|Aa\| = \|a\| \cdot \|Aa\| = \|Aa\| = \|\varphi_a(A)\|$ for all $A \in X_a$.

(b) Let $F \in \mathfrak{B}(X_a, \mathbb{C})$ and define $G : \mathcal{H} \to \mathbb{C}$ with $G(f) = F(\varphi_a^{-1}(f))$ for all $f \in \mathcal{H}$. Then, $G \in \mathfrak{B}(\mathcal{H}, \mathbb{C})$. Thus, there exists $x_F \in \mathcal{H}$ such that $G(f) = \langle f, x_F\rangle$ for all $f \in \mathcal{H}$. If we put $f = Aa = \varphi_a(A) \in \mathcal{H}$ then we get $\langle Aa, x_F\rangle = F(\varphi_a^{-1}(\varphi_a(A))) = F(A)$ for all $A \in X_a$. If $\left\langle Aa, x_F'\right\rangle = \left\langle Aa, x_F''\right\rangle$ i.e. $\left\langle Aa, x_F' - x_F''\right\rangle = 0$ for all $A \in X_a$ then for the operator $A_{x_F'-x_F''}(y) = \langle y, a\rangle (x_F' - x_F'')$, $y \in \mathcal{H}$, we have $A_{x_F'-x_F''} \in X_a$ and $0 = \left\langle \langle a, a\rangle (x_F' - x_F''), x_F' - x_F''\right\rangle = \|a\|^2 \cdot \|x_F' - x_F''\|^2$ i.e. $x_F' = x_F''$. To prove the rest of the part **(b)**, for $x \in \mathcal{H}$ we have $|F_x(A)| \leqslant \|Aa\| \cdot \|x\| = \|A\| \cdot \|x\|$ for all $A \in X_a$. This implies $\|F_x\| \leqslant \|x\|$. For the operator $A_x(y) = \langle y, a\rangle\, x$, $y \in \mathcal{H}$, we have $A_x \in X_a$, $\|A_x\| = \|a\| \cdot \|x\| = \|x\|$ and $\|F_x\| \geqslant \frac{|F_x(A_x)|}{\|A_x\|} = \frac{\|x\|^2}{\|x\|} = \|x\|$. Thus, $\|F_x\| = \|x\|$.

(c) Let $\left(A^{(n)}\right)_{n\in\mathbb{N}}$ be the sequence in $L^1_G(\Omega, \mu, X_a)$ such that converges to an element $A = (A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ and let $0 \neq b \in \{a\}^\perp$. For $n \in \mathbb{N}$ we have $A_t^{(n)} b = 0$ for μ-almost all $t \in \Omega$. Thus, we get

$$\left\|A^{(n)} - A\right\|_G = \sup_{\|x\|=\|y\|=1} \int_\Omega \left|\left\langle A_t^{(n)} x - A_t x, y\right\rangle\right| d\mu(t) \geqslant \frac{1}{\|b\|} \sup_{\|y\|=1} \int_\Omega \left|\left\langle A_t b, y\right\rangle\right| d\mu(t) \quad \text{for all} \quad n \in \mathbb{N}.$$

The equality $\lim\limits_{n\to\infty} \left\|A^{(n)} - A\right\|_G = 0$ implies $\int_\Omega \left|\langle A_t b, y\rangle\right| d\mu(t) = 0$ for all $y \in \mathcal{H}$. For fixed $y \in \mathcal{H}$ there is the set $N_{b,y} \in \mathfrak{M}$, $\mu(N_{b,y}) = 0$ such that $\left|\langle A_t b, y\rangle\right| = 0$ for all $t \in \Omega \backslash N_{b,y}$. Let $N_b = \bigcup_{y\in B} N_{b,y}$, where $\mathcal{D}$ is some countable and dense set in the closed unit ball in $\mathcal{H}$. Thus, we have $N_b \in \mathfrak{M}$ and $\mu(N_b) = 0$. Now, for all $t \in \Omega \setminus N_b$ we get $\|A_t b\| = \sup_{y\in\mathcal{D}} \left|\langle A_t b, y\rangle\right| = 0$. Let $N = \bigcup_{b\in\mathfrak{D}} N_b$. Again, $N \in \mathfrak{M}$, $\mu(N) = 0$ and $\|A_t\| = \sup_{b\in\mathcal{D}} \|A_t b\| = 0$ for all $t \in \Omega \setminus N$.. This means $A_t = 0$ on $\{a\}^\perp$ for all $t \in \Omega \setminus N$ i.e. $A = (A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, X_a)$. □

In the next theorem we will prove that the normed space $L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ does not have to be complete in the general case.

Theorem 1.4. *Let $\mathcal{H}$ be a separable Hilbert space and* $\dim(\mathcal{H}) = +\infty$. *The normed space*

$$L^1_G([0,1], \mathcal{B}_{[0,1]}, m, \mathfrak{B}(\mathcal{H}))$$

is not Banach.

Proof. Let $e \in \mathcal{H}$ be the unit vector and let X_e be the Banach space from the Lemma 1.3. If $A_t \in X_e$ for $t \in [0,1]$, then weak* measurability and weak measurability of the family $(A_t)_{t\in[0,1]}$ coincide by part **(b)** of Lemma 1.3. For $x \in \mathcal{H}$ there is a unique $a_x \in \{e\}^\perp$ such that $x = \langle x, e\rangle\, e + a_x$. Thus, for weakly measurable family $(A_t)_{t\in[0,1]}$ in X_e we have the next computation:

$$\begin{aligned} \|(A_t)_{t\in[0,1]}\|_G &= \sup_{\|x\|=\|y\|=1} \int_{[0,1]} |\langle A_t x, y\rangle|\, dm(t) = \sup_{\|x\|=\|y\|=1} \int_{[0,1]} |\langle x, e\rangle \langle A_t e, y\rangle + \langle A_t a_x, y\rangle|\, dm(t) \\ &= \sup_{\|x\|=1} |\langle x, e\rangle| \cdot \sup_{\|y\|=1} \int_{[0,1]} |\langle A_t e, y\rangle|\, dm(t) = \sup_{\substack{F\in X_e^* \\ \|F\|=1}} \int_{[0,1]} |F(A_t)|\, dm(t) = \|(A_t)_{t\in[0,1]}\|_P. \end{aligned} \tag{10}$$

The space X_e is reflexive by part **(a)** of Lemma 1.3. Thus, $\|(A_t)_{t\in[0,1]}\|_P < +\infty$ for weakly measurable family $(A_t)_{t\in[0,1]}$ in X_e implies $(A_t)_{t\in[0,1]} \in P_1([0,1], \mathcal{B}_{[0,1]}, m, X_e)$. Due to (10) we have the equality

$$P_1([0,1], \mathcal{B}_{[0,1]}, m, X_e) = L^1_G([0,1], \mathcal{B}_{[0,1]}, m, X_e).$$

According to the part **(a)** of Lemma 1.3, the normed space X_e is closed in $\mathfrak{B}(\mathcal{H})$. Using $\dim(\mathcal{H}) = +\infty$, from the Theorem 1.2 we have that the normed space $P_1([0,1], \mathcal{B}_{[0,1]}, m, X_e)$ is not Banach. Therefore, there is a Cauchy sequence $\left(A^{(n)}\right)_{n\in\mathbb{N}}$ in $P_1([0,1],\mathcal{B}_{[0,1]},m,X_e)$ such that this sequence does not converge in $P_1([0,1],\mathcal{B}_{[0,1]},m,X_e)$. From (10) it follows that the sequence $\left(A^{(n)}\right)_{n\in\mathbb{N}}$ is Cauchy in $L^1_G([0,1],\mathcal{B}_{[0,1]},m,X_e)$. The sequence $\left(A^{(n)}\right)_{n\in\mathbb{N}}$ is therefore Cauchy in $L^1_G([0,1],\mathcal{B}_{[0,1]},m,\mathfrak{B}(\mathcal{H}))$ and this sequence does not converge in $L^1_G([0,1],\mathcal{B}_{[0,1]},m,\mathfrak{B}(\mathcal{H}))$ by part **(c)** of Lemma 1.3. Thus, $L^1_G([0,1],\mathcal{B}_{[0,1]},m,\mathfrak{B}(\mathcal{H}))$ is not a Banach space. □

2. Integration of weakly* measurable functions $A:\Omega\to\mathfrak{B}(\mathcal{H})$

We start with a variant of Dominant Convergence Theorem for the separable ideals $\mathfrak{C}_\Phi(\mathcal{H})$. The direct generalization of [13, Lemma 1.2] (where positivity of the operators is assumed) and [17, Theorem 2.16.] (where no positivity of the operators is assumed, but $\Phi=\ell^p$, $p\in[1,+\infty)$) follows.

Theorem 2.1. *Let $\mathfrak{C}_\Phi(\mathcal{H})$ be separable ideal generated by s.n. function Φ and A_n, $A\in\mathfrak{B}(\mathcal{H})$ for all $n\in\mathbb{N}$ and let the sequence $(A_n)_{n\in\mathbb{N}}$ converges weakly to A. If there is an operator $B\in\mathfrak{C}_\Phi(\mathcal{H})$ such that $|A_n|$, $|A_n^*|$, $|A|$, $|A^*|\leqslant B$ for all $n\in\mathbb{N}$ then A_n, $A\in\mathfrak{C}_\Phi(\mathcal{H})$ for all $n\in\mathbb{N}$ and $\lim\limits_{n\to\infty}\|A_n-A\|_\Phi=0$.*

The proof of this theorem is the same as the proof of [17, Theorem 2.16], with minor changes.

In the next theorem we consider measures on the σ-algebra $\mathfrak{M}$ with values in a separable ideal $\mathfrak{C}_\Phi(\mathcal{H})$ generated by an s.n. function Φ. This theorem is a generalization of [13, Theorem 1.3], and we do not assume positivity of the operators.

Theorem 2.2. *Let $(A_t)_{t\in\Omega}$ be weakly* measurable family in $\mathfrak{B}(\mathcal{H})$ such that $(|A_t|)_{t\in\Omega}, (|A_t^*|)_{t\in\Omega}\in L^1_G(\Omega,\mu,\mathfrak{B}(\mathcal{H}))$. Then, we have $(A_t)_{t\in\Omega}\in L^1_G(\Omega,\mu,\mathfrak{B}(\mathcal{H}))$. Furthermore, let $X\in\mathfrak{C}_\Phi(\mathcal{H})$ be an operator such that for all $E\in\mathfrak{M}$ we have $\left|\int_E A_t\,d\mu(t)\right|\leqslant X$ and $\left|\int_E A_t^*\,d\mu(t)\right|\leqslant X$, where $\mathfrak{C}_\Phi(\mathcal{H})$ is the separable ideal. Then, we have:*

- ***(a)*** *The function $\upsilon_A:\mathfrak{M}\to\mathfrak{C}_\Phi(\mathcal{H})$, given by*

$$\upsilon_A(E)=\int_E A_t\,d\mu(t)\quad\text{for all}\quad E\in\mathfrak{M},$$

  is a well-defined $\mathfrak{C}_\Phi(\mathcal{H})$-valued measure on the σ-algebra $\mathfrak{M}$.
- ***(b)*** *Let $p\in[1,+\infty)$ and $\int_\Omega|A_t|\,d\mu(t)\in\mathfrak{C}_p(\mathcal{H})$. If*

$$\left|\int_E A_t\,d\mu(t)\right|\leqslant\int_E|A_t|\,d\mu(t)\quad\text{for all}\quad E\in\mathfrak{M},\tag{11}$$

  then for all sequences $(E_n)_{n\in\mathbb{N}}$ with mutually disjoint members in $\mathfrak{M}$ we have $\sum_{n=1}^{+\infty}\|\upsilon_A(E_n)\|_p^p<+\infty$.

Proof. For all operators $T\in\mathfrak{B}(\mathcal{H})$ we have Mixed Cauchy-Schwarz inequality (see [5, Chapter 3])

$$|\langle Tf,g\rangle|\leqslant\sqrt{\langle|T|f,f\rangle\cdot\langle|T^*|g,g\rangle}\quad\text{for all}\quad f,g\in\mathcal{H}.\tag{12}$$

Now, from assumed weak* measurability of the family $(A_t)_{t\in\Omega}$ and from (12) we get

$$\int_\Omega|\langle A_tf,f\rangle|\,d\mu(t)\leqslant\int_\Omega\sqrt{\langle|A_t|f,f\rangle\cdot\langle|A_t^*|f,f\rangle}\,d\mu(t)\leqslant\frac12\int_\Omega\langle|A_t|f,f\rangle\,d\mu(t)+\frac12\int_\Omega\langle|A_t^*|f,f\rangle\,d\mu(t)<+\infty$$

for all $f\in\mathcal{H}$. So, we have $(A_t)_{t\in\Omega}\in L^1_G(\Omega,\mu,\mathfrak{B}(\mathcal{H}))$.

- **(a)** From $(A_t)_{t\in\Omega}\in L^1_G(\Omega,\mu,\mathfrak{B}(\mathcal{H}))$ we have that the function $\upsilon_A:\mathfrak{M}\to\mathfrak{B}(\mathcal{H})$ is the weakly countable aditive measure on $\mathfrak{M}$. For all sets $E\in\mathfrak{M}$ we have $\left|\int_E A_t\,d\mu(t)\right|\leqslant X$, $\left|\int_E A_t^*\,d\mu(t)\right|\leqslant X$ and $X\in\mathfrak{C}_\Phi(\mathcal{H})$ and from Dominant Convergence Theorem 2.1 we obtain part **(a)** of our theorem.

(b) From $0 \leqslant \left|\int_E A_t\, d\mu(t)\right| \leqslant \int_\Omega |A_t|\, d\mu(t) \in \mathfrak{C}_p(\mathcal{H})$ we have $\left|\int_E A_t\, d\mu(t)\right| \in \mathfrak{C}_p(\mathcal{H})$ and this implies $\int_E A_t\, d\mu(t) \in \mathfrak{C}_p(\mathcal{H})$ for all $E \in \mathfrak{M}$. Let $N \in \mathbb{N}$ be arbitrary. If we apply the inequality [8, Lemma 2.1, (c)] and the inequality (11) for arbitrary s.n. function Ψ (more precisely, for s.n. functions Ψ for which the expressions below are defined), we obtain

$$\left\|\sum_{k=1}^N \left|\int_{E_k} A_t\, d\mu(t)\right|^p\right\|_\Psi \leqslant \left\|\left[\sum_{k=1}^N \left|\int_{E_k} A_t\, d\mu(t)\right|\right]^p\right\|_\Psi \leqslant \left\|\left[\sum_{k=1}^N \int_{E_k} |A_t|\, d\mu(t)\right]^p\right\|_\Psi = \left\|\left[\int_{\bigsqcup_{k=1}^N E_k} |A_t|\, d\mu(t)\right]^p\right\|_\Psi .$$

In particular, for $\Psi = \ell^1$, the above inequalities become

$$\begin{aligned}\sum_{k=1}^N \|v_A(E_k)\|_p^p &= \sum_{k=1}^N \left\|\int_{E_k} A_t\, d\mu(t)\right\|_p^p = \sum_{k=1}^N \left\|\left|\int_{E_k} A_t\, d\mu(t)\right|^p\right\|_1 = \left\|\sum_{k=1}^N \left|\int_{E_k} A_t\, d\mu(t)\right|^p\right\|_1 \\ &\leqslant \left\|\left[\int_{\bigsqcup_{k=1}^N E_k} |A_t|\, d\mu(t)\right]^p\right\|_1 = \left\|\int_{\bigsqcup_{k=1}^N E_k} |A_t|\, d\mu(t)\right\|_p^p \leqslant \left\|\int_\Omega |A_t|\, d\mu(t)\right\|_p^p < +\infty.\end{aligned} \tag{13}$$

The last inequality follows from $(|A_t|)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ and $\int_\Omega |A_t|\, d\mu(t) \in \mathfrak{C}_p(\mathcal{H})$. We now have the inequality $\sum_{k=1}^{+\infty} \|v_A(E_k)\|_p^p < +\infty$ if $N \to +\infty$ in (13).

Theorem is proved. □

As we have already said, if $A_t \geqslant 0$ for all $t \in \Omega$, we obtain [13, Theorem 1.3] from the proven theorem.

In the following, we will deal with families of integrable functions. The concept of uniform integrability plays an important role here. The set $S \subseteq L^1(\Omega, \mathfrak{M}, \mu)$ is *uniformly integrable* if for $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in S$ the conditions $E \in \mathfrak{M}$ and $\mu(E) < \delta$ imply $\left|\int_E f\, d\mu\right| < \varepsilon$. For the definition of uniform integrability and other related properties, see [16, Chapter 6, Exercise 10].

We are introducing a new definition.

Definition 2.3. *Let $\mathscr{A}^{(n)} = \left(A_t^{(n)}\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ for all $n \in \mathbb{N}$. The set $\{\mathscr{A}^{(n)} : n \in \mathbb{N}\}$ is weakly* uniformly integrable if for $\varepsilon > 0$ exists $\delta > 0$ such that for all $n \in \mathbb{N}$ the conditions $E \in \mathfrak{M}$ and $\mu(E) < \delta$ imply $\left\|\int_E A_t^{(n)} d\mu(t)\right\| < \varepsilon$.*

For the family $\mathscr{A}^{(n)} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ we define $v_n(E) = \int_E A_t^{(n)}\, d\mu(t)$ for all $E \in \mathfrak{M}$. In accordance with the introduced notation, we have the following theorem.

Theorem 2.4. *Let $\mathscr{A}^{(n)} = \left(A_t^{(n)}\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ be a family of positive operators such that $\liminf_{n\to\infty} \|v_n(\Omega)\| < +\infty$ and let the sequence $\left(A_t^{(n)}\right)_{n\in\mathbb{N}}$ converges weakly to $A_t \in \mathfrak{B}(\mathcal{H})$ for all $t \in \Omega$.*

(a) *We have $(A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ and the next inequality holds:*

$$\left\|\int_\Omega A_t\, d\mu(t)\right\| \leqslant \liminf_{n\to\infty} \left\|\int_\Omega A_t^{(n)} d\mu(t)\right\|.$$

(b) *Let $\mu(\Omega) < +\infty$ and let the set $\{\mathscr{A}^{(n)} : n \in \mathbb{N}\}$ be weakly* uniformly integrable. Then, the function $v_A : \Omega \to \mathfrak{B}(\mathcal{H})$ defined by $v_A(E) = \int_E A_t\, d\mu(t)$, $E \in \mathfrak{M}$, is a well-defined $\mathfrak{B}(\mathcal{H})$-valued measure on $\mathfrak{M}$.*

Proof. We have $A_t \geqslant 0$ due to $\langle A_t f, f\rangle = \lim_{n\to\infty} \left\langle A_t^{(n)} f, f\right\rangle \geqslant 0$, which is satisfied for all $t \in \Omega$ and all $f \in \mathcal{H}$.

(a) From $A_t^{(n)} \geqslant 0$ for all $t \in \Omega$ follows $v_n(\Omega) \geqslant 0$ for all $n \in \mathbb{N}$. The family $(A_t)_{t\in\Omega}$ is weakly* measurable, because we have $\langle A_t f, g\rangle = \lim_{n\to\infty} \left\langle A_t^{(n)} f, g\right\rangle$ for all $t \in \Omega$ and $f, g \in \mathcal{H}$. The condition $\liminf_{n\to\infty} \|v_n(\Omega)\| < +\infty$ implies $(A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$, which follows from Fatou lemma. Indeed, we have

$$\int_\Omega \langle A_t f, f\rangle\, d\mu(t) \leqslant \liminf_{n\to\infty} \int_\Omega \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) = \liminf_{n\to\infty} \langle v_n(\Omega) f, f\rangle \leqslant \|f\|^2 \liminf_{n\to\infty} \|v_n(\Omega)\| < +\infty.$$

for all vectors $f \in \mathcal{H}$. From the above calculation we have $\langle v_A(\Omega)f, f\rangle \leqslant \liminf_{n\to\infty} \|v_n(\Omega)\| \langle f, f\rangle$ for all $f \in \mathcal{H}$ and due to $v_A(\Omega) \geqslant 0$ we get $\|v_A(\Omega)\| \leqslant \liminf_{n\to\infty} \|v_n(\Omega)\|$.

(b) From the part **(a)**, we have that the function v_A is a well-defined, weakly countable additive measure on $\mathfrak{M}$. Let $N \in \mathbb{N}$ be arbitrary and let $(E_n)_{n\in\mathbb{N}}$ be the sequence of mutually disjoint members in $\mathfrak{M}$. Again, from Fatou lemma and the elementary properties of supremum and infimum, we obtain the following chain of equalities and inequalities

$$
\begin{aligned}
\left\| v_A\left(\bigsqcup_{n=1}^{+\infty} E_n\right) - v_A\left(\bigsqcup_{k=1}^{N} E_k\right)\right\| &= \left\| \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} A_t \, d\mu(t)\right\| \\
&= \sup_{\|f\|=1} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \langle A_t f, f\rangle \, d\mu(t) \\
&= \sup_{\|f\|=1} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \lim_{n\to\infty} \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) \\
&\leqslant \sup_{\|f\|=1} \liminf_{n\to\infty} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) \\
&= \sup_{\|f\|=1} \sup_{m\in\mathbb{N}} \inf_{\mathbb{N}\ni n\geqslant m} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) \\
&= \sup_{m\in\mathbb{N}} \sup_{\|f\|=1} \inf_{\mathbb{N}\ni n\geqslant m} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) \\
&\leqslant \sup_{m\in\mathbb{N}} \inf_{\mathbb{N}\ni n\geqslant m} \sup_{\|f\|=1} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) \\
&\leqslant \sup_{m\in\mathbb{N}} \inf_{\mathbb{N}\ni n\geqslant m} \sup_{\|f\|=1} \left\langle \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} A_t^{(n)} d\mu(t) f, f\right\rangle \\
&= \sup_{m\in\mathbb{N}} \inf_{\mathbb{N}\ni n\geqslant m} \left\| \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} A_t^{(n)} d\mu(t)\right\| = \liminf_{n\to\infty} \left\| \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} A_t^{(n)} d\mu(t)\right\|.
\end{aligned}
$$

From $\mu(\Omega) < +\infty$ follows $\lim_{N\to\infty} \mu\left(\bigsqcup_{k=N+1}^{+\infty} E_k\right) = 0$ and due to the assumed weakly* uniform integrability of the set $\{\mathscr{A}^{(n)} : n \in \mathbb{N}\}$, the last limit inferior can be enough small as $N \to \infty$. Thus, we obtain $v_A\left(\bigsqcup_{n=1}^{+\infty} E_n\right) = \sum_{n=1}^{+\infty} v_A(E_n)$, where the last series converge in the norm of the space $\mathcal{B}(\mathcal{H})$.

The theorem is proved. □

In the next theorem we give sufficient conditions for the weakly* uniform integrability of the set $\{\mathscr{A}^{(n)} : n \in \mathbb{N}\}$. We will use the same notation as in the Theorem 2.4.

Theorem 2.5. *Let $\mu(\Omega) < +\infty$ and $\mathscr{A}^{(n)} = \left(A_t^{(n)}\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ consisting of positive operators and let $\left(A_t^{(n)}\right)_{n\in\mathbb{N}}$ converges weakly to $A_t \in \mathfrak{C}_\infty(\mathcal{H})$ for all $t \in \Omega$. If we have $v_A(\Omega) \in \mathfrak{C}_\infty(\mathcal{H})$, $A_t^{(n)} \leqslant A_t^{(n+1)}$ for all $n \in \mathbb{N}$ and all $t \in \Omega$ and $\sup_{n\in\mathbb{N}} \|v_n(\Omega)\| < +\infty$ then the set $\{\mathscr{A}^{(n)} : n \in \mathbb{N}\}$ is weakly* uniformly integrable.*

Proof. We have $(A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ due to the part **(a)** of the Theorem 2.4. For all $E \in \mathfrak{M}$ we have

$$
\lim_{n\to\infty} \langle v_n(E)f, f\rangle = \lim_{n\to\infty} \int_E \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) = \int_E \lim_{n\to\infty} \left\langle A_t^{(n)} f, f\right\rangle d\mu(t) = \int_E \langle A_t f, f\rangle \, d\mu(t) = \langle v_A(E)f, f\rangle
$$

due to the Monotone Convergence Theorem. The sequence $(v_n(E))_{n\in\mathbb{N}}$ therefore converges weakly to $v_A(E)$. From $0 \leqslant v_n(E) \leqslant v_A(E) \leqslant v_A(\Omega) \in \mathfrak{C}_\infty(\mathcal{H})$ it follows that $v_n(E) \in \mathfrak{C}_\infty(\mathcal{H})$ for all $n \in \mathbb{N}$ and all $E \in \mathfrak{M}$. Moreover, by [13, Lemma 1.2] or by the Theorem 2.1 we have $\lim_{n\to\infty} \|v_n(E) - v_A(E)\| = 0$ for all $E \in \mathfrak{M}$.

From $\mu(\Omega) < +\infty$, $v_A(\Omega) \in \mathfrak{C}_\infty(\mathcal{H})$ and $A_t \in \mathfrak{C}_\infty(\mathcal{H})$ for all $t \in \Omega$, due to [13, Theorem 2.3, (b)] we have that for $\varepsilon > 0$ there exists $\delta > 0$ such that $E \in \mathfrak{M}$ and $\mu(E) < \delta$ imply $\|v_A(E)\| < \varepsilon$. Let $\varepsilon > 0$ and choose $\delta > 0$ from mentioned theorem and $E \in \mathfrak{M}$ such that $\mu(E) < \delta$. There is the number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\|v_n(\Omega) - v_A(\Omega)\| < \varepsilon$ for all $\mathbb{N} \ni n \geqslant n_0$. Now, for all $\mathbb{N} \ni n \geqslant n_0$ we have $\|v_n(E)\| \leqslant \|v_n(E) - v_A(E)\| + \|v_A(E)\| \leqslant \|v_n(\Omega) - v_A(\Omega)\| + \|v_A(E)\| \leqslant \varepsilon + \varepsilon = 2\varepsilon$.

Moreover, according to [13, Theorem 2.3, (b)], the finite set $\{\mathscr{A}^{(n)} : n \in \{1, \ldots, N\}\}$ is weakly* uniformly integrable, because we have $A_t^{(n)} \in \mathfrak{C}_\infty(\mathcal{H})$ and $v_n(\Omega) \in \mathfrak{C}_\infty(\mathcal{H})$ for all $t \in \Omega$ and $n \in \mathbb{N}$. Thus, the set $\{\mathscr{A}^{(n)} : n \in \mathbb{N}\}$ is weakly* uniformly integrable. □

We deal with further applications of [13, Theorem 2.3], which connects the compactness of the operator A_t for μ-almost all $t \in \Omega$ and the compactness of the operator $\int_\Omega A_t \, d\mu(t)$. The following example shows that we cannot obtain the result from [13, Theorem 2.3, (a)] if the measure is not finite.

Example 2.6. *Let $\Omega = \mathbb{N}$, $\mathfrak{M} = \mathcal{P}(\mathbb{N})$ and μ be a counting measure on $\mathbb{N}$. Let $(e_n)_{n\in\mathbb{N}}$ be the orthonormal basis of the Hilbert space $\mathcal{H}$ and we define the mapping $A : \mathbb{N} \to \mathfrak{B}(\mathcal{H})$ with $A_n(f) = \langle f, e_n\rangle e_n$ for all $n \in \mathbb{N}$ and all $f \in \mathcal{H}$. We have $A_n \in \mathfrak{C}_\infty(\mathcal{H})$ for all $n \in \mathbb{N}$. Furthermore, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $E \in \mathfrak{M}$, $\mu(E) < \delta$ implies $\left\|\int_E A_n \, d\mu(n)\right\| < \varepsilon$. Indeed, for δ we can take $\delta = \frac{1}{2}$. Thus, $\left\|\int_E A_n \, d\mu(n)\right\| = 0 < \varepsilon$. We have $(A_n)_{n\in\mathbb{N}} \in L^1_G(\mathbb{N}, \mu, \mathfrak{B}(\mathcal{H}))$ and $\left\langle \int_\mathbb{N} A_n \, d\mu(n) f, f\right\rangle = \int_\mathbb{N} \langle A_n f, f\rangle \, d\mu(n) = \sum_{n=1}^{+\infty} \langle A_n f, f\rangle = \sum_{n=1}^{+\infty} |\langle f, e_n\rangle|^2 = \|f\|^2 = \langle If, f\rangle$ for all $f \in \mathcal{H}$. This implies $\int_\mathbb{N} A_n \, d\mu(n) = I$. From $\dim(\mathcal{H}) = +\infty$ we have $\int_\mathbb{N} A_n \, d\mu(n) = I \notin \mathfrak{C}_\infty(\mathcal{H})$.* △

Definition 2.7. *The family $(A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ satisfies the condition $\mathcal{AC}_\mu$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $E \in \mathfrak{M}$ and $\mu(E) < \delta$ imply $\left\|\int_E A_t \, d\mu(t)\right\| < \varepsilon$.*

The next example shows that we can omit the condition $\mathcal{AC}_\mu$ in [13, Theorem 3.1, (a)]. Also, in the following example, we constructed Gelfand integrable function $A : \Omega \to \mathfrak{B}(\mathcal{H})$ that is not Pettis integrable.

Example 2.8. *Let $(e_n)_{n\geqslant 1}$ be some orthonormal basis of Hilbert space $\mathcal{H}$ and function $\varphi_n : (-1, 1) \to \mathbb{C}$ is defined by $\varphi_n = n \cdot \chi_{(0,\frac{1}{n})} - n \cdot \chi_{(-\frac{1}{n},0)}$ for all $n \in \mathbb{N}$ and for all $t \in (-1, 1)$ we define the function $A : (-1, 1) \to \mathfrak{B}(\mathcal{H})$ by*

$$A_t(f) = \sum_{n=1}^{+\infty} \varphi_n(t) \cdot \langle f, e_n\rangle e_n \quad \text{for all vectors} \quad f \in \mathcal{H}. \tag{14}$$

For fixed $t \in (-1, 1)$, we have the most finite members of nonzero operators in (14), thus $A_t \in \mathfrak{C}_\infty(\mathcal{H})$ for all $t \in (-1, 1)$. Using the equalities $\int_{(-1,0)} \varphi_n(t) \, dm(t) = -1$ and $\int_{(0,1)} \varphi_n(t) \, dm(t) = 1$ we get

$$\begin{aligned}\int_{(-1,1)} \langle A_t f, f\rangle \, dm(t) &= \int_{(-1,0)} \langle A_t f, f\rangle \, dm(t) + \int_{(0,1)} \langle A_t f, f\rangle \, dm(t) \\ &= \sum_{n=1}^{+\infty} \left(\int_{(-1,0)} \varphi_n(t) \, dm(t)\right) |\langle f, e_n\rangle|^2 + \sum_{n=1}^{+\infty} \left(\int_{(0,1)} \varphi_n(t) \, dm(t)\right) |\langle f, e_n\rangle|^2 = -\|f\|^2 + \|f\|^2 = 0\end{aligned}$$

for all $f \in \mathcal{H}$ and $n \in \mathbb{N}$. Thus, $(A_t)_{t\in(-1,1)} \in L^1_G((-1, 1), m, \mathfrak{B}(\mathcal{H}))$ and $\int_{(-1,1)} A_t \, dm(t) = 0 \in \mathfrak{C}_\infty(\mathcal{H})$. Choose $\varepsilon = 1$ and $\delta > 0$ arbitrary, $N \in \left(\frac{1}{\delta}, +\infty\right) \cap \mathbb{N}$ and define the set $E_N = \left(0, \frac{1}{N}\right)$. From $E_N \subset (0, +\infty)$ we have

$$\begin{aligned}\left\langle \int_{E_N} A_t \, dm(t) f, f\right\rangle &= \sum_{n=1}^{+\infty} \left(\int_{E_N} \varphi_n(t) \, dm(t)\right) |\langle f, e_n\rangle|^2 \\ &= \sum_{n=1}^{+\infty} \left(n \int_{(0,\frac{1}{N})} \chi_{(0,\frac{1}{n})}(t) \, dm(t)\right) |\langle f, e_n\rangle|^2 = \sum_{n=1}^{+\infty} n \min\left\{\frac{1}{n}, \frac{1}{N}\right\} |\langle f, e_n\rangle|^2 \quad \text{for all} \quad f \in \mathcal{H}.\end{aligned} \tag{15}$$

Thus, we have $m(E_N) = \frac{1}{N} < \delta$ *and from* (15) *we get*

$$\left\| \int_{E_N} A_t \, dm(t) \right\| = \sup_{\|f\|=1} \left\langle \int_{E_N} A_t \, dm(t) f, f \right\rangle \geqslant \sum_{n=1}^{+\infty} n \cdot \min\left\{ \frac{1}{n}, \frac{1}{N} \right\} |\langle e_N, e_n \rangle|^2 = 1. \tag{16}$$

Note that $n \cdot \min\left\{\frac{1}{n}, \frac{1}{N}\right\} = \min\left\{1, \frac{n}{N}\right\} \leqslant 1$ *for all* $n \in \mathbb{N}$. *Thus,* (15) *and Bessel equality implies* $\left\| \int_{E_N} A_t \, dm(t) \right\| \leqslant 1$ *and together with the inequality* (16) *we obtain* $\left\| \int_{E_N} A_t \, dm(t) \right\| = 1$. *The condition* $\mathcal{AC}_\mu$ *fails for the function* $A : (-1, 1) \to \mathcal{B}(\mathcal{H})$ *and consequently* $A \notin P_1((-1, 1), m, \mathcal{B}(\mathcal{H}))$ *(see [4, Theorem 5, Chapter 2]).* △

The examples 2.6, 2.8 and [13, Example 2.2] motivate us to the following theorem, in which we establish a connection between the Gelfand integral and the Spectral integral, in the case that all operators A_t come from the same spectral measure.

Let $(X, \mathfrak{N})$ be (another) measurable space and let $E : \mathfrak{N} \to \mathcal{B}(\mathcal{H})$ be spectral measure. Then, for $f \in \mathcal{H}$ we define a measure $\eta_f : \mathfrak{N} \to [0, +\infty]$ with the expression $\eta_f(\delta) = \langle E(\delta)f, f\rangle$, $\delta \in \mathfrak{N}$. The measure η_f is finite for all $f \in \mathcal{H}$, because we have $\eta_f(X) = \langle E(X)f, f\rangle = \langle E(X)^2 f, f\rangle = \|E(X)f\|^2 = \|If\|^2 = \|f\|^2$.

For $\varphi \in L^\infty(X, \mathfrak{N}, E)$ we have that the spectral integral $\int_X \varphi(x) \, dE(x)$ exists in $\mathcal{B}(\mathcal{H})$ and

$$\left\langle \int_X \varphi(x) \, dE(x) f, f \right\rangle = \int_X \varphi(x) \, d\eta_f(x) \quad \text{for all} \quad f \in \mathcal{H}. \tag{17}$$

Integration with respect to the spectral measure E is isometric*-isomorphism of the Banach algebra $L^\infty(X, \mathfrak{N}, E)$ with some closed and normal subalgebra in $\mathcal{B}(\mathcal{H})$. Thus, we have the property

$$\sup_{x \in X} \operatorname{ess} |\varphi(x)| = \left\| \int_X \varphi(x) \, dE(x) \right\| \quad \text{for all} \quad \varphi \in L^\infty(X, \mathfrak{N}, E). \tag{18}$$

Theorem 2.9. *Let* $\mathcal{H}$ *be a Hilbert space,* $(\Omega, \mathfrak{M}, \mu)$ *a* σ*-finite measurable space and* $(X, \mathfrak{N})$ *another measurable space with the spectral measure* $E : \mathfrak{N} \to \mathcal{B}(\mathcal{H})$. *Also, let* $\varphi : \Omega \times X \to \mathbb{C}$ *be the function with the following properties:*

- ***(a)*** *The function* $X \ni x \mapsto \varphi(t, x) \in \mathbb{C}$ *belongs to* $L^\infty(X, \mathfrak{N}, E)$ *for all* $t \in \Omega$.
- ***(b)*** $\varphi \in L^1(\Omega \times X, \mathfrak{M} \otimes \mathfrak{N}, \mu \otimes \eta_f)$ *for all* $f \in \mathcal{H}$.
- ***(c)*** *The function* $X \ni x \mapsto \int_\Omega \varphi(t, x) \, d\mu(t) \in \mathbb{C}$ *belongs to* $L^\infty(X, \mathfrak{N}, E)$.

Then, the operator $A_t = \int_X \varphi(t, x) \, dE(x)$ *exists for all* $t \in \Omega$, $(A_t)_{t \in \Omega} \in L^1_G(\Omega, \mathfrak{M}, \mu, \mathcal{B}(\mathcal{H}))$ *and we have*

$$\int_\Omega A_t \, d\mu(t) = \int_X \int_\Omega \varphi(t, x) \, d\mu(t) \, dE(x). \tag{19}$$

Furthermore, the operator $\int_\Omega A_t \, d\mu(t)$ *is normal and*

$$\left\| \int_\Omega A_t \, d\mu(t) \right\| = \sup_{x \in X} \operatorname{ess} \left| \int_\Omega \varphi(t, x) \, d\mu(t) \right|. \tag{20}$$

Proof. From the assumption **(a)** we have that the Spectral integral $A_t = \int_X \varphi(t, x) \, dE(x)$ exists in $\mathcal{B}(\mathcal{H})$ for all $t \in \Omega$. Let $f \in \mathcal{H}$ be arbitrary. From **(b)**, by applying Fubini theorem and (17), we have that the function

$$\Omega \ni t \mapsto \langle A_t f, f\rangle = \left\langle \int_X \varphi(t, x) \, dE(x) f, f \right\rangle = \int_X \varphi(t, x) \, d\eta_f(x) \in \mathbb{C}$$

belongs to $L^1(\Omega, \mathfrak{M}, \mu)$. This implies $(A_t)_{t \in \Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$. Again, from Fubini theorem we have

$$\int_{\Omega \times X} \varphi(t, x) \, d(\mu \otimes \eta_f) = \int_\Omega \left(\int_X \varphi(t, x) \, d\eta_f(x) \right) d\mu(t) = \int_\Omega \langle A_t f, f\rangle \, d\mu(t). \tag{21}$$

From the assumption **(b)** we get that the function $X \ni x \mapsto \int_\Omega \varphi(t,x)\,d\mu(t) \in \mathbb{C}$ exists for almost all $x \in X$ with respect to the measure η_f. Using the assumption **(c)** we can apply (17) and from (21) we obtain

$$\left\langle \int_X \left(\int_\Omega \varphi(t,x)\,d\mu(t) \right) dE(x) f, f \right\rangle = \int_X \left(\int_\Omega \varphi(t,x)\,d\mu(t) \right) d\eta_f(x) = \int_\Omega \langle A_t f, f\rangle\,d\mu(t) = \left\langle \int_\Omega A_t\,d\mu(t) f, f \right\rangle.$$

As $f \in \mathcal{H}$ was arbitrary, we have the formula (19).

The operator $\int_\Omega A_t\,d\mu(t)$ is normal in $\mathcal{B}(\mathcal{H})$ and this follows directly from the proven formula (19) and the mentioned property that the integration with respect to the spectral measure E is isometric*-isomorphism of the Banach algebra $L^\infty(X,\mathfrak{N},E)$ and some closed, normal subalgebra of $\mathcal{B}(\mathcal{H})$.

By the condition **(c)** we can apply (18) and obtain (20). □

In the following, we will look at other possible formulas for representations of the Gelfand integral $\int_\Omega f(t)A_t\,d\mu(t)$ if the members of the family $(A_t)_{t\in\Omega}$ have special properties. To be more precise, we can derive the formula for the representation of the Gelfand integral $\int_\Omega f(t)A_t\,d\mu(t)$ as a weak sum of series of operators in $\mathcal{B}(\mathcal{H})$ if the set Ω is partitioned as a countable $\mathfrak{M}$-measurable partition and if the operator-valued function $A : \Omega \to \mathcal{B}(\mathcal{H})$ has a constant value on each member of this partition.

Furthermore, in the next theorem we give a sufficient condition for the mentioned formula to be a sum in the norm of the separable ideal $\mathcal{C}_\Phi(\mathcal{H})$. We do not assume $(A_t)_{t\in\Omega} \in L^1_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$

Theorem 2.10. *Let $(A_t)_{t\in\Omega}$ be the family in $\mathcal{B}(\mathcal{H})$ such that $\sup_{t\in\Omega}\|A_t\| < +\infty$ and $f \in L^1(\Omega,\mathfrak{M},\mu)$. If $(E_n)_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{M}$, $\bigsqcup_{n=1}^{+\infty} E_n = \Omega$ and if $\{B_k : k \in \mathbb{N}\}$ is the subset of $\mathcal{B}(\mathcal{H})$ such that $A_t = B_k$ for all $t \in E_k$, then $(f(t)A_t)_{t\in\Omega} \in L^1_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$ and we have*

$$\int_\Omega f(t)A_t\,d\mu(t) = \sum_{k=1}^{+\infty} \left[\int_{E_k} f(t)\,d\mu(t) \right] B_k, \tag{22}$$

where the series in (22) converges weakly. Moreover, if we have $f(t)A_t \geqslant 0$ for all $t \in \Omega$ and if $\int_\Omega f(t)A_t\,d\mu(t) \in \mathcal{C}_\Phi(\mathcal{H})$ then the series in (22) converges in the norm $\|\cdot\|_\Phi$ of the separable ideal $\mathcal{C}_\Phi(\mathcal{H})$.

Proof. The family $(f(t)A_t)_{t\in\Omega}$ belongs to $L^1_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$ because for all $x,y \in \mathcal{H}$ we have

$$\begin{aligned}\int_\Omega |\langle f(t)A_t x, y\rangle|\,d\mu(t) &= \sum_{k=1}^{+\infty} \int_{E_k} |\langle A_t x, y\rangle| \cdot |f(t)|\,d\mu(t) = \sum_{k=1}^{+\infty} |\langle B_k x, y\rangle| \int_{E_k} |f(t)|\,d\mu(t) \\ &\leqslant \sup_{t\in\Omega}\|A_t\|\cdot\|x\|\cdot\|y\| \sum_{k=1}^{+\infty} \int_{E_k} |f(t)|\,d\mu(t) = \sup_{t\in\Omega}\|A_t\|\cdot\|x\|\cdot\|y\|\cdot\|f\|_1 < +\infty.\end{aligned}$$

The function $\mathfrak{M} \ni E \mapsto \int_E f(t)\langle A_t x, y\rangle\,d\mu(t) \in \mathbb{C}$ is complex measure on $\mathfrak{M}$ for fixed $x,y \in \mathcal{H}$. Thus, we have

$$\left\langle \int_\Omega f(t)A_t\,d\mu(t)x, y \right\rangle = \int_\Omega f(t)\langle A_t x, y\rangle\,d\mu(t) = \sum_{k=1}^{+\infty} \int_{E_k} f(t)\langle A_t x, y\rangle\,d\mu(t) = \lim_{n\to\infty} \left\langle \left(\sum_{k=1}^{n} \left(\int_{E_k} f(t)\,d\mu(t) \right) B_k \right) x, y \right\rangle,$$

which confirms the formula (22). Moreover, if $f(t)A_t \geqslant 0$ for all $t \in \Omega$ and if $\int_\Omega f(t)A_t\,d\mu(t) \in \mathcal{C}_\Phi(\mathcal{H})$ then [13, Theorem 1.3, (a)] implies that the function $\mathfrak{M} \ni E \mapsto \int_E f(t)A_t\,d\mu(t)$ is a $\mathcal{C}_\Phi(\mathcal{H})$-valued measure. By (7) we get $\int_\Omega f(t)A_t\,d\mu(t) = \sum_{k=1}^{+\infty} \int_{E_k} f(t)A_t\,d\mu(t) = \sum_{k=1}^{+\infty} \int_{E_k} f(t)B_k\,d\mu(t) = \sum_{k=1}^{+\infty} \left(\int_{E_k} f(t)\,d\mu(t) \right) B_k$ in the norm $\|\cdot\|_\Phi$ of the separable ideal $\mathcal{C}_\Phi(\mathcal{H})$. □

Let $A_t \in \mathcal{C}_\infty(\mathcal{H})$ for μ-almost all $t \in \Omega$. If we have $(A_t)_{t\in\Omega} \in L^1_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$ and if the function v_A associated to the family $(A_t)_{t\in\Omega}$ is $\mathcal{B}(\mathcal{H})$-valued measure on $\mathfrak{M}$, then we have the same conclusion as in [13, Theorem 2.3, (a)], if measure μ is σ-finite.

Theorem 2.11. *Let μ be a σ-finite measure on $\mathfrak{M}$ and let $(A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ such that $A_t \in \mathfrak{C}_\infty(\mathcal{H})$ for μ-almost all $t \in \Omega$. If the function $v_A : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ defined by $v_A(E) = \int_E A_t \, d\mu(t)$, $E \in \mathfrak{M}$, is a $\mathcal{B}(\mathcal{H})$-valued measure and if the family $(A_t)_{t\in\Omega}$ satisfies $\mathcal{AC}_\mu$ condition then $\int_\Omega A_t \, d\mu(t) \in \mathfrak{C}_\infty(\mathcal{H})$.*

Proof. There is a sequence $(E_n)_{n\in\mathbb{N}}$ with mutually disjoint members from $\mathfrak{M}$ with the properties $\mu(E_n) < +\infty$ for all $n \in \mathbb{N}$ and $\bigsqcup_{n\in\mathbb{N}} E_n = \Omega$. Applying [13, Theorem 2.3, (a)] to each E_n gives us $v_A(E_n) = \int_{E_n} A_t \, d\mu(t) \in \mathfrak{C}_\infty(\mathcal{H})$ for all $n \in \mathbb{N}$. The function v_A is a $\mathcal{B}(\mathcal{H})$-valued measure on $\mathfrak{M}$, i.e. $\int_\Omega A_t \, d\mu(t) = v_A(\Omega) = \sum_{n=1}^{+\infty} v_A(E_n) = \sum_{n=1}^{+\infty} \int_{E_n} A_t \, d\mu(t)$, where the last convergence is in the norm of the space $\mathcal{B}(\mathcal{H})$. The theorem follows due to the fact that the ideal $\mathfrak{C}_\infty(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ is closed with respect to the operator norm. □

Note that in the Example 2.6 the function $v_A(E) = \int_E A_n \, d\mu(n)$, $E \in \mathfrak{M}$, is not a $\mathcal{B}(\mathcal{H})$-valued measure on $\mathfrak{M} = \mathcal{P}(\mathbb{N})$, although it is finitely additive. Indeed, if v_A is a $\mathcal{B}(\mathcal{H})$-valued measure on the σ-algebra $\mathfrak{M} = \mathcal{P}(\mathbb{N})$, then it follows from Theorem 2.11 that $I \in \mathfrak{C}_\infty(\mathcal{H})$. This is a contradiction with $\dim(\mathcal{H}) = +\infty$.

Let $(A_t)_{t\in\Omega}$ be the family of operators from $\mathcal{B}(\mathcal{H})$. The following theorem gives a sufficient condition for the convergence of the sequence of Gelfand integrals $\left(\int_\Omega f_n(t) A_t \, d\mu(t)\right)_{n\in\mathbb{N}}$ in the norm of the space $\mathcal{B}(\mathcal{H})$ to $\int_\Omega f(t) A_t \, d\mu(t)$, where $(f_n)_{n\in\mathbb{N}}$ and f are functions from $L^2(\Omega, \mathfrak{M}, \mu)$, with some additional properties.

Theorem 2.12. *Let $\mu(\Omega) < +\infty$ and $f_n \in L^2(\Omega, \mathfrak{M}, \mu)$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for μ-almost all $t \in \Omega$. If $\sup_{n\in\mathbb{N}} \|f_n\|_2 < +\infty$ and if the set of functions $\Omega \ni t \mapsto \|A_t x\|^2 \in \mathbb{R}$, $\|x\| \leqslant 1$, is uniformly integrable, then both families of operators $(f_n(t)A_t)_{t\in\Omega}$ and $(f(t)A_t)_{t\in\Omega}$ belongs to $L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and we have*

$$\lim_{n\to\infty} \left\| \int_\Omega f_n(t) A_t \, d\mu(t) - \int_\Omega f(t) A_t \, d\mu(t) \right\| = 0. \tag{23}$$

Proof. For $n \in \mathbb{N}$ and all $x, y \in \mathcal{H}$, Cauchy-Schwarz inequality implies

$$\int_\Omega |\langle f_n(t) A_t x, y\rangle| \, d\mu(t) \leqslant \sqrt{\int_\Omega |f_n(t)|^2 \, d\mu(t)} \cdot \sqrt{\int_\Omega |\langle A_t x, y\rangle|^2 \, d\mu(t)} \leqslant \|f_n\|_2 \sqrt{\int_\Omega \|A_t x\|^2 \, d\mu(t)} \cdot \|y\| < +\infty,$$

Thus, we get $(f_n(t)A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ for all $n \in \mathbb{N}$. Let $L := \sup_{n\in\mathbb{N}} \|f_n\|_2 < +\infty$. Fatou lemma implies $f \in L^2(\Omega, \mathfrak{M}, \mu)$. Indeed, we have

$$\|f\|_2^2 = \int_\Omega |f(t)|^2 \, d\mu(t) = \int_\Omega \lim_{n\to\infty} |f_n(t)|^2 \, d\mu(t) \leqslant \liminf_{n\to\infty} \int_\Omega |f_n(t)|^2 \, d\mu(t) = \liminf_{n\to\infty} \|f_n\|_2^2 \leqslant L^2.$$

Thus, we have $\int_\Omega |\langle f(t) A_t x, y\rangle| \, d\mu(t) < +\infty$, $x, y \in \mathcal{H}$. This implies $(f(t)A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and we get

$$\left\| \int_\Omega f_n(t) A_t \, d\mu(t) - \int_\Omega f(t) A_t \, d\mu(t) \right\| \leqslant \sup_{\|x\|=\|y\|=1} \int_\Omega |f_n(t) - f(t)| \cdot |\langle A_t x, y\rangle| \, d\mu(t) \quad \text{for all} \quad n \in \mathbb{N}. \tag{24}$$

Let $\varepsilon > 0$ be arbitrary and choose $\delta = \delta(\varepsilon) > 0$ from the definition of the uniform integrability of the set of functions $\Omega \ni t \mapsto \|A_t x\|^2 \in \mathbb{R}$, $\|x\| \leqslant 1$. Using Egoroff theorem we can find a set $\mathfrak{M} \ni E_\delta \subseteq \Omega$ such that

$$\lim_{n\to\infty} \sup_{t\in E_\delta} |f_n(t) - f(t)| = 0 \quad \text{and} \quad \mu(\Omega \setminus E_\delta) < \delta.$$

For all $x \in \mathcal{H}$ we have $\int_\Omega \|A_t x\|^2 \, d\mu(t) < +\infty$. Thus, $(A_t^* A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$. From (8) we have

$$M := \sup_{\|x\|=1} \int_\Omega \langle A_t^* A_t x, x\rangle \, d\mu(t) = \sup_{\|x\|=1} \int_\Omega \|A_t x\|^2 \, d\mu(t) < +\infty.$$

By applying the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sup_{\|x\|=\|y\|=1}\int_{E_\delta}|f_n(t)-f(t)|\cdot|\langle A_t x,y\rangle|\,d\mu(t) &\leqslant \sup_{t\in E_\delta}|f_n(t)-f(t)|\sup_{\|x\|=\|y\|=1}\int_{E_\delta}|\langle A_t x,y\rangle|\,d\mu(t)\\ &\leqslant \sup_{t\in E_\delta}|f_n(t)-f(t)|\sup_{\|x\|=1}\int_{E_\delta}\|A_t x\|\,d\mu(t)\\ &\leqslant \sup_{t\in E_\delta}|f_n(t)-f(t)|\sqrt{\sup_{\|x\|=1}\int_{E_\delta}\|A_t x\|^2\,d\mu(t)}\cdot\sqrt{\mu(E_\delta)}\\ &\leqslant \sqrt{M\cdot\mu(\Omega)}\sup_{t\in E_\delta}|f_n(t)-f(t)| \quad \text{for all} \quad n\in\mathbb{N}.\end{aligned} \tag{25}$$

The triangle inequality in $L^2(\Omega,\mathfrak{M},\mu)$ implies $\|f_n-f\|_2 \leqslant \|f_n\|_2+\|f\|_2 \leqslant 2L$ for all $n\in\mathbb{N}$. Using the Cauchy–Schwarz inequality and the uniform integrability of the set of functions $\Omega \ni t \mapsto \|A_t x\|^2 \in \mathbb{R}$, $\|x\| \leqslant 1$, combined with the inequality $\mu(\Omega\setminus E_\delta)<\delta$, we obtain

$$\begin{aligned}\sup_{\|x\|=\|y\|=1}\int_{\Omega\setminus E_\delta}|f_n(t)-f(t)|\cdot|\langle A_t x,y\rangle|\,d\mu(t) &\leqslant \sup_{\|x\|=\|y\|=1}\sqrt{\int_{\Omega\setminus E_\delta}|f_n(t)-f(t)|^2\,d\mu(t)}\cdot\sqrt{\int_{\Omega\setminus E_\delta}|\langle A_t x,y\rangle|^2\,d\mu(t)}\\ &\leqslant \|f_n-f\|_2\sup_{\|x\|=\|y\|=1}\sqrt{\int_{\Omega\setminus E_\delta}|\langle A_t x,y\rangle|^2\,d\mu(t)}\\ &\leqslant 2L\sup_{\|x\|=1}\sqrt{\int_{\Omega\setminus E_\delta}\|A_t x\|^2\,d\mu(t)} \leqslant 2L\sqrt{\varepsilon} \quad \text{for all} \quad n\in\mathbb{N}.\end{aligned} \tag{26}$$

Now, from (24), (25) and (26), we get

$$\left\|\int_\Omega f_n(t)A_t\,d\mu(t)-\int_\Omega f(t)A_t\,d\mu(t)\right\| \leqslant \sqrt{M\cdot\mu(\Omega)}\sup_{t\in E_\delta}|f_n(t)-f(t)|+2L\sqrt{\varepsilon} \quad \text{for all} \quad n\in\mathbb{N}.$$

Using the equality $\lim_{n\to\infty}\sup_{t\in E_\delta}|f_n(t)-f(t)|=0$ and $\mu(\Omega)<+\infty$ we have

$$\limsup_{n\to\infty}\left\|\int_\Omega f_n(t)A_t\,d\mu(t)-\int_\Omega f(t)A_t\,d\mu(t)\right\| \leqslant 2L\sqrt{\varepsilon} \quad \text{for all} \quad \varepsilon>0.$$

This implies the equality (23). □

Note that $(A_t)_{t\in\Omega}\in L^1_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$ was not assumed in the previous theorem.

The condition of uniform integrability of the set of functions $\Omega\ni t\mapsto\|A_t x\|^2\in\mathbb{R}$, $\|x\|\leqslant 1$, is trivially satisfied if $\mu(\Omega)<+\infty$ and $\sup_{t\in\Omega}\|A_t\|<+\infty$.

We can easy see that the set $\{f_n : n\in\mathbb{N}\}$ from the Theorem 2.12 is uniformly integrable. Indeed, from $\mu(\Omega)<+\infty$ and $f_n\in L^2(\Omega,\mathfrak{M},\mu)$ we have $f_n\in L^1(\Omega,\mathfrak{M},\mu)$ for all $n\in\mathbb{N}$. If we assume that the set $\{f_n : n\in\mathbb{N}\}$ is not uniformly integrable, then we can find $\varepsilon_0>0$ and a sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathfrak{M}$, such that $\lim_{n\to\infty}\mu(E_n)=0$ and for all $n\in\mathbb{N}$ there is $m(n)\in\mathbb{N}$ such that $\left|\int_{E_n} f_{m(n)}(t)\,d\mu(t)\right|\geqslant\varepsilon_0$. Then, we have

$$\varepsilon_0 \leqslant \left|\int_{E_n} f_{m(n)}(t)\,d\mu(t)\right| \leqslant \sqrt{\int_{E_n}|f_{m(n)}(t)|^2\,d\mu(t)}\cdot\sqrt{\mu(E_n)} \leqslant \sup_{n\in\mathbb{N}}\|f_n\|_2\cdot\sqrt{\mu(E_n)} \quad \text{for all} \quad n\in\mathbb{N}.$$

Now, we obtain the contradiction from $\sup_{n\in\mathbb{N}}\|f_n\|_2<\infty$ and $\lim_{n\to\infty}\mu(E_n)=0$.

In the next example, we will show that the condition $\mu(\Omega)<+\infty$ in the Theorem 2.12 cannot be omitted. For the sequence $(f_n)_{n\in\mathbb{N}}$ we will choose the one that is uniformly integrable.

Example 2.13. *Let $\Omega = [1, +\infty)$. We define $A : [1, +\infty) \to \mathcal{B}(\mathcal{H})$ by $A_t(f) = \langle f, e_{\lfloor t \rfloor}\rangle e_{\lfloor t \rfloor}$ for all $f \in \mathcal{H}$ and $t \in [1, +\infty)$, where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in a Hilbert space $\mathcal{H}$. For $n \in \mathbb{N}$ we have $\chi_{[n,n+1)}(t)A_t(f) = \langle f, e_n\rangle e_n$, if $t \in [n, n+1)$ and $\chi_{[n,n+1)}(t)A_t(f) = 0$ if $t \notin [n, n+1)$. Consequently,*

$$\int_{[1,+\infty)} \langle \chi_{[n,n+1)}(t)A_t(f), f\rangle \, dm(t) = \int_{[n,n+1)} |\langle f, e_n\rangle|^2 \, dm(t) = |\langle f, e_n\rangle|^2 \quad \text{for all} \quad f \in \mathcal{H}. \tag{27}$$

Now, we have $(\chi_{[n,n+1)}(t)A_t)_{t\in[1,+\infty)} \in L^1_G([1, +\infty), m, \mathcal{B}(\mathcal{H}))$. By Bessel equality we obtain

$$\int_{[1,+\infty)} \|A_t f\|^2 \, dm(t) = \sum_{n=1}^{+\infty} \int_{[n,n+1)} \|A_t f\|^2 \, dm(t) = \sum_{n=1}^{+\infty} |\langle f, e_n\rangle|^2 = \|f\|^2 \quad \text{for all} \quad f \in \mathcal{H}. \tag{28}$$

Furthermore, for all Lebesgue-measurable sets $E \subseteq [1, +\infty)$, $m(E) < +\infty$ and all $f \in \mathcal{H}$, $\|f\| \leqslant 1$ we have

$$\int_E \|A_t f\|^2 \, dm(t) = \sum_{n=1}^{+\infty} |\langle f, e_n\rangle|^2 m(E \cap [n, n+1)) \leqslant \|f\|^2 \sum_{n=1}^{+\infty} m(E \cap [n, n+1)) = \|f\|^2 \cdot m(E) \leqslant m(E).$$

This means that the set of functions $[1, +\infty) \ni t \mapsto \|A_t f\|^2$, $\|f\| \leqslant 1$, is uniformly integrable. Using (27) we get

$$\left\| \int_{[1,+\infty)} \chi_{[n,n+1)}(t)A_t \, dm(t) \right\| = \sup_{\|f\|=1} \int_{[1,+\infty)} \langle \chi_{[n,n+1)}(t)A_t(f), f\rangle \, dm(t) = \sup_{\|f\|=1} |\langle f, e_n\rangle|^2 = \|e_n\|^2 = 1$$

for all $n \in \mathbb{N}$. This shows that (23) does not hold. Using (27) we have $(A_t)_{t\in[1,+\infty)} \in L^1_G([1, +\infty), m, \mathcal{B}(\mathcal{H}))$ and $\int_{[1,+\infty)} A_t \, dm(t) = I$. From (28) also follows $(A_t^* A_t)_{t\in[1,+\infty)} \in L^1_G([1, +\infty), m, \mathcal{B}(\mathcal{H}))$ and $\int_{[1,+\infty)} A_t^* A_t \, dm(t) = I$. △

The following theorem is a consequence of the previous statement for the convergence of the sequence of Gelfand integrals $\left(\int_{E_n} A_t \, d\mu(t)\right)_{n\in\mathbb{N}}$ in the norm of the space $\mathcal{B}(\mathcal{H})$ to the operator $\int_E A_t \, d\mu(t)$, where the sequence of sets $(E_n)_{n\in\mathbb{N}}$ in $\mathfrak{M}$ converges to a set $E \in \mathfrak{M}$.

Corollary 2.14. *Let $\mu(\Omega) < +\infty$ and let the set of functions $\Omega \ni t \mapsto \|A_t x\|^2$, $\|x\| \leqslant 1$, is uniformly integrable. If $(E_n)_{n\in\mathbb{N}}$ is the sequence in $\mathfrak{M}$ such that there exists the set limit $E = \lim_{n\to\infty} E_n$, then the operators $\int_{E_n} A_t \, d\mu(t)$ and $\int_E A_t \, d\mu(t)$ exist in $\mathcal{B}(\mathcal{H})$ and we have the equality*

$$\lim_{n\to\infty} \left\| \int_{E_n} A_t \, d\mu(t) - \int_E A_t \, d\mu(t) \right\| = 0. \tag{29}$$

Proof. From $E = \lim_{n\to\infty} E_n$ we have $E \in \mathfrak{M}$. Denote $f_n = \chi_{E_n}$ for all $n \in \mathbb{N}$ and $f = \chi_E$. For all $n \in \mathbb{N}$ we have $\|f_n\|_2 = \mu(E_n) \leqslant \mu(\Omega) < +\infty$, i.e. $\sup_{n\in\mathbb{N}} \|f_n\|_2 < +\infty$ and $\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \chi_{E_n}(t) = \chi_E(t) = f(t)$ for μ-almost all $t \in \Omega$. Also, we have $f \in L^2(\Omega, \mathfrak{M}, \mu)$. All assumptions of the Theorem 2.12 are satisfied and this implies $(\chi_{E_n}(t)A_t)_{t\in\Omega}, (\chi_E(t)A_t)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and from (23) we directly get (29). □

Similar problems were discussed in [12, Lemma 2.1].

3. Integration of weakly* measurable functions $A : \Omega \to \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$

The function $\Omega \ni t \mapsto \bigoplus_{n=1}^{+\infty} A_t^{(n)} \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ is weakly* measurable if and only if the function $\Omega \ni t \mapsto A_t^{(n)} \in \mathcal{B}(\mathcal{H}_n)$ is weakly* measurable for all $n \in \mathbb{N}$ and this follows directly from the definition (1) of the inner product in the Hilbert space $\bigoplus_{n=1}^{+\infty} \mathcal{H}_n$.

The following theorem is a reformulation of the necessary and sufficient conditions for the function $\Omega \ni t \mapsto \bigoplus_{n=1}^{+\infty} A_t^{(n)} \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ to belongs to the space $L^1_G\left(\Omega, \mu, \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)\right)$. In this theorem, we also consider the relationship between the operators $\int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t)$ and $\bigoplus_{n=1}^{+\infty} \int_E A_t^{(n)} d\mu(t)$ for $E \in \mathfrak{M}$, if $\int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t)$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$.

Theorem 3.1. *Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be the sequence of Hilbert spaces and $\left(A_t^{(n)}\right)_{t\in\Omega}$ be the family in $\mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$. The operator $\int_\Omega \bigoplus_{n=1}^{+\infty} A_t^{(n)}\, d\mu(t)$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ if and only if we have both of the following conditions:*

- **(a)** $\sup_{n\in\mathbb{N}} \left\|A_t^{(n)}\right\| < +\infty$ *for μ-almost all $t \in \Omega$.*
- **(b)** *The function $\Omega \ni t \mapsto \sum_{n=1}^{+\infty} \left\langle A_t^{(n)} f_n, f_n\right\rangle \in \mathbb{C}$ belongs to the space $L^1(\Omega, \mathfrak{M}, \mu)$ for all $(f_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$.*

Furthermore, under the conditions **(a)** *and* **(b)** *we have*

$$\sup_{n\in\mathbb{N}} \left\| \int_E A_t^{(n)}\, d\mu(t) \right\| < +\infty \quad \text{for all} \quad E \in \mathfrak{M}, \tag{30}$$

the operator $\bigoplus_{n=1}^{+\infty} \int_E A_t^{(n)} d\mu(t)$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ and we have the equality

$$\int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) = \bigoplus_{n=1}^{+\infty} \int_E A_t^{(n)} d\mu(t). \tag{31}$$

Proof. Condition **(a)** is equivalent to the existence of the operator $\bigoplus_{n=1}^{+\infty} A_t^{(n)}$ in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ for μ-almost all $t \in \Omega$. By the definition (1) of the inner product in the Hilbert space $\bigoplus_{n=1}^{+\infty} \mathcal{H}_n$ the condition **(b)** is equivalent with the fact that the function $\Omega \ni t \mapsto \left\langle \left(\bigoplus_{n=1}^{+\infty} A_t^{(n)}\right)(f_n)_{n\in\mathbb{N}}, (f_n)_{n\in\mathbb{N}}\right\rangle = \sum_{n=1}^{+\infty} \left\langle A_t^{(n)} f_n, f_n\right\rangle \in \mathbb{C}$ belongs to the space $L^1(\Omega, \mathfrak{M}, \mu)$ for all vectors $(f_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$. Thus, the conditions **(a)** and **(b)** are equivalent with $\left(\bigoplus_{n=1}^{+\infty} A_t^{(n)}\right)_{t\in\Omega} \in L^1_G\left(\Omega, \mu, \mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)\right)$.

From the condition **(b)** it follows that the operator $\int_\Omega A_t^{(n)}\, d\mu(t)$ exists in $\mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$, because we can take $f_n \in \mathcal{H}_n$ arbitrary and $f_k = 0$ for all $k \in \mathbb{N} \setminus \{n\}$ in **(b)**, and obtain that the scalar function $\Omega \ni t \mapsto \left\langle A_t^{(n)} f_n, f_n\right\rangle \in \mathbb{C}$ belongs to $L^1(\Omega, \mathfrak{M}, \mu)$ for all $f_n \in \mathcal{H}_n$.

Now we will prove (30) under the conditions **(a)** and **(b)**. Let $E \in \mathfrak{M}$ and $k \in \mathbb{N}$ arbitrary. Using the equalities (4) and (1) we have

$$\begin{aligned}
\left\| \int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) \right\| &= \sup_{\|(f_n)_{n\in\mathbb{N}}\| = \|(g_n)_{n\in\mathbb{N}}\| = 1} \left| \left\langle \left(\int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) \right) (f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \right\rangle \right| \\
&= \sup_{\|(f_n)_{n\in\mathbb{N}}\| = \|(g_n)_{n\in\mathbb{N}}\| = 1} \left| \int_E \left\langle \left(\bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) \right) (f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \right\rangle \right| \\
&= \sup_{\|(f_n)_{n\in\mathbb{N}}\| = \|(g_n)_{n\in\mathbb{N}}\| = 1} \left| \int_E \sum_{n=1}^{+\infty} \left\langle A_t^{(n)} f_n, g_n \right\rangle d\mu(t) \right| \geqslant \left| \int_E \left\langle A_t^{(k)} x_k, y_k \right\rangle d\mu(t) \right| = \left| \left\langle \int_E A_t^{(k)}\, d\mu(t) x_k, y_k \right\rangle \right|
\end{aligned}$$

for all unit vectors $x_k, y_k \in \mathcal{H}_k$. Thus, we have

$$\left\| \int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) \right\| \geqslant \sup_{\|x_k\| = \|y_k\| = 1} \left| \left\langle \int_E A_t^{(k)}\, d\mu(t) x_k, y_k \right\rangle \right| = \left\| \int_E A_t^{(k)}\, d\mu(t) \right\|.$$

This implies

$$\sup_{k\in\mathbb{N}} \left\| \int_E A_t^{(k)}\, d\mu(t) \right\| \leqslant \left\| \int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)} d\mu(t) \right\| < +\infty \quad \text{for all} \quad E \in \mathfrak{M}.$$

We proved the inequality (30). Therefore, the operator $\bigoplus_{n=1}^{+\infty} \int_E A_t^{(n)} d\mu(t)$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$.

Now, we will show the equality (31). Again, let $k \in \mathbb{N}$ be a fixed number. We define the projection map $T_k : \bigoplus_{n=1}^{+\infty} \mathcal{H}_n \to \mathcal{H}_k$ with $T_k((x_n)_{n\in\mathbb{N}}) = x_k$ for all $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$. From $\|T_k((x_n)_{n\in\mathbb{N}})\|^2 = \|x_k\|^2 \leqslant$

$\sum_{n=1}^{+\infty}\|x_n\|^2 = \|(x_n)_{n\in\mathbb{N}}\|^2$ for all $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty}\mathcal{H}_n$ follows $T_k \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n, \mathcal{H}_k\right)$. Based on the second and third property in (6), we have equalities

$$T_k \int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)}\,d\mu(t) = \int_E T_k \bigoplus_{n=1}^{+\infty} A_t^{(n)}\,d\mu(t) = \int_E A_t^{(k)} T_k\,d\mu(t) = \int_E A_t^{(k)}\,d\mu(t) T_k = T_k \bigoplus_{n=1}^{+\infty}\int_E A_t^{(n)}\,d\mu(t).$$

From the above equalities we have that the vector on the k-th position in $\left(\int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)}\,d\mu(t)\right)(f_m)_{m\in\mathbb{N}}$ is equal to the vector on the k-th position in the sequence $\left(\bigoplus_{n=1}^{+\infty}\int_E A_t^{(n)}\,d\mu(t)\right)(f_m)_{m\in\mathbb{N}}$ for all $(f_m)_{m\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty}\mathcal{H}_n$. Since $k \in \mathbb{N}$ was arbitrary, we obtain the equality (31). □

Under the conditions **(a)** and **(b)** of the Theorem 3.1, the equality (31) implies

$$\left\|\int_E \bigoplus_{n=1}^{+\infty} A_t^{(n)}\,d\mu(t)\right\| = \left\|\bigoplus_{n=1}^{+\infty}\int_E A_t^{(n)}\,d\mu(t)\right\| = \sup_{n\in\mathbb{N}}\left\|\int_E A_t^{(n)}\,d\mu(t)\right\| \quad \text{for all} \quad E \in \mathfrak{M}. \tag{32}$$

It is possible that the operator $\bigoplus_{n=1}^{+\infty}\int_\Omega A_t^{(n)}\,d\mu(t)$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$ but $\int_\Omega \bigoplus_{n=1}^{+\infty} A_t^{(n)}\,d\mu(t)$ does not exist. This is shown in the next example.

Example 3.2. *Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces, $\Omega = (-1,1)$ and $\varphi_n = -n\chi_{(-1,0)} + n\chi_{(0,1)}$ for $n \in \mathbb{N}$. We define $A_t^{(n)} = \varphi_n(t)I_n$ for all $t \in (-1,1)$ and $n \in \mathbb{N}$. Due to (7) we have $\int_{(-1,1)} A_t^{(n)}\,dm(t) = \int_{(-1,1)}\varphi_n(t)\,dm(t)I_n = 0 \in \mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$. Thus, we get $\bigoplus_{n=1}^{+\infty}\int_{(-1,1)} A_t^{(n)}\,dm(t) = 0 \in \mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$. On the other hand, for $t \in (-1,0)$ we have $\sup_{n\in\mathbb{N}}\left\|A_t^{(n)}\right\| = \sup_{n\in\mathbb{N}}\|\varphi_n(t)I_n\| = +\infty$ i.e. the operator $\bigoplus_{n=1}^{+\infty} A_t^{(n)}$ does not exist. Thus, $\int_{(-1,1)}\bigoplus_{n=1}^{+\infty} A_t^{(n)}\,dm(t)$ does not exist.* △

Let $k \in \mathbb{N}$ be fixed. The next example motivates us to analyse the relation between measures on the σ-algebra $\mathfrak{M}$ taking values in $\mathcal{B}(\mathcal{H}_k)$ and $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$.

Example 3.3. *Let $\Omega = (0,1]$ and let $\mathfrak{M}$ be Lebesgue σ-algebra on $(0,1]$ and $f_n = n\chi_{(0,\frac{1}{n}]}$ for all $n \in \mathbb{N}$. Then $\int_{(0,1]} f_n(t)\,dm(t) = 1$ and due to (7) we have $\int_E f_n(t)I_n\,dm(t) = \left(\int_E f_n(t)\,dm(t)\right)I_n$ for all $n \in \mathbb{N}$ and $E \in \mathfrak{M}$. For $n \in \mathbb{N}$ we define $v_n : \mathfrak{M} \to \mathcal{B}(\mathcal{H}_n)$ with $v_n(E) = \int_E f_n(t)I_n\,dm(t) = \left(\int_E f_n(t)\,dm(t)\right)I_n = n\cdot m\left(E \cap \left(0,\frac{1}{n}\right]\right)I_n$ for all $E \in \mathfrak{M}$. The function $\mathfrak{M} \ni E \mapsto n\cdot m\left(E\cap\left(0,\frac{1}{n}\right]\right) \in [0,1]$ is positive, finite measure on $\mathfrak{M}$, thus the function v_n is a $\mathcal{B}(\mathcal{H}_n)$-valued measure on $\mathfrak{M}$ for all $n \in \mathbb{N}$. We have $\left(\bigoplus_{n=1}^{+\infty} f_n(t)I_n\right)_{t\in(0,1]} \in L^1_G\left((0,1], m, \mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)\right)$. Indeed, $\sup_{n\in\mathbb{N}}\|f_n(t)I_n\| = \sup_{n\in\mathbb{N}} n\chi_{(0,\frac{1}{n}]}(t) < +\infty$ for all $t \in (0,1]$. This implies that the operator $\bigoplus_{n=1}^{+\infty} f_n(t)I_n$ exists in $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$ for all $t \in (0,1]$. Also, for $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty}\mathcal{H}_n$ we have*

$$\int_{(0,1]}\left|\left\langle\left(\bigoplus_{n=1}^{+\infty} f_n(t)I_n\right)(x_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}\right\rangle\right| dm(t) \leqslant \sum_{n=1}^{+\infty}\|x_n\|^2\int_{(0,1]}|f_n(t)|\,dm(t) = \sum_{n=1}^{+\infty}\|x_n\|^2 < +\infty.$$

Thus, we can apply the Theorem 3.1 and define the function $v : \mathfrak{M} \to \mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$ by

$$v(E) = \int_E \bigoplus_{n=1}^{+\infty} f_n(t)I_n\,dm(t) = \bigoplus_{n=1}^{+\infty}\left(\int_E f_n(t)\,dm(t)\right)I_n = \bigoplus_{n=1}^{+\infty} v_n(E) \quad \textit{for all} \quad E \in \mathfrak{M}.$$

The function $v : \mathfrak{M} \to \mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$ is weakly countable additive vector measure on $\mathfrak{M}$, but this function is not $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty}\mathcal{H}_n\right)$-valued measure. To see that, denote $E_k = \left(\frac{1}{k+1},\frac{1}{k}\right] \in \mathfrak{M}$, $k \in \mathbb{N}$. Then we have $\bigsqcup_{k=1}^{+\infty} E_k = (0,1]$ and

$v((0,1]) = \bigoplus_{n=1}^{+\infty} I_n$. *For fixed number $N \in \mathbb{N}$ we have the following calculations:*

$$\left\| v((0,1]) - v\left(\bigsqcup_{k=1}^{N} E_k\right) \right\| = \left\| \bigoplus_{n=1}^{+\infty} I_n - \bigoplus_{n=1}^{+\infty} \left(\int_{\bigsqcup_{k=1}^{N} E_k} f_n(t)\, dm(t) \right) I_n \right\| = \left\| \bigoplus_{n=1}^{+\infty} \left(1 - \left(\int_{\bigsqcup_{k=1}^{N} E_k} f_n(t)\, dm(t) \right) \right) I_n \right\|$$
$$= \sup_{n\in\mathbb{N}} \left| 1 - \int_{\left(\frac{1}{N+1},1\right]} f_n(t)\, dm(t) \right| \geqslant \left| 1 - \int_{\left(\frac{1}{N+1},1\right]} f_N(t)\, dm(t) \right| = \frac{N}{N+1}.$$

It follows that the series $\sum_{n=1}^{+\infty} v(E_n)$ does not converge to $v\left(\bigsqcup_{n=1}^{+\infty} E_k\right)$ in the norm of the space $\mathcal{B}\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$. △

Note that in the Example 3.3 the sequence $(f_n)_{n\in\mathbb{N}}$ was not uniformly integrable.

In the next theorem, we assume $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{N}$. For a given sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathfrak{M}$-measurable functions and a family $(A_t)_{t\in\Omega}$ in $\mathcal{B}(\mathcal{H})$ such that $\left(\bigoplus_{n=1}^{+\infty} f_n(t)A_t\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\ell^2(\mathcal{H})))$ we define the functions $v_n : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ and $v : \mathfrak{M} \to \mathcal{B}(\ell^2(\mathcal{H}))$ with the expressions $v_n(E) = \int_E f_n(t)A_t\, d\mu(t)$ and $v(E) = \int_E \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t)$, respectively. We will investigate sufficient conditions for the function v to be $\mathcal{B}(\ell^2(\mathcal{H}))$-valued measure on $\mathfrak{M}$. If the function v is $\mathcal{B}(\ell^2(\mathcal{H}))$-valued measure on $\mathfrak{M}$, then the function v_n is well-defined $\mathcal{B}(\mathcal{H})$-valued measure on $\mathfrak{M}$ for all $n \in \mathbb{N}$. Indeed, the family $(f_n(t)A_t)_{t\in\Omega}$ belongs to $L^1_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$. According to the Theorem 3.1 we have the equalities

$$v(E) = \int_E \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t) = \bigoplus_{n=1}^{+\infty} \int_E f_n(t)A_t\, d\mu(t) = \bigoplus_{n=1}^{+\infty} v_n(E) \quad \text{for all} \quad E \in \mathfrak{M}. \tag{33}$$

Due to (33), for the sequence $(E_n)_{n\in\mathbb{N}}$, $E_n \cap E_m = \emptyset$, $n \neq m$ in $\mathfrak{M}$ we have

$$\lim_{N\to\infty} \left\| v_n\left(\bigsqcup_{k=1}^{+\infty} E_k\right) - v_n\left(\bigsqcup_{k=1}^{N} E_k\right) \right\| = \lim_{N\to\infty} \left\| v_n\left(\bigsqcup_{k=N+1}^{+\infty} E_k\right) \right\| \leqslant \lim_{N\to\infty} \sup_{n\in\mathbb{N}} \left\| v_n\left(\bigsqcup_{k=N+1}^{+\infty} E_k\right) \right\| = \lim_{N\to\infty} \left\| \bigoplus_{n=1}^{+\infty} v_n\left(\bigsqcup_{k=N+1}^{+\infty} E_k\right) \right\|$$
$$= \lim_{N\to\infty} \left\| v\left(\bigsqcup_{k=N+1}^{+\infty} E_k\right) \right\| = \lim_{N\to\infty} \left\| v\left(\bigsqcup_{k=1}^{+\infty} E_n\right) - v\left(\bigsqcup_{k=1}^{N} E_k\right) \right\| = 0.$$

Thus, the function v_n is $\mathcal{B}(\mathcal{H})$-valued measure for all $n \in \mathbb{N}$.

Theorem 3.4. *Let $\mu(\Omega) < +\infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{M}$-measurable functions and let $(A_t)_{t\in\Omega}$ be a family in $\mathcal{B}(\mathcal{H})$. If the set of functions $\Omega \ni t \mapsto \|A_t\| f_n(t) \in \mathbb{C}$, $n \in \mathbb{N}$, is uniformly integrable and $\left(\bigoplus_{n=1}^{+\infty} f_n(t)A_t\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\ell^2(\mathcal{H})))$, then the function $v : \mathfrak{M} \to \mathcal{B}(\ell^2(\mathcal{H}))$ in (33) is well-defined $\mathcal{B}(\ell^2(\mathcal{H}))$-valued measure on $\mathfrak{M}$.*

Proof. From $\left(\bigoplus_{n=1}^{+\infty} f_n(t)A_t\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathcal{B}(\ell^2(\mathcal{H})))$ it follows that the integral $\int_E f_n(t)A_t\, d\mu(t)$ exists in $\mathcal{B}(\mathcal{H})$ for every $n \in \mathbb{N}$ and $E \in \mathfrak{M}$. Let $E \in \mathfrak{M}$ be arbitrary and let $(E_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{M}$ with mutually disjoint members and $E = \bigsqcup_{n=1}^{+\infty} E_n$. We can apply the equality (32) and obtain

$$\left\| v\left(\bigsqcup_{k=1}^{+\infty} E_k\right) - v\left(\bigsqcup_{k=1}^{N} E_k\right) \right\| = \left\| \int_E \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t) - \int_{\bigsqcup_{k=1}^{N} E_k} \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t) \right\|$$
$$= \left\| \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t) \right\| = \left\| \bigoplus_{n=1}^{+\infty} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} f_n(t)A_t\, d\mu(t) \right\| \tag{34}$$
$$= \sup_{n\in\mathbb{N}} \left\| \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} f_n(t)A_t\, d\mu(t) \right\| \leqslant \sup_{n\in\mathbb{N}} \int_{\bigsqcup_{k=N+1}^{+\infty} E_k} |f_n(t)| \cdot \|A_t\|\, d\mu(t)$$

for all $N \in \mathbb{N}$. From $\mu(\Omega) < +\infty$ we have the equality $\lim_{N\to\infty} \mu\left(\bigsqcup_{k=N+1}^{+\infty} E_k\right) = 0$. The assumption of uniform integrability of the set of functions $\Omega \ni t \mapsto \|A_t\| \cdot f_n(t) \in \mathbb{C}$, $n \in \mathbb{N}$ implies the uniform integrability of the set of functions $\Omega \ni t \mapsto \|A_t\| \cdot |f_n(t)| \in \mathbb{R}$, $n \in \mathbb{N}$. Thus, the last supremum in (34) can be small enough if $N \to \infty$. This implies that the function v is a $\mathcal{B}(\ell^2(\mathcal{H}))$-valued measure. □

In the next theorem we consider sufficient conditions for the compactness of the Gelfand integrals of a weakly* integrable function $\Omega \ni t \mapsto \bigoplus_{n=1}^{+\infty} f_n(t)A_t \in \mathcal{B}(\ell^2(\mathcal{H}))$, where $(f_n)_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{M}$-measurable functions with some additional properties.

First we need [3, Theorem 2.2], which is given without proof in this paper. For the comfort of reading, we give our proof, which is based on the fact that an operator on the Hilbert space is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.

Lemma 3.5. *Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be the sequence of Hilbert spaces, $A_n \in \mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \|A_n\| < +\infty$. Then, we have $\bigoplus_{n=1}^{+\infty} A_n \in \mathfrak{C}_\infty\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ if and only if $A_n \in \mathfrak{C}_\infty(\mathcal{H}_n)$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \|A_n\| = 0$.*

Proof. ($\Rightarrow$:) Fix $k \in \mathbb{N}$. The sequence $\left(x_n^{(k)}\right)_{n\in\mathbb{N}}$ in $\mathcal{H}_k$ converges weakly to 0 iff the sequence $\left(y_n^{(k)}\right)_{n\in\mathbb{N}}$, where $y_n^{(k)} = \left(0, 0, \ldots, 0, x_n^{(k)}, 0, \ldots\right)$ (at the k-th position is $x_n^{(k)}$), converges weakly to $(0, 0, \ldots, 0, \ldots) \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$. From $\bigoplus_{n=1}^{+\infty} A_n \in \mathfrak{C}_\infty\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ we get $\lim_{n\to\infty} \left\|A_k x_n^{(k)}\right\| = \lim_{n\to\infty} \left\|\left(\bigoplus_{m=1}^{+\infty} A_m\right) y_n^{(k)}\right\| = 0$. Therefore, $A_k \in \mathfrak{C}_\infty(\mathcal{H}_k)$.

Now, we will prove $\lim_{n\to\infty} \|A_n\| = 0$. Assume that there exists $\varepsilon > 0$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\|A_{n_k}\| \geqslant 2\varepsilon$ for all $k \in \mathbb{N}$. There exists $x_{n_k} \in \mathcal{H}_{n_k}$, $\|x_{n_k}\| = 1$, such that $\|A_{n_k} x_{n_k}\| \geqslant \varepsilon$ for all $k \in \mathbb{N}$. If we form the vectors $v^{(k)} = (0, 0, \ldots, 0, x_{n_k}, 0, \ldots) \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$ (at the n_k-th position is x_{n_k}) we obtain that the sequence $\left(v^{(k)}\right)_{k\in\mathbb{N}}$ in $\bigoplus_{n=1}^{+\infty} \mathcal{H}_n$ converges weakly to $0 \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$. Indeed, if $y = (y_n)_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$, then $\lim_{n\to\infty} \|y_n\| = 0$ due to the $\sum_{n=1}^{+\infty} \|y_n\|^2 < +\infty$. Thus, we have $\left|\left\langle v^{(k)}, y\right\rangle\right| = |\langle x_{n_k}, y_{n_k}\rangle| \leqslant \|x_{n_k}\| \cdot \|y_{n_k}\| = \|y_{n_k}\| \to 0$ as $k \to \infty$. By the assumption $\bigoplus_{n=1}^{+\infty} A_n \in \mathfrak{C}_\infty\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$ we have that $\lim_{k\to\infty} \left\|\left(\bigoplus_{n=1}^{+\infty} A_n\right) v^{(k)}\right\| = 0$ i.e. $\lim\limits_{k\to\infty} \|A_{n_k} x_{n_k}\| = 0$. Contradiction with $\|A_{n_k} x_{n_k}\| \geqslant \varepsilon > 0$ for all $k \in \mathbb{N}$.

($\Leftarrow$:) Let $\left(a^{(n)}\right)_{n\in\mathbb{N}}$ be the sequence in $\bigoplus_{m=1}^{+\infty} \mathcal{H}_m$, where $a^{(n)} = \left(a_k^{(n)}\right)_{k\in\mathbb{N}}$, such that converges weakly to 0 and let $N \in \mathbb{N}$ be fixed. There exists $M > 0$ such that $\left\|a^{(n)}\right\| \leqslant M$ for all $n \in \mathbb{N}$. Then, we have $\left\|\left(\bigoplus_{k=1}^{+\infty} A_k\right) a^{(n)}\right\|^2 = \sum_{k=1}^{N} \left\|A_k a_k^{(n)}\right\|^2 + \sum_{k=N+1}^{+\infty} \left\|A_k a_k^{(n)}\right\|^2 \leqslant \sum_{k=1}^{N} \left\|A_k a_k^{(n)}\right\|^2 + M^2 \sup_{\mathbb{N}\ni k\geqslant N+1} \|A_k\|^2$. The sequence $\left(a_k^{(n)}\right)_{n\in\mathbb{N}}$ in $\mathcal{H}_k$ converges weakly to 0 for all $k \in \mathbb{N}$. From $A_k \in \mathfrak{C}_\infty(\mathcal{H}_k)$ for $k \in \{1, \ldots, N\}$ we get $\lim_{n\to\infty} \sum_{k=1}^{N} \left\|A_k a_k^{(n)}\right\|^2 = 0$. Thus, $\limsup_{n\to\infty} \left\|\left(\bigoplus_{k=1}^{+\infty} A_k\right) a^{(n)}\right\|^2 \leqslant M^2 \sup_{\mathbb{N}\ni k\geqslant N+1} \|A_k\|^2$ for all $N \in \mathbb{N}$. If $N \to \infty$ we get $\limsup_{n\to\infty} \left\|\left(\bigoplus_{k=1}^{+\infty} A_k\right) a^{(n)}\right\|^2 \leqslant M^2 \lim_{N\to\infty} \sup_{\mathbb{N}\ni k\geqslant N+1} \|A_k\|^2 = M^2 \limsup_{k\to\infty} \|A_k\|^2 = M^2 \lim_{k\to\infty} \|A_k\|^2 = 0$. So, we proved $\bigoplus_{n=1}^{+\infty} A_n \in \mathfrak{C}_\infty\left(\bigoplus_{n=1}^{+\infty} \mathcal{H}_n\right)$. □

Theorem 3.6. *Let $\mu(\Omega) < +\infty$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{M}$-measurable functions and $\lim\limits_{n\to\infty} f_n(t) = 0$ for μ-almost all $t \in \Omega$. Let $g : \Omega \to \mathbb{C}$ be $\mathfrak{M}$-measurable function such that $|f_n(t)| \leqslant g(t)$ for all $n \in \mathbb{N}$ and μ-almost all $t \in \Omega$. If the family $(A_t)_{t\in\Omega}$ is weakly* measurable and the function $\Omega \ni t \mapsto \|A_t\| \cdot g(t) \in \mathbb{C}$ belongs to $L^1(\Omega, \mathfrak{M}, \mu)$ and $A_t \in \mathfrak{C}_\infty(\mathcal{H})$ for μ-almost all $t \in \Omega$, then $\left(\bigoplus_{n=1}^{+\infty} f_n(t)A_t\right)_{t\in\Omega} \in L_G^1(\Omega, \mu, \mathcal{B}(\ell^2(\mathcal{H})))$ and $\int_\Omega \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t) \in \mathfrak{C}_\infty(\ell^2(\mathcal{H}))$.*

Proof. From the assumption of the theorem, for all $E \in \mathfrak{M}$ we have

$$\begin{aligned}
\left\|\int_E \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t)\right\| &= \sup_{\|(x_n)_{n\in\mathbb{N}}\|=\|(y_n)_{n\in\mathbb{N}}\|=1} \left|\int_E \sum_{n=1}^{+\infty} f_n(t)\langle A_t x_n, y_n\rangle\, d\mu(t)\right| \\
&\leqslant \sup_{\|(x_n)_{n\in\mathbb{N}}\|=\|(y_n)_{n\in\mathbb{N}}\|=1} \int_E \sum_{n=1}^{+\infty} g(t) \cdot \|A_t x_n\| \cdot \|y_n\|\, d\mu(t) \\
&\leqslant \sup_{\|(x_n)_{n\in\mathbb{N}}\|=\|(y_n)_{n\in\mathbb{N}}\|=1} \int_E g(t) \cdot \sqrt{\sum_{n=1}^{+\infty} \|A_t x_n\|^2} \cdot \sqrt{\sum_{n=1}^{+\infty} \|y_n\|^2}\, d\mu(t) \leqslant \int_E g(t) \cdot \|A_t\|\, d\mu(t).
\end{aligned}$$

The above calculations implies $\left(\bigoplus_{n=1}^{+\infty} f_n(t)A_t\right)_{t\in\Omega} \in L^1_G(\Omega, \mu, \mathfrak{B}(\ell^2(\mathcal{H})))$. Furthermore, we have $f_n(t)A_t \in \mathfrak{C}_\infty(\mathcal{H})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|f_n(t)A_t\| = \lim_{n\to\infty} |f_n(t)| \cdot \|A_t\| = 0$ for μ-almost all $t \in \Omega$. By Lemma 3.5 we get $\bigoplus_{n=1}^{+\infty} f_n(t)A_t \in \mathfrak{C}_\infty(\ell^2(\mathcal{H}))$ for μ-almost all $t \in \Omega$. The mapping $\mathfrak{M} \ni E \mapsto \int_E g(t) \cdot \|A_t\|\, d\mu(t) \in [0, +\infty)$ is a positive measure on $\mathfrak{M}$, absolutely continuous with respect to the measure μ. Thus, the conditions of [13, Theorem 2.3, (a)] are satisfied and we obtain $\int_\Omega \bigoplus_{n=1}^{+\infty} f_n(t)A_t\, d\mu(t) \in \mathfrak{C}_\infty(\ell^2(\mathcal{H}))$. □

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