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Generalized inverses in $\mathbb{Z}[x]/(vx + x^2)$

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Abstract. In this paper, we give a $*$-ring $\mathbb{Z}[x]/(vx + x^2)$, where $v \in \mathbb{Z}$ and $*$ is defined as $(a_1 + a_2x)^* = a_1 - va_2 - a_2x$, where $a_1, a_2 \in \mathbb{Z}$. Mainly, some classical generalized inverses are considered in this ring, such as regular inverses, group inverses, Moore-Penrose inverses and so on. Furthermore, it's proven that this ring is isomorphic to a special second-order matrix ring.

1. Introduction

Throughout this article, $\mathbb{Z}$ is the ring of integers, $\mathbb{Z}_+$ is the semi-ring of non-negative integers and R is a ring with identity 1. Let $*$ be an involution on R, that is the involution $*$ satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. We call R a $*$-ring if there exists an involution on R. Let R be a $*$-ring. An element $a \in R$ is said to be Hermitian if $a^* = a$, the set of all Hermitian elements of R is denoted by R^{Her} [1]. An element $e \in R$ satisfies $e^2 = e$, then e is called an idempotent element, the set of all idempotent elements of R is denoted by $E(R)$. If $e \in E(R) \cap R^{Her}$, then e is a projection, the set of all projections of R is denoted by R^{proj}. If each element $e \in E(R)$ satisfies $ea = ae$ for any $a \in R$, then R is called an Abel ring.

Recalling the following equations:

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax,\ (4)\ (xa)^* = xa,$$

$$(5)\ ax = xa,\ (6)\ a^k = a^{k+1}x, \text{ for some } k \geq 1,\ (7)\ xa^2 = a,\ (8)\ ax^2 = x.$$

An element $a \in R$ is regular if there exists $x \in R$ satisfying Eq.(1). In this case, x is called the regular inverse (inner inverse or 1-inverse) of a, and is denoted by $a^{(1)}$ (or a^-), the set of all regular inverses of a is denoted by $a\{1\}$, the set of all regular elements of R is denoted by R^{reg}. We use the symbol $U(R)$ to denote the set of all invertible elements of R. Clearly, $U(R) \subseteq R^{reg}$.

The Drazin inverse of $a \in R$ [2] is the element $x \in R$ which satisfies Eq.(2), (5), (6). The element x above is unique if it exists and is denoted by a^D. The least such k is called the index of a, and denoted by $ind(a)$. In particular, when $ind(a) = 1$, the Drazin inverse of a is called the group inverse of a and is denoted by $a^\#$. The set of all Drazin (resp. group) invertible elements of R is denoted by R^D (resp. $R^\#$).

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For any element a in a $*$-ring R is Moore-Penrose invertible, if there is an element x which is the unique solution to Eq.(1), (2), (3), (4). Such solution x is called the Moore-Penrose inverse of a and is denoted by $a^{\dagger}$, the set of all Moore-Penrose invertible elements of R is denoted by $R^{\dagger}$ [3]. An element $a \in R$ is EP if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$, the set of all EP elements of R is denoted by R^{EP} [1]. An element $a \in R^{EP}$ is SEP if $a^* = a^{\dagger}$, the set of all SEP elements of R is denoted by R^{SEP} [4]. An element a in a $*$-ring R is core invertible, if there exists $x \in R$ satisfying Eq.(1), (2), (3), (7), (8), the set of all core invertible elements of R is denoted by $R^{\textcircled{\#}}$ [5].

In [6], Cao et al. studied the generalized inverses in the quotient ring $\mathbb{Z}[x]/(x+x^2)$. Let R be a ring which is free as a $\mathbb{Z}$-module, a $\mathbb{Z}_+$-basis of R is a basis $B = \{b_i\}_{i\in I}$ such that $b_ib_j = \Sigma_{k\in I} c_{ij}^k b_k$, where $c_{ij}^k \in \mathbb{Z}_+$. A $\mathbb{Z}_+$-ring is a ring with a fixed $\mathbb{Z}_+$-basis and with identity 1 which is a non-negative linear combination of the basis elements [7]. $\mathbb{Z}_+$-ring has important significance in the study of representation theory in Hopf Algebras. In fact, this quotient ring $\mathbb{Z}[x]/(x+x^2)$ is one important example of $\mathbb{Z}_+$-rings. In this paper, we continue to study the problems of generalized inverses in quotient rings. In the following sections, the quotient ring $\mathbb{Z}[x]/(vx+x^2)$ is considered, where $v \in \mathbb{Z}$. Specifically, in section 2, we define $(a_1+a_2x)^* = a_1 - va_2 - a_2x$, where $a_1, a_2 \in \mathbb{Z}$, hence, this quotient ring is a $*$-ring. And then in $\mathbb{Z}[x]/(vx+x^2)$, some classical generalized inverses are considered in this ring, such as regular inverses, group inverses, Moore-Penrose inverses and so on. At the end of the paper, we construct a special second-order matrix ring, and find that $\mathbb{Z}[x]/(vx+x^2)$ is isomorphic to this ring. Similarly, we get $\mathbb{Z}[x]/(x^2)$ is isomorphic to another special second-order matrix ring.

2. Generalized inverses in $\mathbb{Z}[x]/(vx + x^2)$

Firstly, We define $*$ as $(a_1+a_2x)^* = a_1 - va_2 - a_2x$, where $a_1, a_2 \in \mathbb{Z}$ in the quotient ring $\mathbb{Z}[x]/(f(x))$, where $f(x) = vx + x^2$, $v \in \mathbb{Z}$. In what follows, it is proven that the quotient ring is a $*$-ring.

Proposition 2.1. *The quotient ring $\mathbb{Z}[x]/(vx+x^2)$ is a $*$-ring with $*$ is defined as $(a_1+a_2x)^* = a_1 - va_2 - a_2x$, where $v, a_1, a_2 \in \mathbb{Z}$.*

Proof. Let $a = a_1 + a_2x$, $b = b_1 + b_2x \in \mathbb{Z}[x]/(vx+x^2)$, where $a_i, b_i, v \in \mathbb{Z}, i = 1, 2$. Then, by the definition of $*$, $(a_1+a_2x)^* = a_1 - va_2 - a_2x$, it follows that

$$(a^*)^* = ((a_1+a_2x)^*)^* = (a_1 - va_2 - a_2x)^* = a_1 - va_2 + va_2 + a_2x = a.$$

Moreover, by a computation, we have

$$\begin{aligned}(a+b)^* &= ((a_1+b_1)+(a_2+b_2)x)^* = (a_1+b_1) - v(a_2+b_2) - (a_2+b_2)x \\ &= (a_1 - va_2 - a_2x) + (b_1 - vb_2 - b_2x) = a^* + b^*.\end{aligned}$$

At last, by $x^2 = -vx$, it is not difficult to check that

$$\begin{aligned}(ab)^* &= ((a_1+a_2x)(b_1+b_2x))^* = (a_1b_1 + a_1b_2x + a_2b_1x + a_2b_2x^2)^* \\ &= (a_1b_1 + a_1b_2x + a_2b_1x - va_2b_2x)^* \\ &= ((a_1b_1) + (a_1b_2 + a_2b_1 - va_2b_2)x)^* \\ &= a_1b_1 - v(a_1b_2 + a_2b_1 - va_2b_2) - (a_1b_2 + a_2b_1 - va_2b_2)x \\ &= (a_1 - va_2 - a_2x)(b_1 - vb_2 - b_2x) \\ &= (b_1 - vb_2 - b_2x)(a_1 - va_2 - a_2x) \\ &= b^*a^*.\end{aligned}$$

Hence, it is proven that the quotient ring $\mathbb{Z}[x]/(vx+x^2)$ is a $*$-ring, where $v \in \mathbb{Z}$. □

In the following, some classical generalized inverses in this quotient ring are considered, we firstly start with the Hermitian elements.

Proposition 2.2. $(\mathbb{Z}[x]/(vx+x^2))^{Her} = \mathbb{Z}$, *where $v \in \mathbb{Z}$.*

Proof. Let $a = a_1 + a_2x \in (\mathbb{Z}[x]/(vx + x^2))^{Her}$, where $a_1, a_2, v \in \mathbb{Z}$. As $a^* = a$, then we have $a^* = (a_1 + a_2x)^* = a_1 - va_2 - a_2x = a_1 + a_2x$. It follows that

$$\begin{cases} a_1 - va_2 = a_1 \\ -a_2 = a_2, \end{cases}$$

and thus $\begin{cases} a_1 = k \\ a_2 = 0, \end{cases}$ for any $k \in \mathbb{Z}$. It gives that $(\mathbb{Z}[x]/(vx + x^2))^{Her} = \mathbb{Z}$, where $v \in \mathbb{Z}$. □

Next, the idempotent elements in ∗-ring $\mathbb{Z}[x]/(vx + x^2)$ will be considered.

Proposition 2.3. $E(\mathbb{Z}[x]/(vx + x^2))=\begin{cases} \{0,1\}, & v \neq \pm 1 \\ \{0,1,-x,1+x,\}, & v = 1 \\ \{0,1,x,1-x\}, & v = -1. \end{cases}$

Proof. Let $a = a_1 + a_2x \in E(\mathbb{Z}[x]/(vx + x^2))$, where $a_1, a_2, v \in \mathbb{Z}$. Then, by $a^2 = a$, we have $a^2 = (a_1 + a_2x)^2 = a_1^2 + 2a_1a_2x + a_2^2x^2 = a_1^2 + 2a_1a_2x - va_2^2x = a_1 + a_2x$, which implies that $\begin{cases} a_1^2 = a_1 \\ 2a_1a_2 - va_2^2 = a_2. \end{cases}$

(1) When $a_1 = 0$, it gives $-va_2^2 = a_2$.

If $a_2 = 0$, we have $a = 0$.

If $a_2 \neq 0$, we have $-va_2 = 1$. As $a_2, v \in \mathbb{Z}$, it implies that $v = 1$, $a_2 = -1$ or $v = -1$, $a_2 = 1$. Thus, in this case, when $v = 1$, we get $a = -x$. When $v = -1$, we get $a = x$.

(2) When $a_1 \neq 0$, it gives that $a_1 = 1$ and $a_2 = va_2^2$.

If $a_2 = 0$, we get $a = 1$.

If $a_2 \neq 0$, we get $va_2 = 1$. As $a_2, v \in \mathbb{Z}$, it implies that $v = a_2 = 1$ or $v = a_2 = -1$. Thus, in this case, when $v = 1$, we have $a = 1 + x$. When $v = -1$, we have $a = 1 - x$.

Hence, we can obtain $E(\mathbb{Z}[x]/(vx + x^2))=\begin{cases} \{0,1\}, & v \neq \pm 1 \\ \{0,1,-x,1+x,\}, & v = 1 \\ \{0,1,x,1-x\}, & v = -1. \end{cases}$ □

From the definition of projections, we know that $(\mathbb{Z}[x]/(vx+x^2))^{proj} = (\mathbb{Z}[x]/(vx+x^2))^{Her} \cap E(\mathbb{Z}[x]/(vx+x^2))$, where $v \in \mathbb{Z}$. Immediately, we have the following corollary.

Corollary 2.4. $(\mathbb{Z}[x]/(vx + x^2))^{proj} = \{0, 1\}$, *where* $v \in \mathbb{Z}$.

From the above results, all Hermitian elements, idempotent elements, projections are found in $\mathbb{Z}[x]/(vx+x^2)$. In what follows, we consider other generalized inverses in $\mathbb{Z}[x]/(vx + x^2)$.

Lemma 2.5. $(\mathbb{Z}[x]/(vx + x^2))^{reg}=\begin{cases} \{0,-1,1\}, & v \neq 0,\pm 1,\pm 2 \\ \{0,-1,1,-x,x,-1-x,1+x,-1-2x,1+2x\}, & v = 1 \\ \{0,-1,1,-x,x,1-x,-1+x,1-2x,-1+2x\}, & v = -1 \\ \{0,-1+kx,1+kx\}, & v = 0 \\ \{0,-1,1,-1-x,1+x\}, & v = 2 \\ \{0,-1,1,-1+x,1-x\}, & v = -2, \end{cases}$ *where* $k \in \mathbb{Z}$.

Proof. Let $a = a_1 + a_2x \in (\mathbb{Z}[x]/(vx + x^2))^{reg}$, where $a_1, a_2 \in \mathbb{Z}$. Then there exists $b = b_1 + b_2x \in \mathbb{Z}[x]/(vx + x^2)$, where $b_1, b_2 \in \mathbb{Z}$, such that

$$\begin{aligned} aba = a^2b &= (a_1 + a_2x)^2(b_1 + b_2x) = (a_1^2 + 2a_1a_2x + a_2^2x^2)(b_1 + b_2x) \\ &= (a_1^2 + 2a_1a_2x - va_2^2x)(b_1 + b_2x) = (a_1^2 + (2a_1a_2 - va_2^2)x)(b_1 + b_2x) \\ &= a_1^2b_1 + a_1^2b_2x + 2a_1a_2b_1x - va_2^2b_1x + 2a_1a_2b_2x^2 - va_2^2b_2x^2 \\ &= a_1^2b_1 + a_1^2b_2x + 2a_1a_2b_1x - va_2^2b_1x - 2va_1a_2b_2x + v^2a_2^2b_2x \\ &= a_1^2b_1 + (a_1^2b_2 + 2a_1a_2b_1 - va_2^2b_1 - 2va_1a_2b_2 + v^2a_2^2b_2)x \\ &= a_1 + a_2x = a. \end{aligned}$$

This implies that $\begin{cases} a_1^2 b_1 = a_1 \\ a_1^2 b_2 + 2a_1 a_2 b_1 - v a_2^2 b_1 - 2v a_1 a_2 b_2 + v^2 a_2^2 b_2 = a_2. \end{cases}$ Immediately, we can obtain $\begin{cases} a_1(1 - a_1 b_1) = 0 \\ v a_2(v a_2 b_2 - 2a_1 b_2 - a_2 b_1) = a_2 \end{cases}$

Case I: When $a_1 = 0$, then we have $va_2(va_2b_2 - a_2b_1) = a_2$.

(1) If $a_2 = 0$, then it implies that $a = 0$ and $0\{1\} = \mathbb{Z}[x]/(vx + x^2)$.

(2) If $a_2 \neq 0$, then it gives $va_2(vb_2 - b_1) = 1$. So we can obtain $va_2 = vb_2 - b_1 = 1$ or $va_2 = vb_2 - b_1 = -1$.

(i) When $va_2 = vb_2 - b_1 = 1$, it follows that $v = a_2 = 1$ or $v = a_2 = -1$.

If $v = a_2 = 1$, then $b_2 - b_1 = 1$, we get $a = x$, $b = k + (1 + k)x$, where $k \in \mathbb{Z}$ and $x\{1\} = \{k + (1 + k)x | k \in \mathbb{Z}\}$.

If $v = a_2 = -1$, then it gives $-b_2 - b_1 = 1$, we can obtain $a = -x$ and $b = k - (1 + k)x$, where $k \in \mathbb{Z}$ and $-x\{1\} = \{k - (1 + k)x | k \in \mathbb{Z}\}$.

(ii) When $va_2 = vb_2 - b_1 = -1$, we can see that $v = 1, a_2 = -1$ or $v = -1, a_2 = 1$.

If $v = 1, a_2 = -1$, then $b_2 - b_1 = -1$, we get $a = -x$, $b = k + (k-1)x$, where $k \in \mathbb{Z}$ and $-x\{1\} = \{k + (k-1)x | k \in \mathbb{Z}\}$.

If $v = -1, a_2 = 1$, then $-b_2 - b_1 = -1$, we get $a = x$, $b = k + (1-k)x$, where $k \in \mathbb{Z}$ and $x\{1\} = \{k + (1-k)x | k \in \mathbb{Z}\}$.

Case II: When $a_1 \neq 0$, then $1 = a_1 b_1$. So $a_1 = b_1 = 1$ or $a_1 = b_1 = -1$.

(1) If $a_1 = b_1 = 1$, then $va_2(va_2b_2 - 2b_2 - a_2) = -b_2 - a_2$. We can get $b_2(va_2 - 1)^2 = a_2(va_2 - 1)$.

(i) When $va_2 - 1 = 0$, then $v = a_2 = 1$ or $v = a_2 = -1$.

If $v = a_2 = 1$, then $a = 1 + x$, $b = 1 + kx$, where $k \in \mathbb{Z}$, $1 + x\{1\} = \{1 + kx | k \in \mathbb{Z}\}$.

If $v = a_2 = -1$, then $a = 1 - x$, $b = 1 + kx$, where $k \in \mathbb{Z}$, $1 - x\{1\} = \{1 + kx | k \in \mathbb{Z}\}$.

(ii) When $va_2 - 1 \neq 0$, then $b_2(va_2 - 1) = a_2$.

If $v = 0$, then $-b_2 = a_2$. Hence, $a = 1 + kx$, $b = 1 - kx$, where $k \in \mathbb{Z}$, $1 + kx\{1\} = \{1 - kx | k \in \mathbb{Z}\}$.

If $v \neq 0$, then $(vb_2 - 1)(va_2 - 1) = 1$. So $vb_2 = va_2 = 2$ or $vb_2 = va_2 = 0$.

When $vb_2 = va_2 = 2$, then $v = 1, b_2 = a_2 = 2$ or $v = -1, b_2 = a_2 = -2$ or $v = 2, b_2 = a_2 = 1$ or $v = -2$, $b_2 = a_2 = -1$. If $v = 1, b_2 = a_2 = 2$, then $a = 1 + 2x$, $b = 1 + 2x$, $1 + 2x\{1\} = \{1 + 2x\}$. If $v = -1, b_2 = a_2 = -2$, then $a = 1 - 2x$, $b = 1 - 2x$, $1 - 2x\{1\} = \{1 - 2x\}$. If $v = 2, b_2 = a_2 = 1$, then $a = 1 + x$, $b = 1 + x$, $1 + x\{1\} = \{1 + x\}$. If $v = -2, b_2 = a_2 = -1$, then $a = 1 - x$, $b = 1 - x$, $1 - x\{1\} = \{1 - x\}$.

When $vb_2 = va_2 = 0$, according to the assumption $v \neq 0$, we can get $b_2 = a_2 = 0$, then it implies that $a = 1$, $b = 1$, $1\{1\} = 1$.

(2) If $a_1 = b_1 = -1$, then $va_2(va_2b_2 + 2b_2 + a_2) = -b_2 - a_2$. We can get $b_2(va_2 + 1)^2 = -a_2(va_2 + 1)$.

(i) When $va_2 + 1 = 0$, then $v = 1, a_2 = -1$ or $v = -1, a_2 = 1$.

If $v = 1, a_2 = -1$, then $a = -1 - x$, $b = -1 + kx$, where $k \in \mathbb{Z}$, $-1 - x\{1\} = \{-1 + kx | k \in \mathbb{Z}\}$.

If $v = -1, a_2 = 1$, then $a = -1 + x$, $b = -1 + kx$, where $k \in \mathbb{Z}$, $-1 + x\{1\} = \{-1 + kx | k \in \mathbb{Z}\}$.

(ii) When $va_2 + 1 \neq 0$, then $b_2(va_2 + 1) = -a_2$.

If $v = 0$, then $b_2 = -a_2$. Hence, $a = -1 + kx$, $b = -1 - kx$, where $k \in \mathbb{Z}$, $-1 + kx\{1\} = \{-1 - kx | k \in \mathbb{Z}\}$.

If $v \neq 0$, then $(vb_2 + 1)(va_2 + 1) = 1$. So $vb_2 = va_2 = -2$ or $vb_2 = va_2 = 0$.

When $vb_2 = va_2 = -2$, then $v = 1, b_2 = a_2 = -2$ or $v = -1, b_2 = a_2 = 2$ or $v = 2, b_2 = a_2 = -1$ or $v = -2$, $b_2 = a_2 = 1$. If $v = 1, b_2 = a_2 = -2$, then $a = -1 - 2x$, $b = -1 - 2x$, $-1 - 2x\{1\} = \{-1 - 2x\}$. If $v = -1, b_2 = a_2 = 2$, then $a = -1 + 2x$, $b = -1 + 2x$, $-1 + 2x\{1\} = \{-1 + 2x\}$. If $v = 2, b_2 = a_2 = -1$, then $a = -1 - x$, $b = -1 - x$, $-1 - x\{1\} = \{-1 - x\}$. If $v = -2, b_2 = a_2 = 1$, then $a = -1 + x$, $b = -1 + x$, $-1 + x\{1\} = \{-1 + x\}$.

When $vb_2 = va_2 = 0$, according to the assumption $v \neq 0$, we can get $b_2 = a_2 = 0$, then $a = -1$, $b = -1$, $-1\{1\} = -1$.

To sum up, we can get

$$(\mathbb{Z}[x]/(vx + x^2))^{reg} = \begin{cases} \{0, -1, 1\}, & v \neq 0, \pm 1, \pm 2 \\ \{0, -1, 1, -x, x, -1 - x, 1 + x, -1 - 2x, 1 + 2x\}, & v = 1 \\ \{0, -1, 1, -x, x, 1 - x, -1 + x, 1 - 2x, -1 + 2x\}, & v = -1 \\ \{0, -1 + kx, 1 + kx\}, & v = 0 \\ \{0, -1, 1, -1 - x, 1 + x\}, & v = 2 \\ \{0, -1, 1, -1 + x, 1 - x\}, & v = -2. \end{cases} \quad \square$$

We have found all regular elements in $\mathbb{Z}[x]/(vx + x^2)$. Obviously, $U(\mathbb{Z}[x]/(vx + x^2))$, $(\mathbb{Z}[x]/(vx + x^2))^{\#}$, $(\mathbb{Z}[x]/(vx + x^2))^{\dagger}$, $(\mathbb{Z}[x]/(vx + x^2))^{EP}$, $(\mathbb{Z}[x]/(vx + x^2))^{SEP}$ are all subsets of $(\mathbb{Z}[x]/(vx + x^2))^{reg}$. It is natural to consider the following theorem.

Theorem 2.6. $U(\mathbb{Z}[x]/(vx+x^2))=\begin{cases}\{-1,1\}, & v\neq 0,\pm1,\pm2\\ \{-1,1,-1-2x,1+2x\}, & v=1\\ \{-1,1,1-2x,-1+2x\}, & v=-1\\ \{-1+kx,1+kx\}, & v=0\\ \{-1,1,-1-x,1+x\}, & v=2\\ \{-1,1,-1+x,1-x\}, & v=-2,\end{cases}$ *where* $k\in\mathbb{Z}$.

Proof. It is easy to know that $U(\mathbb{Z}[x]/(vx+x^2))\subseteq(\mathbb{Z}[x]/(vx+x^2))^{reg}$, where $v\in\mathbb{Z}$. And it is clear that 0 is not invertible, $1^{-1}=1$ and $(-1)^{-1}=-1$.

(1) When $v\neq 0,\pm1,\pm2$, then $U(\mathbb{Z}[x]/(vx+x^2))=\{-1,1\}$.

(2) When $v=1$, $U(\mathbb{Z}[x]/(vx+x^2))=\{-1,1,1+2x,-1-2x\}$, and $(1+2x)^{-1}=1+2x$, $(-1-2x)^{-1}=-1-2x$ by [6, Corollary 3.5].

(3) When $v=-1$, we find that $(1-2x)^{-1}=1-2x$, $(-1+2x)^{-1}=-1+2x$. Now, we need to prove $-x,x,1-x,-1+x$ are not invertible. For any $a=a_1+a_2x\in\mathbb{Z}[x]/(vx+x^2)$, where $a_1,a_2\in\mathbb{Z}$. Then we have $x(a_1+a_2x)=a_1x+a_2x^2=(a_1+a_2)x\neq 1$ and $(-1+x)(a_1+a_2x)=-a_1-a_2x+a_1x+a_2x^2=-a_1+a_1x\neq 1$. Hence, $x,-1+x$ are not invertible. Similarly, we can prove $-x,1-x$ are not invertible. Therefore, when $v=-1$, we can obtain $U(\mathbb{Z}[x]/(vx+x^2))=\{-1,1,1-2x,-1+2x\}$.

(4) When $v=0$, we can get $(1+kx)^{-1}=1-kx$, $(-1+kx)^{-1}=-1-kx$. Hence, in this case, $U(\mathbb{Z}[x]/(vx+x^2))=\{-1+kx,1+kx\}$, where $k\in\mathbb{Z}$.

(5) When $v=2$, we obtain $(1+x)^{-1}=1+x$, $(-1-x)^{-1}=-1-x$. Hence, in this case, $U(\mathbb{Z}[x]/(vx+x^2))=\{-1,1,-1-x,1+x\}$.

(6) When $v=-2$, we find that $(-1+x)^{-1}=-1+x$, $(1-x)^{-1}=1-x$. Hence, in this case, $U(\mathbb{Z}[x]/(vx+x^2))=\{-1,1,-1+x,1-x\}$. □

In the following, two classical generalized inverses (group inverse and Moore-Penrose inverse), will be considered in the $*$-ring $\mathbb{Z}[x]/(vx+x^2)$. It should be noted here that the $*$-ring $\mathbb{Z}[x]/(vx+x^2)$ is commutative, then it is not difficult to see that $(\mathbb{Z}[x]/(vx+x^2))^{\#}=(\mathbb{Z}[x]/(vx+x^2))^{reg}$. However, for the convenience of the discussion concerning on EP elements, we will provide the following theorem.

Theorem 2.7. *The following hold:*

(1) $(\mathbb{Z}[x]/(vx+x^2))^{\#}=\begin{cases}\{0,-1,1\}, & v\neq 0,\pm1,\pm2\\ \{0,-1,1,-x,x,-1-x,1+x,-1-2x,1+2x\}, & v=1\\ \{0,-1,1,-x,x,1-x,-1+x,1-2x,-1+2x\}, & v=-1\\ \{0,-1+kx,1+kx\}, & v=0\\ \{0,-1,1,-1-x,1+x\}, & v=2\\ \{0,-1,1,-1+x,1-x\}, & v=-2,\end{cases}$ *where* $k\in\mathbb{Z}$.

(2) $(\mathbb{Z}[x]/(vx+x^2))^{\dagger}=\begin{cases}\{0,-1,1\}, & v\neq 0,\pm1,\pm2\\ \{0,-1,1,-1-2x,1+2x\}, & v=1\\ \{0,-1,1,1-2x,-1+2x\}, & v=-1\\ \{0,-1+kx,1+kx\}, & v=0\\ \{0,-1,1,-1-x,1+x\}, & v=2\\ \{0,-1,1,-1+x,1-x\}, & v=-2,\end{cases}$ *where* $k\in\mathbb{Z}$.

Proof. (1) It is clear that $\mathbb{Z}[x]/(vx+x^2)$ is commutative, so $(\mathbb{Z}[x]/(vx+x^2))^{\#}=(\mathbb{Z}[x]/(vx+x^2))^{reg}$, where $v\in\mathbb{Z}$. Next, we will provide all group inverses of every group invertible element. It is easy to check $0^{\#}=0$. Also, we have that for any $a\in U(\mathbb{Z}[x]/(vx+x^2))\subseteq(\mathbb{Z}[x]/(vx+x^2))^{\#}$ and $a^{\#}=a^{-1}$.

(i) When $v\neq 0,\pm1,\pm2$, then $(\mathbb{Z}[x]/(vx+x^2))^{\#}=\{0,-1,1\}$.

(ii) When $v=1$, from [6, Proposition 3.6], we can get $(\mathbb{Z}[x]/(vx+x^2))^{\#}=\{0,-1,1,-x,x,-1-x,1+x,1+2x,-1-2x\}$. Moreover, for every group invertible element, we have $x^{\#}=x$, $(-x)^{\#}=-x$, $(1+x)^{\#}=1+x$, $(-1-x)^{\#}=-1-x$, $(1+2x)^{\#}=1+2x$ and $(-1-2x)^{\#}=-1-2x$.

(iii) When $v=-1$, by the proof of Theorem 2.6, we know that $1-2x$ and $-1+2x$ are invertible, and it gives that $(1-2x)^{\#}=(1-2x)^{-1}=1-2x$, $(-1+2x)^{\#}=(-1+2x)^{-1}=-1+2x$. We only need to consider $(-x)^{\#},x^{\#},(1-x)^{\#}$ and $(-1+x)^{\#}$.

First, from Lemma 2.5, $x^\#$ has the form $k + (1 - k)x$, where $k \in \mathbb{Z}$. Then $(k + (1 - k)x)x(k + (1 - k)x) = (kx + x^2 - kx^2)(k + (1 - k)x) = x(k + (1 - k)x) = x$. Hence, $k = 0$, $x^\# = x$. Similarly, we can get $(-x)^\# = -x$.

Next, by Lemma 2.5, $(-1 + x)^\#$ has the form $-1 + kx$, where $k \in \mathbb{Z}$. Then $(-1 + kx)(-1 + x)(-1 + kx) = (1 - x)(-1 + kx) = -1 + kx + x - kx^2 = -1 + x$. Hence, $k = 1$, $(-1 + x)^\# = -1 + x$. Similarly, we can get $(1 - x)^\# = 1 - x$. Therefore, $(\mathbb{Z}[x]/(vx + x^2))^\# = \{0, -1, 1, -x, x, 1 - x, -1 + x, 1 - 2x, -1 + 2x\}$.

(iv) When $v = 0$, by the proof of Theorem 2.6, we know that $1+kx$ and $-1+kx$ are invertible, where $k \in \mathbb{Z}$, so $(1+kx)^\# = (1+kx)^{-1} = 1-kx$, $(-1+kx)^\# = (-1+kx)^{-1} = -1-kx$. Hence, $(\mathbb{Z}[x]/(vx+x^2))^\# = \{0, -1+kx, 1+kx\}$.

(v) When $v = 2$, by Theorem 2.6, we have $1+x$ and $-1-x$ are invertible, therefore $(1+x)^\# = (1+x)^{-1} = 1+x$, $(-1 - x)^\# = (-1 - x)^{-1} = -1 - x$. Hence, $(\mathbb{Z}[x]/(vx + x^2))^\# = \{0, -1, 1, -1 - x, 1 + x\}$.

(vi) When $v = -2$, by Theorem 2.6, we get that $-1 + x$ and $1 - x$ are invertible, so we get $(-1 + x)^\# = (-1 + x)^{-1} = -1 + x$, $(1 - x)^\# = (1 - x)^{-1} = 1 - x$. Hence, $(\mathbb{Z}[x]/(vx + x^2))^\# = \{0, -1, 1, -1 + x, 1 - x\}$.

(2) It is easy to check $0^\dagger = 0$. Also, we know that for any $a \in U(\mathbb{Z}[x]/(vx + x^2)) \subseteq (\mathbb{Z}[x]/(vx + x^2))^\dagger \subseteq (\mathbb{Z}[x]/(vx + x^2))^{reg}$ and $a^\dagger = a^{-1}$, where $v \in \mathbb{Z}$.

(i) When $v \neq 0, \pm 1, \pm 2$, then $(\mathbb{Z}[x]/(vx + x^2))^\dagger = \{0, -1, 1\}$.

(ii) When $v = 1$, from [6, Proposition 3.6], we can get $(\mathbb{Z}[x]/(vx + x^2))^\dagger = \{0, -1, 1, 1 + 2x, -1 - 2x\}$. $(1 + 2x)^\dagger = 1 + 2x$, $(-1 - 2x)^\dagger = -1 - 2x$.

(iii) When $v = -1$, it is easy to see $\{0, -1, 1, 1 - 2x, -1 + 2x\} \subseteq (\mathbb{Z}[x]/(vx + x^2))^\dagger$. In fact, $\mathbb{Z}[x]/(vx + x^2)$ is commutative and $-x, x, 1 - x, -1 + x$ are group invertible, we only need to consider equation (3), however,

$$(x^2)^* = x^* = 1 - x \neq x = x^2,$$

$$((1 - x)^2)^* = (1 - x)^* = x \neq 1 - x = (1 - x)^2.$$

Therefore, $(\mathbb{Z}[x]/(vx + x^2))^\dagger = \{0, -1, 1, 1 - 2x, -1 + 2x\}$.

(iv) When $v = 0$, by the proof of Theorem 2.6, we know that $1+kx$ and $-1+kx$ are invertible, where $k \in \mathbb{Z}$, so $(1 + kx)^\dagger = (1 + kx)^{-1} = 1 - kx$, $(-1 + kx)^\dagger = (-1 + kx)^{-1} = -1 - kx$. $(\mathbb{Z}[x]/(vx + x^2))^\dagger = \{0, -1 + kx, 1 + kx\}$.

(v) When $v = 2$, by Theorem 2.6, we get that $1 + x$ and $-1 - x$ are invertible, therefore $(1 + x)^\dagger = (1 + x)^{-1} = 1 + x$, $(-1 - x)^\dagger = (-1 - x)^{-1} = -1 - x$. $(\mathbb{Z}[x]/(vx + x^2))^\dagger = \{0, -1, 1, -1 - x, 1 + x\}$.

(vi) When $v = -2$, by Theorem 2.6, we know that $-1 + x$ and $1 - x$ are invertible, so we get $(-1 + x)^\dagger = (-1 + x)^{-1} = -1 + x$, $(1 - x)^\dagger = (1 - x)^{-1} = 1 - x$. $(\mathbb{Z}[x]/(vx + x^2))^\dagger = \{0, -1, 1, -1 + x, 1 - x\}$. □

Theorem 2.8. *The following hold:*

(1) $(\mathbb{Z}[x]/(vx + x^2))^{EP} = \begin{cases} \{0, -1, 1\}, & v \neq 0, \pm 1, \pm 2 \\ \{0, -1, 1, -1 - 2x, 1 + 2x\}, & v = 1 \\ \{0, -1, 1, 1 - 2x, -1 + 2x\}, & v = -1 \\ \{0, -1 + kx, 1 + kx\}, & v = 0 \\ \{0, -1, 1, -1 - x, 1 + x\}, & v = 2 \\ \{0, -1, 1, -1 + x, 1 - x\}, & v = -2, \end{cases}$ *where* $k \in \mathbb{Z}$.

(2) $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \begin{cases} \{0, -1, 1\}, & v \neq 0 \\ \{0, -1 + kx, 1 + kx\}, & v = 0, \end{cases}$ *where* $k \in \mathbb{Z}$.

Proof. (1) From Theorem 2.7, we only have to check the elements whose group inverse is equal to its Moore-Penrose inverse.

(2) (i) When $v \neq 0, \pm 1, \pm 2$, $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \{0, -1, 1\}$.

(ii) When $v = 1$, from [6, Proposition 3.7], we can get $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \{0, -1, 1\}$.

(iii) When $v = -1$, in fact, $(1-2x)^\dagger = 1-2x \neq -1+2x = (1-2x)^*$, $(-1+2x)^\dagger = -1+2x \neq 1-2x = (-1+2x)^*$. Hence, $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \{0, -1, 1\}$.

(iv) When $v = 0$, we can get $(-1 + kx)^\dagger = -1 - kx = (-1 + kx)^*$, $(1 + kx)^\dagger = 1 - kx = (1 + kx)^*$. Hence, $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \{0, -1 + kx, 1 + kx\}$, where $k \in \mathbb{Z}$.

(v) When $v = 2$, it is easy to see $(-1 - x)^\dagger = -1 - x \neq 1 + x = (-1 - x)^*$, $(1 + x)^\dagger = 1 + x \neq -1 - x = (1 + x)^*$. Hence, $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \{0, -1, 1\}$.

(vi) When $v = -2$, in fact, $(-1+x)^\dagger = -1+x \neq 1-x = (-1+x)^*$, $(1-x)^\dagger = 1-x \neq -1+x = (1-x)^*$. Hence, $(\mathbb{Z}[x]/(vx + x^2))^{SEP} = \{0, -1, 1\}$. □

It is clear that $\mathbb{Z}[x]/(vx+x^2)$ is commutative, where $v \in \mathbb{Z}$, so $(\mathbb{Z}[x]/(vx+x^2))^{\#} = (\mathbb{Z}[x]/(vx+x^2))^{EP}$. Immediately, we can give the following corollary.

Corollary 2.9. $(\mathbb{Z}[x]/(vx+x^2))^{\#} = \begin{cases} \{0,-1,1\}, & v \neq 0, \pm 1, \pm 2 \\ \{0,-1,1,-1-2x,1+2x\}, & v=1 \\ \{0,-1,1,1-2x,-1+2x\}, & v=-1 \\ \{0,-1+kx,1+kx\}, & v=0 \\ \{0,-1,1,-1-x,1+x\}, & v=2 \\ \{0,-1,1,-1+x,1-x\}, & v=-2, \end{cases}$ *where* $k \in \mathbb{Z}$.

3. Isomorphism of $\mathbb{Z}[x]/(vx+x^2)$

In this section, by constructing the second-order matrix ring, we find that $\mathbb{Z}[x]/(vx+x^2)$ is isomorphic to the special second-order matrix ring, where $v \in \mathbb{Z}$. Let $T_2^{(v)}(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{Z} \right\}$. For any $A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \in T_2^{(v)}(\mathbb{Z})$, we define addition and multiplication of $T_2^{(v)}(\mathbb{Z})$ as

$$A+B = \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ 0 & a_1+b_1 \end{pmatrix} \text{and } AB = \begin{pmatrix} a_1b_1 & a_1b_2+a_2b_1-va_2b_2 \\ 0 & a_1b_1 \end{pmatrix}.$$

It is not difficult to check that $T_2^{(v)}(\mathbb{Z})$ is a ring.

Proposition 3.1. *$T_2^{(v)}(\mathbb{Z})$ is a $*-$ ring, where $*$ is defined as* $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a-vb & -b \\ 0 & a-vb \end{pmatrix}$.

Proof. Assume that $A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \in T_2^{(v)}(\mathbb{Z})$, then by a computation, we can obtain

$$(A^*)^* = \begin{pmatrix} a_1-va_2 & -a_2 \\ 0 & a_1-va_2 \end{pmatrix}^* = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} = A.$$

Moreover, we can see that

$$\begin{aligned} (A+B)^* &= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ 0 & a_1+b_1 \end{pmatrix}^* = \begin{pmatrix} a_1+b_1-v(a_2+b_2) & -(a_2+b_2) \\ 0 & a_1+b_1-v(a_2+b_2) \end{pmatrix} \\ &= \begin{pmatrix} a_1-va_2 & -a_2 \\ 0 & a_1-va_2 \end{pmatrix}^* + \begin{pmatrix} b_1-vb_2 & -b_2 \\ 0 & b_1-vb_2 \end{pmatrix}^* = A^*+B^*. \end{aligned}$$

Further, by computations, we can obtain

$$\begin{aligned} (AB)^* &= \begin{pmatrix} a_1b_1 & a_1b_2+a_2b_1-va_2b_2 \\ 0 & a_1b_1 \end{pmatrix}^* \\ &= \begin{pmatrix} a_1b_1-v(a_1b_2+a_2b_1-va_2b_2) & -(a_1b_2+a_2b_1-va_2b_2) \\ 0 & a_1b_1-v(a_1b_2+a_2b_1-va_2b_2) \end{pmatrix} \\ &= \begin{pmatrix} b_1-vb_2 & -b_2 \\ 0 & b_1-vb_2 \end{pmatrix} \begin{pmatrix} a_1-va_2 & -a_2 \\ 0 & a_1-va_2 \end{pmatrix} \\ &= B^*A^*. \end{aligned}$$

Hence, $T_2^{(v)}(\mathbb{Z})$ is a $*-$ ring. □

Definition 3.2. *Let R_1 and R_2 be two involution rings. We say R_1 and R_2 are involution-isomorphic, if there exists a ring isomorphism f such that $f(a^*) = (f(a))^\star$.*

Theorem 3.3. *$\mathbb{Z}[x]/(vx+x^2)$ and $T_2^{(v)}(\mathbb{Z})$ are involution-isomorphic.*

Proof. We define the map f: $\mathbb{Z}[x]/(vx+x^2) \to T_2^{(v)}(\mathbb{Z})$ as this form:

$$a_1 + a_2x \mapsto \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}.$$

It is clear that f is bijective. Then for any $a = a_1 + a_2x$ and $b = b_1 + b_2x \in \mathbb{Z}[x]/(vx+x^2)$, $a_i, b_i \in \mathbb{Z}$, it is easy to find that $f(a_1 + a_2x + b_1 + b_2x) = f(a_1 + a_2x) + f(b_1 + b_2x)$. Moreover, we can check that

$$\begin{aligned} f((a_1 + a_2x)(b_1 + b_2x)) &= f(a_1b_1 + (a_1b_2 + a_2b_1)x + a_2b_2x^2) \\ &= f(a_1b_1 + (a_1b_2 + a_2b_1 - va_2b_2)x) \\ &= \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 - va_2b_2 \\ 0 & a_1b_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \\ &= f(a_1 + a_2x)f(b_1 + b_2x). \end{aligned}$$

Further, it can be found that $f((a_1 + a_2x)^*) = (f(a_1 + a_2x))^*$. Hence, $\mathbb{Z}[x]/(vx+x^2)$ and $T_2^{(v)}(\mathbb{Z})$ are involution-isomorphic. □

Especially, when $v = 0$, the following result can be concluded.

Corollary 3.4. *$\mathbb{Z}[x]/(x^2)$ and $T_2^{(0)}(\mathbb{Z})$ are involution-isomorphic.*

References

[1] D. Mosić, Generalized inverses. Faculty of Sciences and Mathematics, University of Niš, Niš (2018).
[2] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506–514.
[3] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406–413.
[4] R.J. Zhao, H. Yao, J. C. Wei, Characterizations of partial isometries and two special kinds of EP elements, Czecho. Math. J. 70 (2020) 539–551.
[5] D.S. Rakić, N.C. Dinčić, D.S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115–133.
[6] L. F. Cao, L. You, J. C. Wei, EP elements of $\mathbb{Z}[x]/(x+x^2)$. Filomat 37 (2023) 7467–7478.
[7] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories. Amercian Mathematical Society, Rhode Island (2015).