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Weighted (b, c)-core inverses in semigroups with involution

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Abstract. Let S be a $*$-moniod and let $a, b, c, v, w \in S$. In this paper, we define the (v, w)-weighted (b, c)-core inverse of a. The element a is called (v, w)-weighted (b, c)-core invertible if there exists $x \in S$ such that $xvcawx = x, xvS = bS$ and $Swx = Sc^*$. It is shown that the core inverse, the w-core inverse and the (b, c)-core inverse are special cases of the defined (v, w)-weighted (b, c)-core inverse. Several criteria for the (e, w)-weighted (b, c)-core inverse are given, where e is an invertible Hermitian element. For instance, it is proved that a is (e, w)-weighted (b, c)-core invertible if and only if there exists some $x \in bS$ such that $xecawb = b$, $cawxec = c$ and $(cawx)^* = cawx$ if and only if aw is (b, c)-invertible and c (ca or $cawb$) is $\{e, 1, 3\}$-invertible. The dual (v, w)-weighted (b, c)-core inverse of a is defined by the existence of $y \in S$ satisfying $yvabwy = y$, $yvS = b^*S$ and $Swy = Sc$. Dual results for the dual (v, w)-weighted (b, c)-core inverse are also established. Finally, when S is a unital $*$-ring, the (dual) weighted (b, c)-core inverse is characterized by the direct sum.

1. Introduction

In the last decade, two wider types of classical generalized inverses were introduced in a semigroup S, i.e., the inverse along an element [8] and the (b, c)-inverse [4], which encompass the Moore–Penrose inverse $a^\dagger$ and the Drazin inverse a^D (or the group inverse $a^\#$), see [6, 13] for details. Given any $a, b, c \in S$, the element a is called (b, c)-invertible if there exists some $y \in S$ such that $y \in bSy \cap ySc$, $yab = b$ and $cay = c$. Such an element y is called a (b, c)-inverse of a. It is unique if it exists, and is denoted by $a^{(b,c)}$. We denote by $S^{(b,c)}$ the set of all (b, c)-invertible elements in S. The (b, b)-inverse of a is known as the inverse of a along b, this two inverses coincides with each other. The inverse of a along b is unique if it exists, and is denoted by $a^{\|b}$. By $S^{\|b}$ we denote the set of all invertible elements along b in S.

In the extensive aspect of generalized inverses, lots of articles have been concerned with the weighted version of generalized inverses. As we know, the word "weighted" has been referred to an invertible Hermitian element, see weighted Moore–Penrose inverses [2], weighted Drazin inverses [3] and weighted core inverses [11].

Specially, the weighted Drazin inverse, makes that the definition of the Drazin inverse of a complex square matrix is extended to a rectangular matrix. Besides, in the context of weighted generalized inverses, the importance of the generalized weighted Moore-Penrose inverse is found from its inclusion of the weighted Moore-Penrose inverse, the Moore-Penrose inverse, and an ordinary matrix inverse.

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It is worth mentioning that several articles introduced the "extended weighted" version of generalized inverses in recent studies. For instance, the (v, w)-weighted (b, c)-inverse of tensors was defined in [12], the (v, w)-weighted (b, c)-inverse in semigroups was defined in [7]. Recently, Wu and Zhu [16] introduced the weighted w-core inverse with the weight v in the context of $*$-rings, therein, the weight v is an arbitrary ring element.

The extensive results on the weighted inverse and its wide expansion and application in various mathematical fields (see [9],[10]), which has prompted our investigation into the subject of the weighted inverse.

Let S be a $*$-monoid, that is a monoid S endowed with an involution $* : S \to S$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for every $x, y \in S$.

For any $a, b, c \in S$, the element a is called (b, c)-core invertible [17] if there exists an $x \in S$ such that $caxc = c$, $xS = bS$ and $Sx = Sc^*$. Such an x is called a (b, c)-core inverse of a. It is unique if it exists, and is denoted by $a^{\textcircled{\#}}_{(b,c)}$. As usual, we denote by $S^{\textcircled{\#}}_{(b,c)}$ the set of all (b, c)-core invertible elements in S. It was also shown that a is (b, c)-core invertible if and only if a is (b, c)-invertible and c is {1,3}-invertible if and only if ca is (b, c^*)-invertible.

For any $a, b, c, w \in S$, the (b, c)-core inverse of a extends its classical inverse a^{-1}, core inverse $a^{\textcircled{\#}}$, w-core inverse $a^{\textcircled{\#}}_w$ and Moore–Penrose inverse $a^\dagger$ by taking special b and c. Precisely, a is invertible if and only if it is $(1, 1)$-core invertible, moreover, $a^{-1} = a^{\textcircled{\#}}_{(1,1)}$; a is core invertible if and only if a is (a, a)-core invertible if and only if 1 is (a, a)-core invertible, moreover, $a^{\textcircled{\#}} = aa^{\textcircled{\#}}_{(a,a)} = 1^{\textcircled{\#}}_{(a,a)}$; a is w-core invertible if and only if w is (a, a)-core invertible, moreover, $a^{\textcircled{\#}}_w = w^{\textcircled{\#}}_{(a,a)}$; a is Moore-Penrose invertible if and only if a is (a^*, a^*)-core invertible if and only if a^* is (a, a)-core invertible, moreover, $a^\dagger = a^{\textcircled{\#}}_{(a^*,a^*)}a^* = a^*(a^*)^{\textcircled{\#}}_{(a,a)}$.

And it is known that the (b, c)-inverse and the w-core inverse are special cases of the (b, c)-core inverse. So it is natural to ask whether we could define a class of generalized inverses that unifies the weighted (b, c)-inverse and the weighted w-core inverse.

Inspired by [2, 3, 7, 16, 17], we aim to introduce the weighted version of the (b, c)-core inverse in a $*$-moniod S, i.e., the (v, w)-weighted (b, c)-core inverse, where $v, w \in S$ are arbitrary elements. It extended the notion of the (b, c)-core inverse, the weighted (b, c)-inverse and the weighted w-core inverse.

The paper is organized as follows. In Section 1, the notions of several generalized inverses and motivations are given. In Section 2, we introduce the weighted (b, c)-core inverse and establish the relation with the weighted (b, c)-inverse in S. It is proved in Theorem 2.5 that a is (v, w)-weighted (b, c)-core invertible if and only if ca is (v, w)-weighted (b, c^*)-invertible for any $a, b, c, v, w \in S$. In Section 3, for any $a, b, c, e, w \in S$, several criteria for the (e, w)-weighted (b, c)-core inverse are derived, where e is an invertible Hermitian element. For instance, we show in Theorem 3.2 that a is (e, w)-weighted (b, c)-core invertible if and only if there is some $x \in bS$ such that $xecawb = b$, $cawxec = c$ and $(cawx)^* = cawx$. Dual results for the dual (v, w)-weighted (b, c)-core inverse are derived. In Section 4, when S is a unital $*$-ring (usually denoted by R), the criterion for the (dual) (v, w)-weighted (b, c)-core inverse is given by the direct sum.

Throughout this paper, we suppose that S is a $*$-moniod, i.e., a monoid with an involution $*$. Let us now recall some notions of generalized inverses.

Following [7, Definition 2.2], given any $a, b, c, v, w \in S$, the element a is called (v, w)-weighted (b, c)-invertible if there exists some $y \in S$ such that $yvawy = y$, $yvS = bS$ and $Swy = Sc$. The (v, w)-weighted (b, c)-inverse of a is unique if it exists, and is denoted by $a^{(b,c)}_{(v,w)}$. As is known to all, the $(1, 1)$-weighted (b, c)-inverse is the (b, c)-inverse.

The present author Zhu, Wu and Chen [19] introduced the w-core inverse by making use of three equations in S. For any $a, w \in S$, the element a is called w-core invertible if there exists some $x \in S$ such that $awx^2 = x$, $xawa = a$ and $(awx)^* = awx$. Such an x is called a w-core inverse of a. It is unique if it exists, and is denoted by $a^{\textcircled{\#}}_w$. We denote by $S^{\textcircled{\#}}_w$ the set of all w-core invertible elements in S. It was shown in [19, Theorem 2.6] that $a \in S^{\textcircled{\#}}_w$ if and only if w is invertible along a and a is {1,3}-invertible. Moreover, $a^{\textcircled{\#}}_w = w^{\|a}a^{(1,3)}$.

An element $a \in S$ is called core invertible if it is 1-core invertible or a-core invertible. The standard notion of the core inverse of complex matrices and ring elements can be found in [1, 15]. By $S^{\textcircled{\#}}$ we denote the sets of all core invertible element in S. More results on the w-core inverse can be seen in [19, 20].

2. Definitions and basic properties

We begin this section with the notion of (v, w)-weighted (b, c)-core inverses in a $*$-monoid S.

Definition 2.1. *Let $a, b, c, v, w \in S$. We say that a is (v, w)-weighted (b, c)-core invertible if there exists an $x \in S$ such that* (i) $xvcawx = x$, (ii) $xvS = bS$ *and* (iii) $Swx = Sc^*$. *Such an x is called a (v, w)-weighted (b, c)-core inverse of a.*

It is clear that a is called $(1, 1)$-weighted (b, c)-core invertible if (1) $xcax = x$, (2) $xS = bS$ and (3) $Sx = Sc^*$ for some $x \in S$. From [17], one has that a is (b, c)-core invertible if there exists an $x \in S$ such that (1)′ $caxc = c$, (2) $xS = bS$ and (3) $Sx = Sc^*$.

The following result establishes the equivalence between the equation (1) $xcax = x$ and (1)′ $caxc = c$, under the conditions of (2) and (3).

Proposition 2.2. *Let $a, b, c, x \in S$. The following conditions are equivalent*:

(i) $xcax = x$, $xS = bS$ *and* $Sx = Sc^*$.

(ii) $caxc = c$, $xS = bS$ *and* $Sx = Sc^*$.

(iii) $xcab = b$, $xS = bS$ *and* $Sx = Sc^*$.

Proof. (i) ⇒ (ii) Given (i), then $c^* = tx = t(xcax) = (tx)cax = c^*cax$ for some $t \in S$, and so that $c = (cax)^*c$. This gives $cax = (cax)^*cax = (cax)^*$. Therefore, $c = caxc$, as required.

(ii) ⇒ (i) was proved in [17, Theorem 2.3] (vi) ⇒ (ii).

(i) ⇒ (iii) Since $xS = bS$, one has $b = xs$ for some $x \in S$, and hence $xcab = xcaxs = xs = b$.

(iii) ⇒ (i) As $xS = bS$, then there exists some $y \in S$ such that $x = by = xcaby = xcax$. □ □

We herein say that $(1, 1)$-weighted (b, c)-core invertible element is (b, c)-core invertible, and the (b, c)-core inverse of a is its $(1, 1)$-weighted (b, c)-core inverse. It can be concluded that every (b, c)-core invertible element is a special case of (v, w)-weighted (b, c)-core invertible elements. However, the converse may not be true. See the following example.

Example 2.3. Let $S = M_2(\mathbb{C})$ be the semigroup of all 2 by 2 complex matrices and let the involution $*$ be the transpose. Suppose $a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $b = c = \begin{bmatrix} i & 0 \\ 1 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in S$. Then a is (v, w)-weighted (b, c)-core invertible and $x = \begin{bmatrix} (i+2)/5 & (1-2i)/5 \\ (1-2i)/5 & -(i+2)/5 \end{bmatrix}$ is the (v, w)-weighted (b, c)-core inverse of a. However, $c \notin Sc^*c$, i.e., $c \notin S^{(1,3)}$, and hence $a \notin S^{\circledast}_{(b,c)}$.

As stated in Sect. 1, a is (b, c)-core invertible if and only if ca is (b, c^*)-invertible for any $a, b, c \in S$. A similar characterization between the (v, w)-weighted (b, c)-core inverse and the (v, w)-weighted (b, c)-inverse is given, in virtue of the following lemma.

Lemma 2.4. [7, Proposition 2.3] *Let $a, b, c, v, w \in S$. Then a is (v, w)-weighted (b, c)-invertible if and only if vaw is the (b, c)-invertible.*

Theorem 2.5. *Let $a, b, c, v, w \in S$. Then the following statements are equivalent*:

(i) *a is (v, w)-weighted (b, c)-core invertible.*

(ii) *ca is (v, w)-weighted (b, c^*)-invertible.*

(iii) *$vcaw$ is (b, c^*)-invertible.*

In this case, $(vcaw)^{(b,c^*)}$ is the (v, w)-weighted (b, c)-core inverse of a.

Proof. (i) ⇔ (ii) Suppose a is (v, w)-weighted (b, c)-core invertible. Then $xvcawx = x$, $xvS = bS$ and $Swx = Sc^*$ for some $x \in S$, so that ca is (v, w)-weighted (b, c^*)-invertible. Conversely, as ca is (v, w)-weighted (b, c^*)-invertible, then there is some $y \in S$ such that $yvcawy = y$, $yvS = bS$ and $Swy = Sc^*$ and consequently a is (v, w)-weighted (b, c)-core invertible.

(ii) ⇔ (iii) follows from Lemma 2.4.

One can directly check that $(vcaw)^{(b,c^*)}$ is the (v, w)-weighted (b, c)-core inverse of a. □ □

Taking $v = w = 1$, we have the following corollary.

Corollary 2.6. [17, Theorem 2.9] *Let $a, b, c \in S$. Then a is (b, c)-core invertible if and only if ca is (b, c^*)-invertible. In this case, $a^{\#}_{(b,c)} = (ca)^{(b,c^*)}$.*

Applying Theorem 2.5, we could get the following result, about the uniqueness of the (v, w)-weighted (b, c)-core inverse of a.

Theorem 2.7. *Let $a, b, c, v, w \in S$. If a is (v, w)-weighted (b, c)-core invertible, then it has a unique (v, w)-weighted (b, c)-core inverse.*

The (v, w)-weighted (b, c)-core inverse of a is denoted by $a^{\#}_{(v,w)(b,c)}$. We denote by $S^{\#}_{(v,w)(b,c)}$ the set of all (v, w)-weighted (b, c)-core invertible elements in S.

Again, applying Lemma 2.4 and Theorem 2.5 (i) $\Leftrightarrow$ (ii), the criterion for the $a \in S^{\#}_{(v,w)(b,c)}$ is given in the following result.

Lemma 2.8. [4, Theorem 2.2] *Let $a, b, c \in S$. Then $a \in S^{(b,c)}$ if and only if $b \in Scab$ and $c \in cabS$. In particular, if $b = vcab$ and $c = cabw$ for some $v, w \in S$, then $a^{(b,c)} = bw = vc$.*

Corollary 2.9. *Let $a, b, c, v, w \in S$. Then $a \in S^{\#}_{(v,w)(b,c)}$ if and only if $b \in Sc^*vcawb$ and $c^* \in c^*vcawbS$.*

We next give the concept of the dual (v, w)-weighted (b, c)-core inverse of a and its properties in S.

Definition 2.10. *Let $a, b, c, v, w \in S$. We say that a is dual (v, w)-weighted (b, c)-core invertible if there exists some $y \in S$ satisfying* (i) $yvabwy = y$, (ii) $yvS = b^*S$ *and* (iii) $Swy = Sc$. *Such a y is called a dual (v, w)-weighted (b, c)-core inverse of a.*

For any $a, b, c, v, w \in S$, it could be proved that the dual (v, w)-weighted (b, c)-core inverse of a is unique if it exists, and is denoted by $a_{(v,w)(b,c)^{\#}}$. We denote by $S_{(v,w)(b,c)^{\#}}$ the set of all dual (v, w)-weighted (b, c)-core invertible elements in S.

One observes that the dual $(1, 1)$-weighted (b, c)-core invertibility of a is equivalent to its dual (b, c)-core invertibility. The proof is dual to that of Proposition 2.2.

Theorem 2.11. *Let $a, b, c, v, w \in S$. Then the following statements are equivalent*:

(i) *a is dual (v, w)-weighted (b, c)-core invertible.*

(ii) *ab is (v, w)-weighted (b^*, c)-invertible.*

(iii) *$vabw$ is (b^*, c)-invertible.*

In this case, $a_{(v,w)(b,c)^{\#}} = (vabw)^{(b^*,c)}$.

3. Characterizations for weighted (b, c)-core inverses in a $*$-monoid

In this section, we assume that e and f are both invertible Hermitian elements. In what follows, we mainly investigate (e, w)-weighted (b, c)-core inverses and dual (v, f)-weighted (b, c)-core inverses in a $*$-monoid S.

Following [14], an element $a \in S$ is called weighted Moore–Penrose invertible with weights e, f (abbr. weighted Moore–Penrose invertible) if there exists an element $x \in S$ such that (1) $axa = a$, (2) $xax = x$, (3) $(eax)^* = eax$ and (4) $(fxa)^* = fxa$. Such an x is called a weighted Moore–Penrose inverse of a. It is unique if it exists, and is denoted by $a^{\dagger}_{e,f}$. More broadly, any $x \in S$ satisfying (1) and (3) is called an $\{e, 1, 3\}$-inverse of a, and is denoted by $a^{(1,3)}_e$, and any $x \in S$ satisfying (1) and (4) is called an $\{f, 1, 4\}$-inverse of a, and is denoted by $a^{(1,4)}_f$. The sets of all weighted Moore–Penrose invertible with weights e, f, $\{e, 1, 3\}$-invertible and $\{f, 1, 4\}$-invertible elements in S are denoted by $S^{\dagger}_{e,f}$, $S^{(1,3)}_e$, $S^{(1,4)}_f$, respectively.

We now present an auxiliary lemma, which plays an important role in the sequel.

Lemma 3.1. [18, Propositions 2.1 and 2.2] *Let $a, e, f \in S$. Then*

(i) *a is $\{e, 1, 3\}$-invertible if and only if $a \in Sa^*ea$. Moreover, if $a = xa^*ea$ for some $x \in S$, then x^*e is an $\{e, 1, 3\}$-inverse of a.*

(ii) *a is $\{f, 1, 4\}$-invertible if and only if $a \in af^{-1}a^*S$. Moreover, if $a = af^{-1}a^*y$ for some $y \in S$, then $f^{-1}y^*$ is an $\{f, 1, 4\}$-inverse of a.*

Several criteria for the (e, w)-weighted (b, c)-core inverse are established in Theorem 3.2, which shows that the (e, w)-weighted (b, c)-core inverse of a is characterized by the solution of the system of equations.

For any $a \in S$, following Drazin [5], the left annihilator of a is defined by ${}^0a = \{(p, q) \in S \times S : pa = qa\}$, and the right annihilator of a is defined by $a^0 = \{(r, s) \in S \times S : ar = as\}$. It was proved in [19] that $aS \subseteq bS$ implies ${}^0b \subseteq {}^0a$, and that $Sa \subseteq Sb$ implies $b^0 \subseteq a^0$.

Theorem 3.2. *Let $a, b, c, e, w \in S$. Then the following statements are equivalent*:

(i) $a \in S^{\oplus}_{(e,w)(b,c)}$.

(ii) *There exists some $x \in bS$ such that $xecawb = b$, $cawxec = c$ and $(cawx)^* = cawx$.*

(iii) *There exists some $x \in bS$ such that $xecawx = x$, ${}^0(xe) = {}^0b$ and $(wx)^0 = (c^*)^0$.*

(iv) *There exists some $x \in bS$ such that $xecawx = x$, ${}^0(xe) \subseteq {}^0b$ and $(wx)^0 \subseteq (c^*)^0$.*

(v) *There exists some $x \in S$ such that $xecawx = x$, $xS = bS$ and $Sc^* \subseteq Swx$.*

(vi) *There exists some $x \in S$ such that $xecawx = x$, $xS = bS$ and $(wx)^0 = (c^*)^0$.*

Proof. (i) ⇒ (ii) Suppose that $x \in S$ is the (e, w)-weighted (b, c)-core inverse of a. Then $xecawx = x$, $xeS = bS$ and $Swx = Sc^*$. Consequently, $x = xee^{-1} \in xeS = bS$. Since $Swx = Sc^*$, there exists $y \in S$ such that $c^* = ywx = yw(xecawx) = c^*ecawx$, so that $(cawx)^* = (awx)^*c^* = (awx)^*c^*ecawx = (cawx)^*ecawx$ and $cawx = (cawx)^*$, hence $c = cawxec$. Similarly, we get $b = xet = xecawxet = xecawb$ by $xeS = bS$.

(ii) ⇒ (iii) Since $x \in bS$ and $xecawb = b$, we have $xecawx = x$ and $xeS \subseteq xS \subseteq bS$ implies ${}^0b \subseteq {}^0(xe)$. Also, $xecawb = b$ implies $bS \subseteq (xe)S$, so that ${}^0(xe) \subseteq {}^0b$. We next show $(wx)^0 = (c^*)^0$. From $wx = wxecawx = wxe(cawx)^* = wxe(awx)^*c^*$, it follows that $Swx \subseteq Sc^*$ and $(c^*)^0 \subseteq (wx)^0$. On the other hand, $Sc^* = S(cawxec)^* = S(ec)^*cawx \subseteq Swx$ implies $(wx)^0 \subseteq (c^*)^0$, as required.

(iii) ⇒ (iv) is clear.

(iv) ⇒ (v) As $xecawx = x$, then $wxecawx = wx$ and hence $(1, ecawx) \in (wx)^0 \subseteq (c^*)^0$. We hence have $c^* = c^*ecawx$ and $Sc^* \subseteq Swx$. It next suffices to show that $xS = bS$. Since $x \in bS$, i.e., $xS \subseteq bS$, we only need to prove $bS \subseteq xS$. Note that $xecawxe = xe$, and hence $(1, xecaw) \in {}^0(xe) \subseteq {}^0b$. Then $bS = xecawbS \subseteq xS$.

(v) ⇒ (vi) It follows from $Sc^* \subseteq Swx$ that $(wx)^0 \subseteq (c^*)^0$. Note that $x = xecawx$ and $Sc^* \subseteq Swx$. Then we have $c^* = c^*ecawx$. We have at once $(cawx)^* = (cawx)^*ecawx$, so that $cawx = (cawx)^*$ and $Swx = Swxecawx = Swxe(cawx)^* = Swxe(awx)^*c^* \subseteq Sc^*$. Thus, $(c^*)^0 \subseteq (wx)^0$ and $(c^*)^0 = (wx)^0$.

(vi) ⇒ (i) Note that $xS = xee^{-1}S \in xeS \subseteq xS$. Then $xeS = xS = bS$. It follows from $wx = wxecawx$ that $(ecawx, 1) \in (wx)^0 = (c^*)^0$, so that $c^* = c^*ecawx$, and one has $Sc^* \subseteq Swx$. Once again, $c^* = c^*ecawx$ implies $cawx = (cawx)^*$. So, $Swx = Swxecawx = Swxe(cawx)^* \subseteq Sc^*$. Therefore, a is (e, w)-weighted (b, c)-core invertible. □ □

An element $a \in S$ is regular if there exists $x \in S$ such that $axa = a$. Such an x is called an inner inverse or a $\{1\}$-inverse of a, is denoted by a^-. For any $a, b \in S$ and $aS = bS$ (resp., $Sa = Sb$), if a is regular, then so is b.

An element $q \in S$ is called an idempotent if $q = q^2$. By Theorem 3.2, it follows that $cawxe$ and $xecaw$ are both idempotents.

We next consider to derive the criterion for the (v, w)-weighted (b, c)-core inverse by using the ideal generated by idempotents.

Theorem 3.3. *Let $a, b, c, e, w \in S$. Then the following statements are equivalent*:

(i) $a \in S^{\oplus}_{(e,w)(b,c)}$.

(ii) *There exist idempotents $p, q \in S$ such that $(ep)^* = ep$, $pS = cS = cawS$, $qS = bS$ and $Sq = Scaw$.*

In this case, $a^{\oplus}_{(e,w)(b,c)} = q(caw)^-pe^{-1}$ for any $(caw)^- \in (caw)\{1\}$.

Proof. (i) ⇒ (ii) Suppose that $x \in S$ is the (e,w)-weighted (b,c)-core inverse of a. Then $x \in bS$, $cawx = (cawx)^*$, $cawxec = c$ and $xecawb = b$. Let $p = cawxe$ and $q = xecaw$. Then $p^2 = p$, $q^2 = q$ and $(ep)^* = (ecawxe)^* = e^*(cawx)^*e^* = ecawxe = ep$. Also, we have $p = cawxe \in cS$, $c = cawxec = pc \in pS$ and $c = cawxec \in cawS$, so that $pS = cS = cawS$. From $q = xecaw$, it follows that $qS = xecawS \subseteq bS = xecawbS \subseteq qS$, i.e., $qS = bS$. Similarly, we have $Sq = Sxecaw \subseteq Scaw = Scawxecaw \subseteq Sxecaw = Sq$, as required.

(ii) ⇒ (i) Given $pS = cS = cawS$, we have caw is regular since p is idempotent (hence regular). Also, $Sq = Scaw$ implies $caw = cawq$. Let $x = q(caw)^-pe^{-1}$, where $(caw)^- \in (caw)\{1\}$. Then $x \in qS = bS$, $cawx = cawq(caw)^-pe^{-1} = caw(caw)^-pe^{-1} = (caw(caw)^-p)e^{-1} = pe^{-1} = e^{-1}epe^{-1} = (cawx)^*$ and $xecawb = q(caw)^-pcawb = q(caw)^-(pcaw)b = q(caw)^-(caw)b = qb = b$. We can similarly prove $cawxec = c$, and hence $a \in S^{\text{\textcircled{\#}}}_{(e,w)(b,c)}$ by Theorem 3.2 (ii) ⇒ (i). □ □

In the following result, we establish criteria for the (e,w)-weighted (b,c)-core inverse, extending the statement in [17, Theorem 2.7] for the case $e = w = 1$.

Theorem 3.4. *Let $a,b,c,e,w \in S$. Then the following statements are equivalent:*

(i) $a \in S^{\text{\textcircled{\#}}}_{(e,w)(b,c)}$.

(ii) $aw \in S^{(b,c)}$ *and* $c \in S^{(1,3)}_e$.

(iii) $aw \in S^{(b,c)}$ *and* $ca \in S^{(1,3)}_e$.

(iv) $aw \in S^{(b,c)}$ *and* $cawb \in S^{(1,3)}_e$.

In this case, $a^{\text{\textcircled{\#}}}_{(e,w)(b,c)} = (aw)^{(b,c)}c^{(1,3)}_e e^{-1} = (aw)^{(b,c)}aw(ca)^{(1,3)}_e e^{-1} = b(cawb)^{(1,3)}_e e^{-1}$.

Proof. (i) ⇒ (ii) Suppose that $x \in S$ is the (e,w)-weighted (b,c)-core inverse of a. By Theorem 3.2 (i) ⇒ (ii), we have $x \in bS$, $cawx = (cawx)^*$, $cawxec = c$ and $xecawb = b$. One has at once $b \in Scawb$ and $c \in cawbS$, so that $aw \in S^{(b,c)}$ from Lemma 2.8. Since $c = cawxec = (cawx)^*ec = (awx)^*c^*ec \in Sc^*ec$, we have $c \in S^{(1,3)}_e$ by Lemma 3.1.

(ii) ⇒ (iii) It next suffices to prove $ca \in S^{(1,3)}_e$. Since $aw \in S^{(b,c)}$, $c \in S^{(1,3)}_e$, we have $c \in cawbS$ and $c \in Sc^*ec$, and consequently $ca \in Sc^*eca \subseteq S(cawbS)^*eca \subseteq S(ca)^*eca$, i.e., $ca \in S^{(1,3)}_e$.

(iii) ⇒ (iv) Given $aw \in S^{(b,c)}$ and $ca \in S^{(1,3)}_e$, then $c \in cawbS$ and $ca \in S(ca)^*eca$. So, $cawb \in S(ca)^*ecawb \subseteq Sa^*(cawbS)^*ecawb \subseteq S(cawb)^*ecawb$, i.e., $cawb \in S^{(1,3)}_e$.

(iv) ⇒ (i) Let $x = b(cawb)^{(1,3)}_e e^{-1} \in bS$. Then we have

(1) $cawx = cawb(cawb)^{(1,3)}_e e^{-1} = e^{-1}e(cawb)(cawb)^{(1,3)}_e e^{-1} = (cawx)^*$.

(2) As $aw \in S^{(b,c)}$, then $b \in Scawb$ and hence $b = tcawb$ for some $t \in S$. Thus, $xecawb = b(cawb)^{(1,3)}_e e^{-1}ecawb = tcawb(cawb)^{(1,3)}_e e^{-1}ecawb = tcawb = b$.

(3) $cawxec = c$. Indeed, $cawxec = cawb(cawb)^{(1,3)}_e c = c$ since $c \in cawbS$.

We next give the other formula of $a^{\text{\textcircled{\#}}}_{(e,w)(b,c)}$. Let $x = (aw)^{(b,c)}c^{(1,3)}_e e^{-1}$. Then $x \in bS$ since $(aw)^{(b,c)} \in bS$. Hence, $cawx = caw(aw)^{(b,c)}c^{(1,3)}_e e^{-1} = cc^{(1,3)}_e e^{-1} = e^{-1}ecc^{(1,3)}_e e^{-1} = (cawx)^*$. Similarly, we can get $cawxec = c$ and $xecawb = b$.

Also, one could check that $(aw)^{(b,c)}aw(ca)^{(1,3)}_e e^{-1}$ is the (e,w)-weighted (b,c)-core inverse of a. □ □

Suppose $e = w = 1$ in Theorem 3.4. We have the following criteria for the (b,c)-core inverse.

Corollary 3.5. [17, Theorem 2.7] *Let $a,b,c \in S$. Then the following statements are equivalent:*

(i) $a \in S^{\text{\textcircled{\#}}}_{(b,c)}$.

(ii) $a \in S^{(b,c)}$ *and* $c \in S^{(1,3)}$.

(iii) $a \in S^{(b,c)}$ *and* $ca \in S^{(1,3)}$.

(iv) $a \in S^{(b,c)}$ *and* $cab \in S^{(1,3)}$.

In this case, $a^{\text{\textcircled{\#}}}_{(b,c)} = a^{(b,c)}c^{(1,3)} = a^{(b,c)}a(ca)^{(1,3)} = b(cab)^{(1,3)}$.

It follows from [19] that $a \in S^{\text{\textcircled{\#}}}$ if and only if $a \in S^{\#} \cap S^{(1,3)}$ if and only if a is $(1,1)$-core invertible if and only if a is (a,a)-core invertible, where $S^{\#}$ denotes the set of all group invertible elements in S.

Corollary 3.6. *Let $a \in S$. Then the following statements are equivalent*:
(i) $a \in S^{\text{\textcircled{\#}}}$.
(ii) $a \in S^{\text{\textcircled{\#}}}_{(a,a)}$.
(iii) $a \in S^{\#}$ *and* $a \in S^{(1,3)}$.
(iv) $a \in S^{\#}$ *and* $a^2 \in S^{(1,3)}$.
(v) $a \in S^{\#}$ *and* $a^3 \in S^{(1,3)}$.
In this case, $a^{\text{\textcircled{\#}}}_{(a,a)} = a^{\#}a^{(1,3)} = a^{\#}a(a^2)^{(1,3)} = a(a^3)^{(1,3)}$ *and* $a^{\text{\textcircled{\#}}} = aa^{\text{\textcircled{\#}}}_{(a,a)}$.

By comparing [17, Theorem 2.3] and Theorem 3.2. We easily obtain the following result.

Proposition 3.7. *Let $a, b, c, e, w \in S$. Then $a \in S^{\text{\textcircled{\#}}}_{(e,w)(b,c)}$ if and only if $aw \in S^{\text{\textcircled{\#}}}_{(b,ec)}$.*

Besides, if aw is (b, c)-core invertible rather than (b, ec)-core invertible, it is obvious that Proposition 3.7 may not be true. Next, we consider relationship between the (e, w)-weighted (b, c)-core invertibility of a and the (b, c)-core invertibility of aw.

Proposition 3.8. *Let $a, b, c, e, w \in S$. If $aw \in S^{\text{\textcircled{\#}}}_{(b,c)}$ and $cS \subseteq ecS$, then $a \in S^{\text{\textcircled{\#}}}_{(e,w)(b,c)}$.*

Proof. Given $aw \in S^{\text{\textcircled{\#}}}_{(b,c)}$, by Corollary 3.5, we can have $aw \in S^{(b,c)}$ and $c \in S^{(1,3)}$. As $cS \subseteq ecS$, there exists some $t \in S$ such that $c = ect$. So, $c \in Sc^*c \subseteq S(ect)^*c \subseteq Sc^*ec$. Therefore, $c \in S^{(1,3)}_e$. By Theorem 3.4, we have $a \in S^{\text{\textcircled{\#}}}_{(e,w)(b,c)}$. □ □

In the following result, several criteria for the dual (v, f)-weighted (b, c)-core inverse are given.

Theorem 3.9. *Let $a, b, c, v, f \in S$. Then the following statements are equivalent*:
(i) $a \in S_{(v,f)(b,c)^{\text{\textcircled{\#}}}}$.
(ii) *There exists some $y \in Sc$ such that $bfyvab = b$, $cvabfy = c$ and $(yvab)^* = yvab$.*
(iii) *There exists some $y \in Sc$ such that $yvabfy = y$, ${}^0(yv) = {}^0(b^*)$ and $(fy)^0 = c^0$.*
(iv) *There exists some $y \in Sc$ such that $yvabfy = y$, ${}^0(yv) \subseteq {}^0(b^*)$ and $(fy)^0 \subseteq c^0$.*
(v) *There exists some $y \in S$ such that $yvabfy = y$, $Sfy = Sc$ and $yvS \subseteq b^*S$.*
(vi) *There exists some $y \in S$ such that $yvabfy = y$, $Sfy = Sc$ and ${}^0(yv) = {}^0(b^*)$.*

Theorem 3.10. *Let $a, b, c, v, f \in S$. Then the following statements are equivalent*:
(i) $a \in S_{(v,f)(b,c)^{\text{\textcircled{\#}}}}$.
(ii) *There exist idempotents $p, q \in S$ such that $(qf)^* = qf$, $Sq = Sb = Svab$, $Sp = Sc$ and $pS = vabS$.*
In this case, $a_{(v,f)(b,c)^{\text{\textcircled{\#}}}} = f^{-1}q(vab)^-p$ for any $(vab)^- \in (vab)\{1\}$.

Theorem 3.11. *Let $a, b, c, v, f \in S$. Then the following statements are equivalent*:
(i) $a \in S_{(v,f)(b,c)^{\text{\textcircled{\#}}}}$.
(ii) $va \in S^{(b,c)}, b \in S^{(1,4)}_{f^{-1}}$.
(iii) $va \in S^{(b,c)}, ab \in S^{(1,4)}_{f^{-1}}$.
(iv) $va \in S^{(b,c)}, cafb \in S^{(1,4)}_{f^{-1}}$.
In this case, $a_{(v,f)(b,c)^{\text{\textcircled{\#}}}} = f^{-1}(va)^{(b,c)}b^{(1,4)}_{f^{-1}} = f^{-1}(va)^{(b,c)}va(ab)^{(1,4)}_{f^{-1}} = f^{-1}b(cafb)^{(1,4)}_{f^{-1}}$.

For any $a, b, c, e, f \in S$, as shown in Theorem 3.4, we know that a is (e, f)-weighted (b, c)-core invertible if and only if af is (b, c)-invertible and c (ca or $cafb$) is $\{e, 1, 3\}$-invertible.
Combining with Theorems 3.4 with 3.11, we obtain the following result.

Theorem 3.12. *Let $a, b, c, e, f \in S$. Then the following statements are equivalent*:
(i) $a \in S^{\text{\textcircled{\#}}}_{(e,f)(b,c)} \cap S_{(e,f)(b,c)^{\text{\textcircled{\#}}}}$.
(ii) $af, ea \in S^{(b,c)}$ *and* $cafb \in S^{\dagger}_{(e,f^{-1})}$.

In Theorem 3.12, take $e = f = 1$, one has the equivalence that a is both (b, c)-core and dual (b, c)-core invertible if and only if a is (b, c)-invertible and cab is $\{1, 3\}$-invertible in S, this equivalence was given in the context of $*$-ring in [17, Theorem 3.5].

The weighted w-core inverse [16] was introduced in a unital $*$-ring R. Given any $a, v, w \in R$, the element a is called weighted w-core invertible with the weight v if there exists some $x \in R$ such that $awxvx = x$, $xvawa = a$ and $(awx)^* = awx$. Such an x is called the weighted w-core inverse of a. The weighted w-core inverse of a with the weight v is unique if it exists, and is denoted by $a^{\circledast}_{v,w}$. There is only multiplication in the notion of weighted w-core inverses, and hence it is valid in a $*$-monoid S. The set of all weighted w-core invertible elements with the weight v in S is denoted by $S^{\circledast}_{v,w}$.

It was shown in a $*$-ring [16, Theorem 3.2] that a is w-weighted core invertible with the weight e if and only if w is invertible along a and a is $\{e, 1, 3\}$-invertible. Moreover, $a^{\circledast}_{e,w} = w^{\|a} a_e^{(1,3)} e^{-1}$. This result indeed holds in a $*$-monoid.

In [17, Theorem 2.18], the present author Zhu proved that $a \in S^{\circledast}_w$ if and only if $w \in S^{\circledast}_{(a,a)}$. The following result presents the weighted version of the aforementioned result.

Proposition 3.13. *Let $a, w, e, f \in S$. Then we have*:
(i) $a \in S^{\circledast}_{e,w}$ *if and only if* $w \in S^{\circledast}_{(e,1)(a,a)}$.
(ii) $a \in S^{\circledast}_{e,a^* f}$ *if and only if* $a^* \in S^{\circledast}_{(e,f)(a,a)}$.

Proof. (i) It follows from [16, Theorem 3.2] (i) ⇔ (vi) and Theorem 3.4 that $a \in S^{\circledast}_{e,w}$ if and only if $w \in S^{\|a}$ and $a \in S_e^{(1,3)}$ if and only if $w \in S^{(a,a)}$ and $a \in S_e^{(1,3)}$ if and only if $w \in S^{\circledast}_{(e,1)(a,a)}$.
(ii) can be proved by a similar proof of (i). □ □

4. (Dual) weighted (b, c)-core inverses in $*$-rings

Throughout this section, we assume that R is a unital $*$-ring, that is a ring with the unity 1 and an involution $* : R \to R$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$ for every $a, b \in R$. In what follows, we mainly investigate (dual) weighted (b, c)-core inverses in R.

Let $a \in R$. The right annihilator of a is denoted by $a^0 = \{x \in R : ax = 0\}$ and the left annihilator of a is denoted by ${}^0a = \{y \in R : ya = 0\}$.

A proposition is given, which extends the corresponding result in [17, Lemma 3.1].

Proposition 4.1. *Let $a, b, c, e, w \in R$. Then the following statements are equivalent*:
(i) $c \in cawbR \cap Rc^*ec$.
(ii) $c \in R(cawb)^*ec$.
(iii) $R = R(cawb)^* \oplus {}^0(ec)$.
(iv) $R = R(cawb)^* + {}^0(ec)$.
(v) $R = R(ecawb)^* \oplus {}^0c$.
(vi) $R = R(ecawb)^* + {}^0c$.

Proof. (i) ⇒ (ii) Given $c \in cawbR \cap Rc^*ec$, we have $c \in R(cawbR)^*ec \subseteq R(cawb)^*ec$.

(ii) ⇒ (iii) As $c \in R(cawb)^*ec$, then there exists $t \in R$ such that $c = t(cawb)^*ec = t(awb)^*c^*ec \in Rc^*ec$, so that $awbt^*e$ is an $\{e, 1, 3\}$-inverse of c by Lemma 3.1. Note that $ec = et(awb)^*c^*ec$. Then $1 - et(cawb)^* \in {}^0(ec)$. For any $r \in R$, we have $r = r(1 - et(cawb)^*) + ret(cawb)^* \in {}^0(ec) + R(cawb)^* \in R(cawb)^* + {}^0(ec)$. Hence, $R = R(cawb)^* + {}^0(ec)$. For any $d \in R(cawb)^* \cap {}^0(ec)$, there exists an element $x \in R$ such that $d = x(cawb)^* = x(cc_e{}^{(1,3)}cawb)^* = d(cc_e{}^{(1,3)})^* = decc_e{}^{(1,3)}e^{-1} = 0$ since $dec = 0$. So, $R = R(cawb)^* \oplus {}^0(ec)$.

(iii) ⇒ (iv) is trivial.

(iv) ⇒ (i) Given $R = R(cawb)^* + {}^0(ec)$, then $ec \in Rec \subseteq R(cawb)^*ec \subseteq Rc^*ec$ and thus $c \in Rc^*ec$. Once again, $ec \in Rec \subseteq R(cawb)^*ec$ implies that $ec = s(cawb)^*ec = s(awb)^*c^*ec$ for some $s \in R$, i.e., $awbs^*e$ is an $\{e, 1, 3\}$ of c. We have at once $c = cc_e{}^{(1,3)}c = cawbs^*ec \in cawbR$. So, $c \in cawbR \cap Rc^*ec$.

(i) ⇒ (v) ⇒ (vi) ⇒ (i) can be proved similarly. □ □

Dually, we get following result.

Proposition 4.2. *Let $a, b, c, v, f \in R$. Then the following statements are equivalent*:
(i) $b \in Rcvab \cap bf^{-1}b^*R$.
(ii) $b \in bf^{-1}(cvab)^*R$.
(iii) $R = (vcab)^*R \oplus (bf^{-1})^0$.
(iv) $R = (vcab)^*R + (bf^{-1})^0$.
(v) $R = (vcabf^{-1})^*R \oplus b^0$.
(vi) $R = (vcabf^{-1})^*R + b^0$.

Based on the proposition above, a list of criteria for the (e, w)-weighted (b, c)-core inverse are given by the direct sum.

Theorem 4.3. *Let $a, b, c, e, w \in R$. Then the following statements are equivalent*:
(i) $a \in R^{\#}_{(e,w)(b,c)}$.
(ii) $b \in Rcawb, c \in R(cawb)^*ec$.
(iii) $R = R(cawb)^* \oplus {}^0(ec) = Rcaw \oplus {}^0b$.
(iv) $R = R(cawb)^* + {}^0(ec) = Rcaw + {}^0b$.
(v) $R = R(ecawb)^* \oplus {}^0c = Rcaw \oplus {}^0b$.
(vi) $R = R(ecawb)^* + {}^0c = Rcaw + {}^0b$.
(vii) $R = R(cawb)^* \oplus {}^0(ec) = Rca \oplus {}^0(wb)$.
(viii) $R = R(cawb)^* + {}^0(ec) = Rca + {}^0(wb)$.
(ix) $R = R(ecawb)^* \oplus {}^0c = Rca \oplus {}^0(wb)$.
(x) $R = R(ecawb)^* + {}^0c = Rca + {}^0(wb)$.

Proof. (i) ⇒ (ii) It follows from Theorem 3.4 that $a \in R^{\#}_{(e,w)(b,c)}$ implies that $aw \in R^{(b,c)}$ and $c \in R_e{}^{(1,3)}$. So, we have $b \in Rcawb, c \in cawbR$ and $c \in Rc^*ec$. Hence, $c \in R(cawbR)^*ec \subseteq R(cawb)^*ec$, as required.

(ii) ⇒ (iii) By Proposition 4.1, we have the equivalence $c \in R(cawb)^*ec \Leftrightarrow R = R(cawb)^* \oplus {}^0(ec)$. As $b \in Rcawb$, then there exists an element $y \in R$ such that $b = ycawb$, consequently $1 - ycaw \in {}^0b$. Since $1 = (1 - ycaw) + ycaw$, we have $R = Rcaw + {}^0b$. From $c \in R(cawb)^*ec \subseteq Rc^*ec$, it follows that $c \in R_e^{(1,3)}$ and $awbs^*e$ is an $\{e, 1, 3, \}$-inverse of c. For any $z \in Rcaw \cap {}^0b$, then zb=0, and there is some $g \in R$ such that $z = gcaw = g(cawbs^*ec)aw = zbs^*ecaw = 0$, so that $R = Rcaw \oplus {}^0b$.

(iii) ⇒ (iv) is clear.

(iv) ⇒ (i) Given $R = R(cawb)^* + {}^0(ec)$, then $c \in R_e{}^{(1,3)}$ and $c \in cawbR$ by Proposition 4.1 (iv) ⇒ (i). Also, $R = Rcaw + {}^0b$ implies $b \in Rcawb$, so that $aw \in R^{(b,c)}$. It follows from Theorem 3.4 (ii) ⇒ (i) that $a \in R^{\#}_{(e,w)(b,c)}$.

(i) ⇒ (v) ⇒ (vi) ⇒ (i) and (i) ⇒ (vii) ⇒ (viii) ⇒ (ix) ⇒ (x) ⇒ (i) can be proved similarly. □ □

For any $a, b, c, v, f \in R$, several criteria for the dual (v, f)-weighted (b, c)-core inverse of a are given in the following result.

Theorem 4.4. *Let $a, b, c, v, f \in R$. Then the following statements are equivalent*:
(i) $a \in R_{(v,f)(b,c)^{\#}}$.
(ii) $c \in cvabR, b \in bf^{-1}(cvab)^*R$.
(iii) $R = (vcab)^*R \oplus (bf^{-1})^0 = vabR \oplus c^0$.
(iv) $R = (vcab)^*R + (bf^{-1})^0 = vabR + c^0$.
(v) $R = (vcabf^{-1})^*R \oplus b^0 = vabR \oplus c^0$.
(vi) $R = (vcabf^{-1})^*R + b^0 = vabR + c^0$.
(vii) $R = (vcab)^*R \oplus (bf^{-1})^0 = abR \oplus (cv)^0$.
(viii) $R = (vcab)^*R + (bf^{-1})^0 = abR + (cv)^0$.
(ix) $R = (vcabf^{-1})^*R \oplus b^0 = abR \oplus (cv)^0$.
(x) $R = (vcabf^{-1})^*R + b^0 = abR + (cv)^0$.

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