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Higher order quasi complex Helton class of Hilbert spaces operators

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Abstract. In this paper, we introduce a new class of operators related to a conjugation operator on a Hilbert space, we which called the class of n-quasi complex Helton class of operators with order m. We discuss the most interesting results concerning these classes of operators obtained form the idea of extant results for a number of classes of operators, amongst them Helton class of order m and quasi-Helton class of order m, n-quasi (m, C)-symmetric operators for some conjugation C of $\mathcal{H}$.

1. Introduction and Preliminaries

Throughout this paper, $\mathcal{H}$ stands for an infinite dimension complex Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. Let $A \in \mathcal{B}(\mathcal{H})$, and define

$$\beta_m(A) := \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} A^{*k}A^k. \tag{1}$$

An obvious consequence of (1) is that

$$\beta_{m+1}(A) = A^*\beta_m(A)A - \beta_m(A). \tag{2}$$

A Hilbert space operator $A \in \mathcal{B}(\mathcal{H})$ is said to be an m-isometry if A satisfies the following operator equation

$$\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} A^{*k}A^k = 0, \tag{3}$$

and it is said to be an n-quasi-m-isometry if

$$A^{*n}\beta_m(A)A^n = A^{*n}\Big(\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} A^{*k}A^k\Big)A^n = 0, \tag{4}$$

for some integers $m, n \geq 1$. A detailed study on these classes of Hilbert spaces operators has been done in [3–5, 7, 23, 28, 29] and other references. In [18], J.W. Helton introduced m-symmetric operators for the study

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of Jordan operators. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be n-symmetric or equivalently, an n-symmetry for some positive integer n if and only if A satisfies

$$\alpha_n(A) := \sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}A^{*k}A^{n-k} = 0. \tag{5}$$

Then $\alpha_{n+1}(A) = A^*\alpha_n(A) - \alpha_n(A)A$ holds. Using this, if A is n-symmetric, then A is $(n+k)$-symmetric for each integer $k \geq 1$ (see [33, Proposition 28]). The sum and product of n-symmetric operators was studied in [17] and [33].

Recall from [15] that a conjugation on $\mathcal{H}$ is a map $C : \mathcal{H} \longrightarrow \mathcal{H}$ which is antilinear, involutive ($C^2 = I_{\mathcal{H}}$). Moreover C satisfies the following properties

$$\begin{cases} \langle Cx \mid Cy\rangle = \langle y \mid x\rangle \quad \text{for } all\ x, y \in \mathcal{H}, \\ CAC \in \mathcal{B}(\mathcal{H}) \ \text{ for every } A \in \mathcal{B}(\mathcal{H}), \\ \Big(CAC\Big)^r = CA^rC \quad \text{ for all } r \in \mathbb{N}, \\ \Big(CAC\Big)^* = CA^*C. \end{cases}.$$

See [6] for properties of conjugation operators.

As extensions of the concepts of m-isometric operators on Hilbert spaces, some authors has introduced and study in different papers the following classes of operators.

(1) (m, C)-isometric operator that is an operator $A \in \mathcal{B}(\mathcal{H})$ satisfies

$$\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}A^{*m-k}CA^{m-k}C = 0, \tag{6}$$

for some $m \in \mathbb{N}$ and some conjugation C ([8–10, 27]).

(2) n-quasi-(m, C)-isometric operator that is an operator $A \in \mathcal{B}(\mathcal{Y})$ satisfies

$$A^{*n}\Big(\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}A^{*m-k}CA^{m-k}C\Big)A^n = 0, \tag{7}$$

for some conjugation C and some $n \in \mathbb{N}$ and $m \in \mathbb{N}$ ([22, 26, 27, 32]).

Let A and B be in $\mathcal{B}(\mathcal{H})$. In [19], the authors studied the operator $\Theta(A, B) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ defined by $\Theta(A, B)(X) = AX - XB$. Then

$$\Theta(A, B)^k(I) = \sum_{0\leq j\leq k}(-1)^{k-j}\binom{k}{j}A^jB^{k-j}. \tag{8}$$

In [19], the authors introduced the class of Helton operators as follows: an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be in the *mth* Helton class of B and write $A \in \mathcal{HEL}_m(B)$ if $\Theta(B, A)^m(I) = 0$. (See [20, 21]).

Recently, the author in [1] has introduced the class n-quasi-Helton class of order m as follows. An $B \in \mathcal{B}(\mathcal{H})$ for which there exists an operator A and there are some integers n and m such that

$$A^{*n}\Theta(B; A)^m(I)A^n = A^{*n}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jA^{m-j}\Big)A^n = 0,$$

they say that A belongs to n-quasi- Helton class of B with order m. This is symbolized by: $A \in [nQ \cap \mathcal{HEL}_m\Big(B\Big)$. Our aim in this paper is to consider a generalization of the concepts of Helton class of order m and quasi-Helton class of order m to the concepts of complex Helton class of order m and quasi- complex Helton class of order m. We discuss the most interesting results concerning these classes of operators obtained form the idea of extant results for a number of classes of operators, amongst them Helton class of order m, quasi-Helton class of order m, n-quasi (m, C)-symmetric operators (for some conjugation C of $\mathcal{H}$ ([2]). We invite the reader to refer to the references [11–14, 16, 24, 30, 32] for more details about other interesting classes of operators in this context.

2. Main results

In this section, we present the most important definitions for the classes of operators which we will study and moreover the most important properties

Definition 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$, we say that A belongs complex Helton class of B with order m if there exists a conjugation C such that*

$$\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jCA^{m-j}C = 0 \tag{9}$$

for some positive integer m.

For $B \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{HEL}_m\Big(B, C\Big)$ the set of complex Helton class of B of order m related to a conjugation operator C on $\mathcal{H}$.

Remark 2.2. *We make the following observations*

(1) $A \in \mathcal{HEL}_m\Big(B, C\Big) \Longleftrightarrow CAC \in \mathcal{HEL}_m\Big(B\Big)$.

(2) *If* $CA = AC$ *then* $A \in \mathcal{HEL}_m\Big(B, C\Big) \Longleftrightarrow A \in \mathcal{HEL}_m\Big(B\Big)$.

For $A, B \in \mathcal{B}(\mathcal{H})$ and a conjugation C on $\mathcal{H}$, we set

$$\Phi(B; A, C)^m(I) = \sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jCA^{m-j}C. \tag{10}$$

Lemma 2.3. *Let A and B be in $B(\mathcal{H})$, C be conjugation and k be a positive integer, then*

$$\Phi(B; A, C)^{k+1}(I) = B\Phi(B; A, C)^k(I) - \Phi(B; A, C)^k(I)\Big(CAC\Big). \tag{11}$$

Proof. We use an induction argument to show the statement. For $k = 1$ we have

$$\Phi\Big(B, A, C\Big)^2(I) = B^2 - 2BCAC + CA^2C$$

and

$$\begin{aligned} & B\Phi\Big(B, A, C\Big)(I) - \Phi\Big(A, B, C\Big)(I)(CAC) \\ = \; & B\Big(B - CAC\Big) - \Big(B - CAC\Big)CAC \\ = \; & B^2 - 2B(CAC) + CA^2C. \end{aligned}$$

– Hence equation (11) is true for $k = 1$. Assume that equation (11) is true for k. Then we prove that equation (11) holds for $k + 1$.

In fact, by equation (11) we obtain

$$\begin{aligned} & \Phi\Big(A,B,C\Big)^{k+2}(I) \\ = \;& \Phi\Big(A,B,C\Big)\Big(\Phi\Big(A,B,C\Big)^{k+1}(I)\Big) \\ = \;& B\Big(\Phi\Big(A,B,C\Big)^{k+1}(I)\Big) - \Big(\Phi\Big(A,B,C\Big)^{k+1}(I)\Big)(CAC). \end{aligned}$$

□

Remark 2.4. *The inclusion $\mathcal{HEL}_k(B,C) \subset \mathcal{HEL}_{k+1}(B,C)$ follows from equation (11).*

Remark 2.5. *In [20, Corollary 4.4] it was observed that if $A \in \mathcal{HEL}_m(B)$ then $A-\mu \in \mathcal{HEL}_m(B-\mu)$ for all $\mu \in \mathbb{C}$. In the following proposition we extend this result to $\mathcal{HEL}_m(B,C)$.*

Proposition 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and C be a conjugation operator on $\mathcal{H}$. If $A \in \mathcal{HEL}_m(B,C)$ then $A-\overline{\mu} \in \mathcal{HEL}_m(B-\mu,C)$ for all $\mu \in \mathbb{C}$.*

Proof. According to the statement (1) of Remark 2.2 and [20, Corollary 4.4] we have the following implications.

$$\begin{aligned} A \in \mathcal{HEL}_m(B,C) \quad &\Longrightarrow \quad CAC \in \mathcal{HEL}_m(B) \\ &\Longrightarrow \quad CAC-\mu \in \mathcal{HEL}_m(B-\mu) \\ &\Longrightarrow \quad C\Big(A-\overline{\mu}\Big)C \in \mathcal{HEL}_m(B-\mu) \\ &\Longrightarrow \quad A-\overline{\mu} \in \mathcal{HEL}_m(B-\mu,C). \end{aligned}$$

□

Definition 2.7. *For a positive integers n, m and $A, B \in \mathcal{B}(\mathcal{H})$, we say that A belongs to n-quasi complex Helton class of B with order m if there exists a conjugation C such that*

$$A^{*n}\Phi(B;A,C)^m(I)A^n = A^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n = 0$$

We denote this by $A \in [n\boldsymbol{Q} \cap \mathcal{HEL}_m(B,C)]$.

Remark 2.8. *If $A \in \mathcal{HEL}_m(B,C)$ then $A \in [n\boldsymbol{Q} \cap \mathcal{HEL}_m(B,C)]$.*

Remark 2.9. *The following example shows that there exist two operators A and B and a conjugation C such that $A \in [n\boldsymbol{Q} \cap \mathcal{HEL}_m(B,C)]$ but $A \notin \mathcal{HEL}_m(B,C)$ for some integers n and m. Thus, the set $\mathcal{HEL}_m(B,C)$ is a proper subset of $[n\boldsymbol{Q} \cap \mathcal{HEL}_m(B,C)]$ i.e,.*

$$[\mathcal{HEL}_m(B,C)] \subsetneq [n\boldsymbol{Q} \cap \mathcal{HEL}_m(B,C)].$$

Example 2.10. *Let $\mathcal{H} = \mathbb{C}^3$ and A and B the operators $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Define the conjugation operator $C(u,v,w) = (\overline{w},\overline{v},\overline{u})$ that is $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. A computation shows that*

$$B^2 - 2BCAC + CA^2C \neq 0, \quad A^{*2}\Big(B^2 - 2BCAC + CA^2C\Big)A^2 = 0.$$

Which implies that$A \in [2\boldsymbol{Q} \cap \mathcal{HEL}_2(B,C)]$ but $A \notin [\mathcal{HEL}_2(B,C)]$.

Remark 2.11. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$.*
(1) *Note that*

$$\begin{aligned} & A^{*n}\Big(\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}B^kCA^{m-k}C\Big)A^n \\ = \ & C\Big(CAC\Big)^{*n}\Big(\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\Big(CBC\Big)^kC\Big(CAC\Big)^{m-k}C\Big)\Big(CAC\Big)^nC. \end{aligned}$$

It is clear that $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ if and only if $CAC \in A \in [nQ \cap \mathcal{HEL}_m(CBC, C)]$.

(2) *If $CA = AC$, then $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ if and only if $A \in [nQ \cap \mathcal{HEL}_m(B)]$.*

Proposition 2.12. *Let A and B be in $\mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then the following statements holds.*

(1) *$A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ if and only if $U^*AU \in [nQ \cap \mathcal{HEL}_m\Big(U^*BU, U^*CU\Big)]$, for each unitary operator U.*

(2) *If $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ then $\lambda A \in [nQ \cap \mathcal{HEL}_m(\lambda B, C)]$, for all $\lambda \in \mathbb{C}$*

(3) *If $A_i \in [nQ\cap\mathcal{HEL}_m(B_i, C)]$ for $i = 1, 2$, then $A_1 \oplus A_2 \in [nQ \cap \mathcal{HEL}_m(B_1 \oplus B_2, C \oplus C)]$.*

Proof. (1) We observe that if C is a conjugation operator,then U^*CU is also a conjugation operator. Moreover, a direct calculation gives

$$\begin{aligned} & \Big(U^*AU\Big)^{*n}\Big(\sum_{0\leq k\leq m}(-)^{m-j}\binom{m}{j}\Big(U^*BU\Big)^j\Big(U^*CU\Big)\Big(U^*AU\Big)^{m-j}\Big(U^*CU\Big)\Big)\Big(U^*AU\Big)^n \\ = \ & U^*A^{*n}\Big(\sum_{0\leq k\leq m}(-)^{m-j}\binom{m}{j}B^jCA^{m-j}C\Big)A^nU^n. \end{aligned}$$

Therefore

$$A \in [nQ \cap \mathcal{HIL}_m(\mathcal{B}, C)] \Longleftrightarrow U^*AU \in [nQ \cap \mathcal{HEL}_m(U^*BU), U^*CU].$$

(2) Similarly, the statement (2) follows from the identity

$$\Big(\lambda A^*\Big)^n\Phi(\lambda B; \lambda A, C)^m(I)\Big(\lambda A\Big)^n = \overline{\lambda}^n|\lambda|^{2m}A^{*n}\Phi(B; A, C)^m(I)A^n.$$

(3)

$$\begin{aligned} & \Big(A_1 \oplus A_2\Big)^{*n}\Phi\Big(B_1 \oplus B_2; A_1 \oplus A_2, C \oplus C\Big)^m(I)\Big(A_1 \oplus A_2\Big)^n \\ = \ & {A_1^*}^n\Phi(B_1; A_1, C)^m(I)A_1^n \oplus {A_2^*}^n\Phi(B_2; A_2, C)^m(I)A_2^n \\ = \ & 0. \end{aligned}$$

□

Lemma 2.13. *Let A and B be in $\mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$ such that $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ for some integers $n, m \geqslant 1$. If $\mathcal{R}(A^n)$ is dense, then $A \in \mathcal{HEL}_m(B, C)]$.*

Proof. As $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$, we have

$$A^{*n}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n = 0$$

and therefore

$$\begin{aligned}&\Big\langle A^{*n}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n\omega \mid \omega\Big\rangle = 0\\ \Longrightarrow\ &\Big\langle \sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n\omega \mid A^n\omega\Big\rangle = 0\\ \Longrightarrow\ &\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}B^jCA^{m-j}C = 0 \text{ on } \overline{\mathcal{R}(A^n)} = \mathcal{H}.\end{aligned}$$

Therefore $A \in \mathcal{HEL}_m(B, C)$]. □

Proposition 2.14. *Let* $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$. *If* $\ker(A^*) = \ker(A^{*2})$ *then* $A \in [Q \cap \mathcal{HEL}_m(B, C)]$.

Proof. According to the assumption $\ker(A^*) = \ker(A^{*2})$ we deduce that $\ker(A^*) = \ker(A^{*p})$ for all positive integer p and may write

$$\begin{aligned}A \in [nQ \cap \mathcal{HEL}_m(B, C)] &\Longrightarrow A^{*n}\Phi(B, A, C)^m(I)A^n = 0\\ &\Longrightarrow A^*\Phi(B; A, C)^m(I)A^n = 0 \ (\textit{since} \ker(A^*) = \ker(A^{*n}))\\ &\Longrightarrow A^{*n}\Phi(B; A, C)^{*m}(I)A = 0\\ &\Longrightarrow A^*\Phi(B; A, C)^{*m}(I)A = 0 \ (\textit{since} \ker(A^*) = \ker(A^{*n}))\\ &\Longrightarrow A^*\Phi(B; A, C)^m(I)A = 0\\ &\Longrightarrow A \in [Q \cap \mathcal{HEL}_m(B, C)].\end{aligned}$$

□

Remark 2.15. *The inclusion* $[nQ \cap \mathcal{HEL}_m(B)] \subset [nQ \cap \mathcal{HEL}_{m+1}(B)]$ *does not holds in general with conjugation* C *as shown in the following example.*

Proposition 2.16. *Let* $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ *for some positive integers* n, m *and conjugation* C. *If* $[A^*, B] = [A, CAC] = 0$, *then* $A \in [nQ \cap \mathcal{HEL}_{m+1}(B, C)]$.

Proof. Since $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ for some conjugation C we have $A^{*n}\Phi(B; A, C)^m(I)A^n = 0$. Moreover according to Lemma 2.3

$$\Phi(B : A, C)^{m+1}(I) = B\Phi(B; A, C)^m(I) - \Phi(B; A, C)^m(I)\Big(CAC\Big)$$

and the conditions $[A^*, B] = [A, CAC] = 0$ it follows that

$$\begin{aligned}A^{*n}\Phi(B; A, C)^{m+1}(I)A^n &= A^{*n}\Big(B\Phi(B; A, C)^m(I) - \Phi(B; A, C)^m(I)\Big(CAC\Big)\Big)A^n\\ &= BA^{*n}\Phi(B; A, C)^m(I)A^n - A^{*n}\Phi(B; A, C)^m(I)A^n\Big(CAC\Big)\\ &= 0.\end{aligned}$$

Hence, $A^{*n}\Phi(B; A, C)^{m+1}(I)A^n = 0$ and therefore, $A \in [nQ \cap \mathcal{HEL}_{m+1}(B, C)]$. □

Lemma 2.17. *If* $A \in [nQ \cap \mathcal{HEL}_2(B, C)]$ *such that* $[A, CAC] = [A.B^*] = 0$. *Then*

$$\begin{aligned}&A^{*n}\Phi(B^{m+2}, A^{m+2}, C)^2(I)A^n\\ =\ &BA^{*n}\Phi(B^{m+1}, A^{m+1}, C)^2(I)A^n(CAC) - B^2A^{*n}\Phi(B^m, A^m, C)^2(I)A^n(CA^2C).\end{aligned}$$

Proof. Since $A \in [nQ \cap \mathcal{HEL}_2(B, C)]$ it follows that

$$A^{*n}\Big(B^2 - 2BCAC + CA^2C\Big)A^n = 0$$

and therefore

$$\begin{cases} A^{*n}B^2A^n = A^{*n}\Big(2BCAC - CA^2C\Big)A^n \\ \\ A^{*n}CA^2CA^n = A^{*n}\Big(2BCAC - B^2\Big)A^n. \end{cases}$$

Now for $m = 1$,we get

$$\begin{aligned} A^{*n}\Phi(B^3, A^3, C)^2(I)A^n &= A^{*n}\Big(B^6 - 2B^3CA^3C + CA^6C\Big)A^n \\ &= A^{*n}B^6A^n - 2A^{*n}B^3CA^3C + A^{*n}CA^6CA^n \\ &= B^4A^{*n}\Big(2BCAC - CA^2C\Big)A^n - 2B^3CA^3C \\ &\quad +A^{*n}\Big(2BCAC - B^2\Big)A^nCA^4C \\ &= 2BA^{*n}\Phi(B^2, A^2, C)^2(I)A^nCAC - B^2A^{*n}\Phi(B, A, C)^2(I)A^n(CA^2C). \end{aligned}$$

Assume that

$$\begin{aligned} &A^{*n}\Phi(B^{m+2}, A^{m+2}, C)^2(I)A^n \\ = \;& BA^{*n}\Phi(B^{m+1}, A^{m+1}, C)^2(I)A^n(CAC) - B^2A^{*n}\Phi(B^m, A^m, C)^2(I)A^n(CA^2C) \end{aligned}$$

holds for m. We want to show that

$$\begin{aligned} &A^{*n}\Phi(B^{m+3}, A^{m+3}, C)^2(I)A^n \\ = \;& BA^{*n}\Phi(B^{m+2}, A^{m+2}, C)^2(I)A^n(CAC) - B^2A^{*n}\Phi(B^{m+1}, A^{m+1}, C)^2(I)A^n(CA^2C). \end{aligned}$$

Indeed,

$$\begin{aligned} &A^{*n}\Phi(B^{m+3}, A^{m+3}, C)^2(I)A^n \\ = \;& A^{*n}\Big(B^{2m+6} - 2B^{m+3}CA^{m+3}C + CA^{2m+6}\Big)A^n \\ = \;& A^{*n}B^{2m+6}A^n - 2A^{*n}B^{m+3}CA^{m+3}CA^n + A^{*n}CA^{2m+6}CA^n \\ = \;& B^{2m+4}\underbrace{A^{*n}B^2A^n} - 2A^{*n}B^{m+3}CA^{m+3}CA^n + \underbrace{A^{*n}CA^2CA^n}CA^{2m+4}C \\ = \;& B^{2m+4}A^{*n}\Big(2BCAC - CA^2C\Big)A^n - 2A^{*n}B^{m+3}CA^{m+3}CA^n \\ & +A^{*n}\Big(2BCAC - B^2\Big)A^nCA^{2m+4}C \\ = \;& A^{*n}\Big(2B^{2m+5}CAC - B^{2m+4}CA^2C\Big)A^n - 2A^{*n}B^{m+3}CA^{m+3}CA^n \\ & +A^{*n}\Big(2BCA^{2m+5}C - B^2CA^{2m+4}C\Big)A^n \\ = \;& BA^{*n}\Phi(B^{m+2}, A^{m+2}, C)^2(I)A^nCAC - B^2\Phi(B^{m+1}, A^{m+1}, C)^2(I)A^n(CA^2C). \end{aligned}$$

□

Proposition 2.18. *If $A \in [nQ \cap \mathcal{HEL}_2(B, C)]$ such that $[A, CAC] = [A, B^*] = 0$. Then $A^2 \in [nQ \cap \mathcal{HEL}_2(B^2, C)]$.*

Proof. Since $A \in [nQ \cap \mathcal{HEL}_2(B,C)]$ it follows that

$$A^{*n}\Big(B^2 - 2BCAC + CA^2C\Big)A^n = 0$$

and therefore

$$\begin{cases} A^{*n}B^2A^n = A^{*n}\Big(2BCAC - CA^2C\Big)A^n \\ \\ A^{*n}CA^2CA^n = A^{*n}\Big(2BCAC - B^2\Big)A^n. \end{cases}.$$

On the other hand

$$\begin{aligned} & A^{*n}\Phi(B^2, A^2, C)^2(I)A^n \\ = \;& A^{*n}\Big(B^4 - 2B^2CA^2C + CA^4C\Big)A^n \\ = \;& A^{*n}B^4A^n - 2A^{*n}B^2CA^2CA^n + A^{*n}CA^4CA^n \\ = \;& B^2A^{*n}B^2A^n - 2A^{*n}B^2CA^2CA^n + A^{*n}CA^2CA^n(CA^2C) \\ = \;& B^2A^{*n}\Big(2BCAC - CA^2C\Big)A^n - 2A^{*n}B^2CA^2CA^n + A^{*n}\Big(2BCAC - B^2\Big)A^nCA^2C \\ = \;& A^{*n}\Big(2B^3CAC - B^2CA^2C - 2B^2CA^2C + 2BCA^3C - B^2CA^2C\Big)A^n \\ = \;& BA^{*n}\Big(2B^2 - BCAC - 2BCAC + CA^2C - BCAC\Big)A^nCAC \\ = \;& 2BA^{*n}\Phi(B; A, C)^2(I)A^nCAC \\ = \;& 0. \end{aligned}$$

Thus shows that $A^2 \in [nQ \cap \mathcal{HEL}_2(B^2, C)]$. □

Lemma 2.19. *Let $A \in [nQ \cap \mathcal{HEL}_2(B,C)]$ for some positive integer n and conjugation C. If $[A^*, B] = [A, CAC] = 0$ then $A^2 \in [nQ \cap \mathcal{HEL}_m(B^2, C)]$.*

Proof. According to $A \in [nQ \cap \mathcal{HEL}_m(B,C)]$, we get $A^{*n}\Phi(B; A, C)^2(I)A^n = 0$ and therefore,

$$B^q\Big(A^{*n}\Phi(B; A, C)^2(I)A^n\Big)\Big(CA^{m-q}C\Big) = 0 \;\; for\, q = 0, 1, \cdots, m.$$

From the assumption that $[A^*, B] = [A, CAC] = 0$ and Proposition 2.16 we obtain

$$A^{*n}B^q\Phi(B; A, C)^m(I)CA^{m-q}CA^n = 0 \;\; for\, any\, q = 0, 1, 2, \cdots, m.$$

Hence

$$A^{*2n}\Big(\sum_{0\le q\le m}(-1)^q\binom{m}{q}B^q\Phi(B; A, C)^m(I)CA^{m-q}C\Big)A^{2n} = 0$$

which means that

$$A^{*2n}\Big(\sum_{0\le q\le m}\binom{m}{q}B^q\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)CA^{m-q}C\Big)A^{2n} = 0$$

or

$$A^{*2n}\Big(\sum_{0\le q\le m}\sum_{0\le j\le m}(-1)^j\binom{m}{q}\binom{m}{j}B^{q+j}CA^{2m-(q+j)}C\Big)A^{2n} = 0$$

that is

$$A^{*2n}\Big(\sum_{0\le q\le 2m}\sum_{0\le j\le q}(-1)^j\binom{m}{q-j}\binom{m}{j}B^qCA^{2m-q}C\Big)A^{2n}=0.$$

According to the identity

$$\sum_{0\le j\le q}(-1)^j\binom{m}{q-j}\binom{m}{j}=\begin{cases}0, & if \quad q \text{ is odd}\\ (-1)^r\binom{m}{r}, & if \quad q=2r\end{cases},$$

we get

$$A^{*2n}\Big(\sum_{0\le q\le m}(-1)^q\binom{m}{r}B^{2q}CA^{2m-2q}C\Big)A^{2n}=0$$

which implies that $A^2\in[nQ\cap\mathcal{HEL}_m(B^2,C)]$. □

Theorem 2.20. *Let $A\in[nQ\cap\mathcal{HEL}_m(B,C)]$ for some positive integers n and $m\geqslant 2$. If $[A^*,B]=[A,CAC]=0$ then $A^q\in[nQ\cap\mathcal{HEL}_m(B^q,C)]$ for positive integer q.*

Proof. First case: assume that $A\in[nQ\cap\mathcal{HEL}_2(B,C)]$. We prove by induction that $A^q\in[nQ\cap\mathcal{HEL}_2(B^q,C)]$ for $q=2$, by using Proposition 2.18 we can deduce that $A^2\in[nQ\cap\mathcal{HIL}_2(B^2,C)]$.

Second case: assume that $A^q\in[nQ\cap\mathcal{HEL}_2(B^q,C)]$ for q and we need to prove that $A^{q+1}\in[nQ\cap\mathcal{HEL}_2(B^{q+1},C)]$. In view of Lemma 2.17 we get

$$\begin{aligned}&A^{*n}\Phi(B^{q+1};A^{q+1},C)^2(I)A^n\\=\;&A^{*n}\Big(2B\Phi(B^q;A^q,C)^2(I)A-B^2\Phi(B^{q-1};A^{q-1},C)^2(I)CA^2C\Big)A^n\\=\;&\underbrace{2BA^{*n}\Phi(B^q;A^q,C)^2(I)A^nA}_{=0}-\underbrace{B^2A^{*n}\Phi(B^{q-1};A^{q-1},C)^2(I)A^nCA^2C}_{=0}\\=\;&0.\end{aligned}$$

Therefore, $A^{q+1}\in[nQ\cap\mathcal{HEL}_2(B^{q+1},C)]$ and consequently, $A^q\in[nQ\cap\mathcal{HEL}_2(B^q,C)]$. Now assume that $A\in[nQ\cap\mathcal{HEL}_m(B,C)]$. We have that $A^q\in[nQ\cap\mathcal{HEL}_m(B^q,C)]$ and the result is true for $k=2$. Assume that $A^q\in[nQ\cap\mathcal{HEL}_m(B^q,C)]$. By taking into account Proposition 2.16, we have $[nQ\cap\mathcal{HEL}_m(B,C)]\subset[nQ\cap\mathcal{HEL}_{m+1}(B,C)]$ and we get $A^q\in[nQ\cap\mathcal{HEL}_m(B^q,C)]$. □

The following theorem shows that the n-quasi complex Helton class of an operators is closed in norm.

Theorem 2.21. *If $B\in\mathcal{B}(\mathcal{H})$ and C be a conjugation on H, then $[nQ\cap\mathcal{HEL}_m(B,C)]$ is closed in norm.*

Proof. Suppose that $(A_k)_k$ is a sequence of $[nQ\cap\mathcal{HEL}_m(B,C)]$ such that

$$\lim_{n\to\infty}\|A_k-A\|=0.$$

Since for every positive integer k, A_k is in $[nQ\cap\mathcal{HEL}_m(B,C)]$, we have

$$A_k^{*n}\Phi(B,A_k,C)A_k^n=0.$$

From which it follow that

$$
\begin{aligned}
& \|A_k^{*n}\Phi(B,A_k,C)A_m^n\| = \|A_k^{*n}\Phi(B,A_k,C)A_m^n - A_k^{*n}\Phi(B,A,C)A^n\| \\
= \;& \|A_k^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA_k^{m-j}C\Big)A_m^n - A^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n\| \\
\leqslant\;& \|A_k^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA_k^{m-j}C\Big)A_k^n - A_k^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n\| \\
+\;& \|A_k^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n - A^{*n}\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^jCA^{m-j}C\Big)A^n\| \\
\leqslant\;& \|A_k^{*n}\|\|\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^j\Big(CA_k^{m-j}CA_k^n - CA^{m-j}CA^n\Big)\| \\
+\;& \|(\sum_{0\le j\le m}(-1)^j\binom{m}{j}\Big(A_k^{*n}B^jCA^{m-j}C - A^{*n}B^jCA^{m-j}C\Big)A^n\| \\
\leqslant\;& \|A_k^{*n} - A^n\|\|\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^j\Big(CA_k^{m-j}CA_k^n - CA^{m-j}CA^n\Big)\| \\
+\;& \|A^n\|\|\sum_{0\le j\le m}(-1)^j\binom{m}{j}B^j\Big(CA_k^{m-j}CA_k^n - CA^{m-j}CA^n\Big)\| \\
+\;& \|\Big(\sum_{0\le j\le m}(-1)^j\binom{m}{j}\Big(A_k^{*n}B^jCA^{m-j}C - A^{*n}B^jCA^{m-j}C\Big)A^n\|.
\end{aligned}
$$

Since $A_k^{*n}\Phi(B,A_k,C)A_k^n = 0$ we get by taking $k\to\infty$ that $A^{*n}\Phi(B,A,C)A^n = 0$ and therefore,$A \in [nQ\cap \mathcal{HEL}_m(B,C)]$. □

Theorem 2.22. *If $A_i \in [nQ\cap\mathcal{HEL}_m(B_i,C_i)]$ for $i = 1,2$, then* $\begin{pmatrix} A_1 & W \\ 0 & A_2\end{pmatrix} \in \Big[nQ\cap\mathcal{HEL}_m\Big(\begin{pmatrix} B_1 & W \\ 0 & B_2\end{pmatrix},\begin{pmatrix} C_1 & 0 \\ 0 & C_2\end{pmatrix}\Big)\Big]$ *if and only if*

$$
\begin{cases}
A_1^{*n}\Phi(B_1,A_1,C_1)^m(I)F_n + A_1^{*n}G_mA_2^n = 0 \\
F_n^*\Phi(B_1,A_1,C_1)^m(I)A_1^n = 0 \\
F_n^*\Phi(B_1,A_1,C_1)^m(I)F + F^*G_mA_2^n = 0
\end{cases},
$$

for some operator $W \in \mathcal{B}(\mathcal{H})$, where $F_n = \sum\limits_{0\le i\le n-1} A_1^iWA_2^{n-1-i}$ and

$$
G_k = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\Big\{B_1^j\Big(\sum_{i=0}^{m-j-1}A_1^iWA_2^{m-j-1-i}\Big) + \Big(\sum_{i=0}^{j-1}B_1^iWB_2^{j-1-i}\Big)A_2^{m-j}\Big\}.
$$

Proof.

$$
\begin{pmatrix} A_1 & W \\ 0 & A_2\end{pmatrix} \in \Big[nQ\cap\mathcal{HEL}_m\Big(\begin{pmatrix} B_1 & W \\ 0 & B_2\end{pmatrix}\Big)\Big]
$$

if and only if

$$
\begin{pmatrix} A_1 & W \\ 0 & A_2\end{pmatrix}^{*n}\Bigg[\sum_{0\le j\le m}(-1)^j\binom{m}{j}\begin{pmatrix} B_1 & W \\ 0 & B_2\end{pmatrix}^j\begin{pmatrix} C_1 & 0 \\ 0 & C_2\end{pmatrix}\begin{pmatrix} A_1 & W \\ 0 & A_2\end{pmatrix}^{m-j}\begin{pmatrix} C_1 & 0 \\ 0 & C_2\end{pmatrix}\Bigg]\begin{pmatrix} A_1 & W \\ 0 & A_2\end{pmatrix}^n = 0.
$$

By observing that

$$\begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix}^p = \begin{pmatrix} A_1^p & \sum\limits_{0\le j\le p-1} A_1^j W A_2^{p-1-j} \\ 0 & A_2^p \end{pmatrix}$$

and

$$\begin{pmatrix} B_1 & W \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} B_1^p & \sum\limits_{0\le j\le p-1} B_1^j W B_2^{p-1-j} \\ 0 & B_2^p \end{pmatrix},$$

it follows that

$$\begin{aligned}
&\begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix}^{*n} \left\{ \left[\sum_{0\le j\le m} (-1)^j \binom{m}{j} \begin{pmatrix} B_1 & W \\ 0 & B_2 \end{pmatrix}^j \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix}^{m-j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \right] \right. \\
&\left. \right\} \begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix}^n \\
&= \begin{pmatrix} A_1^n & \sum\limits_{0\le i\le n-1} A_1^i W A_2^{n-1-i} \\ 0 & A_2^n \end{pmatrix}^* \left[\sum_{0\le j\le m} (-1)^j \binom{m}{j} \begin{pmatrix} B_1^j & \sum_{0\le i\le j-1} B_1^i W B_2^{j-1-i} \\ 0 & B_2^j \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \right] \\
&\quad \begin{pmatrix} A_1^{m-j} & \sum\limits_{0\le i\le m-j-1-i} A_1^i W A_2^{k-j-1-i} \\ 0 & A_2^{m-j} \end{pmatrix}^{m-j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} A_1^n & \sum\limits_{0\le i\le n-1} A_1^i W A_2^{n-1-i} \\ 0 & A_2^n \end{pmatrix} \\
&= \begin{pmatrix} A_1^{*n} & 0 \\ \left(\sum\limits_{0\le i\le n-1} A_1^i W B_2^{n-1-i} \right)^* & A_2^{*n} \end{pmatrix} \Bigg\{ \\
&\quad \sum_{0\le j\le m} (-1)^j \binom{m}{j} \begin{pmatrix} B_1^j C_1 A_1^{m-j} C_1 & B_1^j \left(\sum\limits_{0\le i\le m-j-1} A_1^i W A_2^{m-j-1-i} \right) + \left(\sum\limits_{0\le i\le j-1} B_1^i W B_2^{j-1-i} \right) A_2^{m-j} \\ 0 & B_2^j C_2 A_2^{m-j} C_1 \end{pmatrix} \Bigg\} \\
&\quad \begin{pmatrix} A_1^n & \sum_{0\le i\le n-1} A_1^i W A_2^{n-1-i} \\ 0 & A_2^n \end{pmatrix} \\
&= \begin{pmatrix} A_1^{*n} & 0 \\ \left(\sum\limits_{0\le i\le n-1} A_1^i W B_2^{n-1-i} \right)^* & A_2^{*n} \end{pmatrix} \\
&\quad \left[\begin{pmatrix} \Phi(B_1; A_1, C_1)^m(I) & \sum_{j=0}^m (-1)^j \binom{m}{j} \left[B_1^j \left(\sum\limits_{0\le i\le m-j-1} A_1^i W A_2^{m-j-1-i} \right) + \left(\sum\limits_{0\le i\le j-1} B_1^i W B_2^{j-1-i} \right) A_2^{m-j} \right] \\ 0 & \Phi(B_2; A_2, C_2)^m(I) \end{pmatrix} \right] \begin{pmatrix} A_1^n & \sum\limits_{i=0}^{n-1} A_1^i W A_2^{n-1-i} \\ 0 & A_2^n \end{pmatrix} \\
&= \begin{pmatrix} 0 & A_1^{*n} \Phi(B_1; A_1, C_1)^m(I) F_n + A_1^{*n} G_m A_2^n \\ F_n^* \Phi(B_1; A_1, C_1)^m(I) A_1^n & F_n^* \Phi(B_1; A_1, C_1)^m(I) F_n + F_n^* G_m A_2^n \end{pmatrix}.
\end{aligned}$$

□

Theorem 2.23. *Let $A \in [nQ \cap \mathcal{HEL}_m(B,C)]$ where $C = C_1 \oplus C_2 : \overline{\mathcal{R}(A^n)} \bigoplus \ker(A^{*n}) \longrightarrow \overline{R(A^n)} \bigoplus ker(A^{*n})$ be a conjugation operator If $[A,B^*] = 0$ then A, B^* have upper triangular representations*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad B^* = \begin{pmatrix} B_1^* & B_2^* \\ 0 & B_3^* \end{pmatrix}. \, on\, \mathcal{H} = \overline{\mathcal{R}(A^n)} \oplus \ker(A^{*n}).$$

where $A_1 \in \mathcal{HEL}_m(B_1,C_1)$, $A_3^n = 0$ and $[A_1, C_1B_1^*C_1] = 0$.

Proof. According to the decomposition $\mathcal{H} = \overline{\mathcal{R}(A_1^n)} \bigoplus \ker(A_1^{*n})$ and $AB^* = B^*A$ we get that A and B has the upper triangular representations

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad B^* = \begin{pmatrix} B_1^* & B_2^* \\ 0 & B_3^* \end{pmatrix}.$$

$[A,B^*] = 0 \Longrightarrow [A_1, B_1^*] = 0$ On the other hand

$$\begin{aligned} & A \in [nQ \cap \mathcal{HEL}_m(B,C)] = 0 \\ \Longrightarrow & \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{*n} \left(\sum_{0 \le j \le m} (-1)^j \binom{m}{j} \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}^j \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{m-j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \right) \\ & \begin{pmatrix} A_1^* & A_2^* \\ 0 & A_3^* \end{pmatrix}^{m-j} \\ = & \; 0 \\ \Longrightarrow & \begin{pmatrix} A_1^{*n} \left(\sum_{0 \le j \le m} (-1)^j \binom{m}{j} (B_1)^j C (A_1)^{m-j} C \right) A_1^n & \mathcal{X} \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix} = 0 \end{aligned}$$

for some operators $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$. Therefore

$$(A_1)^{*n} \left(\sum_{0 \le j \le m} (-1)^j \binom{m}{j} (B_1)^j C_1 (A_1)^{m-j} C_1 \right) (A_1)^n = 0$$

or equivalently

$$\sum_{0 \le j \le m} (-1)^j \binom{m}{j} B_1^j C_1 A_1^{m-j} C_1 = 0.$$

This means that $A_1 \in [nQ \cap \mathcal{HEL}_m(B_1, C_1)]$ which completed the proof. □

Theorem 2.24. *Let $A_k \in [nQ \cap \mathcal{HEL}_{m_k}(B_k, C)]$ for $k = 1,2$, and $C = C_1 \oplus C_2 : \overline{\mathcal{R}(A_k^n)} \bigoplus \ker(A_k^{*n}) \longrightarrow \overline{\mathcal{R}(A_k^n)} \bigoplus \ker(A_k^{*n})$ is a conjugation operator. If*

$$[A_1, A_2] = [B_1, B_2] = [A_1, B_1^*] = [A_2, B_2^*] = 0.$$

Then $A_1A_2 \in [nQ \cap \mathcal{HEL}_{m_1+m_2-1}\big(B_1B_2, C_1\big)]$.

Proof. From the assumptions we have the following upper triangular representations

$$A_k = \begin{pmatrix} A_{k1} & A_{k2} \\ 0 & A_{k3} \end{pmatrix} \quad B_k^* = \begin{pmatrix} B_{k1}^* & B_{k2}^* \\ 0 & B_{k3}^* \end{pmatrix}. k = 1,2$$

with the decomposition $\mathcal{H} = \overline{R(A_k^n)} \bigoplus ker(A_k^{*n}), k = 1,2.$. Since $A_1 \in [nQ \cap \mathcal{HEL}_{m_1}(B_1, C_1)]$ and $[A_1, B_1^*] = 0$, it follows by Theorem 2.23 that $A_{11} \in [\mathcal{HEL}_{m_1}(B_{11}, C_1)]$. Similarly, since $A_2 \in [nQ \cap \mathcal{HEL}_{m_2}(B_2, C_1)]$ and $[A_1, B_1^*] = 0$, it follows by Theorem 2.23 that $A_{21} \in [\mathcal{HEL}_{m_2}(B_{21}, C_1)]$. According to the statement (1) of Remark 2.2 we have $C_1A_{11}C_1 \in [\mathcal{HEL}_{m_1}(B_{11}]$ and $C_1A_{21}C_1 \in [\mathcal{HEL}_{m_2}(B_{21})]$ and by the assumption $[A_1, A_2] = 0$we have $[C_1A_1C_1, C_1A_2C_1] = 0$. Now, taking into account [11, Theorem 3.1] we deduce that $C_1A_{11}A_{21}C_1 \in [\mathcal{HEL}_{m_1+m_2-1}(B_{11}B_{21})]$ or equivalently $A_{11}A_{21} \in [\mathcal{HEL}_{m_1+m_2-1}(B_{11}B_{21}, C_1)]$ that is

$$\Phi(B_{11}B_{21}; A_{11}A_{21}, C_1)^{m_1+m_2-1}(I) = 0.$$

A simple computation shows that

$$\begin{aligned} & (A_1A_2)^{*n}\left(\sum_{0\leq j\leq m_1+m_2-1}(-1)^j\binom{m_1+m_2-1}{j}(B_1B_2)^jC_1(A_1A_2)^{m_1+m_2-1-j}(C_1)\right)(A_1A_2)^n \\ = & \begin{pmatrix} A_{11}^{*n}A_{21}^{*n} & Z \\ 0 & A_{13}^{*n}A_{23}^{*n} \end{pmatrix}\begin{pmatrix} \Phi(B_{11}B_{21}; A_{11}A_{21}, C_{11}C_{21})^{m_1+m_2-1}(I) & Q \\ 0 & W \end{pmatrix}\begin{pmatrix} A_{11}^nA_{21}^n & Y \\ 0 & 0 \end{pmatrix} \\ = & 0. \end{aligned}$$

Therefore, $A_1A_2 \in [nQ \cap \mathcal{HEL}_{m_1+m_2-1}(B_1B_2, C_1)]$ and the proof is complete. □

Lemma 2.25. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and if C is conjugation on $\mathcal{H}$. The following statements are true.*

(1) $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ *if and only if AotimesI* $\in [nQ \cap \mathcal{HEL}_m(B \otimes I, C \otimes C)]$.

(2) $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ *if and only if* $I \otimes A \in [nQ \cap \mathcal{HEL}_m(I \bigotimes B, C \otimes C)]$.

Proof. According to [10, Lemma 4.5] it follows that $C \otimes C$ is a conjugation operator on $\mathcal{H}\overline{\otimes}\mathcal{H}$. A direct calculation shows that

$$\begin{aligned} & (A \otimes I)^{*n}\Phi(B \otimes I; A \otimes I, C \otimes C)^m(I \otimes I)(A \otimes I)^n \\ = & (A \otimes I)^{*n}\left(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}(B \otimes I)^j(C \otimes C)(A \otimes I)^{m-j}(C \otimes C)\right)(A \otimes I)^n \\ = & A^{*n}\left(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}(B)^jC(A)^{m-j}C\right)A^n \otimes I \\ = & A^{*n}\Phi(B; A, C)^m(I)A^n \otimes I. \end{aligned}$$

□

Theorem 2.26. *Let $A_k \in [nQ \cap \mathcal{HEL}_{m_k}(B_k, C)]$ for $k = 1,2$, and*
$C = C_1 \oplus C_2 : \overline{\mathcal{R}(A_k^n)} \bigoplus \ker(A_k^{*n}) \longrightarrow \overline{\mathcal{R}(A_k^n)} \bigoplus \ker(A_k^{*n})$ *is a conjugation operator.*
If $[A_k, B_k^*] = 0\ \forall k = 1,2$ then $A_1 \otimes A_2 \in [nQ \cap \mathcal{HEL}_{m_1+m_2-1}(B_1 \otimes B_2, C_1 \otimes C_1)]$.

Proof. Let $T_1 = A_1 \otimes I$, $T_2 = I \otimes A_2$ and $S_1 = B_1 \otimes I$, $S_2 = I \otimes B_2$. Since

$$A_1 \otimes A_2 = (A_1 \otimes I)(I \otimes A_2)\, and\, B_1 \otimes B_2 = (B_1 \otimes I)(I \otimes B_2).$$

$$[A_1 \otimes I\; I \otimes A_2] = [B_1 \otimes I\; I \otimes B_2] = [A_1 \otimes I\; B_1^* \otimes I] = [I_1 \otimes A_2\; I \otimes B_2^*] = 0.$$

Then it follows from an application of Lemma 2.25 and Theorem 2.24 that $A_1 \otimes A_2 \in [n\mathcal{Q} \cap \mathcal{HEL}_{m_1+m_2-1}(B_1 \otimes B_2, C_1 \otimes C_1)]$. □

Definition 2.27. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and C is conjugation on $\mathcal{H}$. We say that A belongs to quasi strict Helton class of B with order m and conjugation C if A, B and C satisfy*

$$\begin{cases} A^{*n}\Phi(B, A, C)^m(I)A^n = 0 \\ \\ A^{*n}\Phi(B, A, C)^{m-1}(I)A^n \neq 0 \end{cases},$$

that is $A \in [n\mathcal{Q} \cap \mathcal{HEL}_m(B, C)]$ but $A \notin [n\mathcal{Q} \cap \mathcal{HEL}_{m-1}(B, C)]$.

Below we present an example that fulfills the above definition.

Example 2.28. *Let $H = \mathbb{C}^2$ and A and B the operators* $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. *Let C be a conjugation $C(u, v) = (\overline{v}, \overline{u})$ that is ;* $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
Then, a straightforward calculation shows that

$$\begin{cases} A^*\Phi(B, A, C)^3(I)A = 0 \\ \\ A^*\Phi(B, A, C)^2(I)A \neq 0 \end{cases}.$$

Hence, $A \in [\mathcal{Q} \cap \mathcal{HEL}_3(B, C)]$ and $A \notin [\mathcal{Q} \cap \mathcal{HEL}_2(B, C)]$. Therefore, A belongs to quasi strict Helton class of B with order 3 and conjugation C.

Theorem 2.29. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and C be a conjugation operator on $\mathcal{H}$. Assume that A belongs to quasi strict Helton class of B with order m and conjugation C. If $[A^*, B] = [A, CAC] = 0$ then the family of linear operators*

$$\left\{A^{*n}\Phi(B, A, C)^j(I)A^n,\ j = 0, 1, 2, ...m-1\right\}$$

is linearly independent.

Proof. According to Lemma 2.3 we have for all $j \geq 1$,

$$\Phi(B, A, C)^j = B\Phi(B, A, C)^{j-1}(I) - \Phi(B, A, C)^{j-1}(I)(CAC).$$

We my write

$$\begin{aligned} A^{*n}\Phi(B, A, C)^m(I)A^n &= A^{*n}\Big(B\Phi(B, A, C)^{m-1}(I) - \Phi(B, A, C)^{m-1}(I)(CAC)\Big)A^n \\ &= BA^{*n}\Phi(B, A, C)^{m-1}(I)A^n - A^{*n}\Phi(B, A, C)^{m-1}(I)A^n(CAC). \end{aligned}$$

Let $\gamma_j \in \mathbb{C}$ for $j = 1, \cdots, m-1$ such that

$$\sum_{0 \leq j \leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^j(I)A^n = 0. \tag{12}$$

Multiplying the equation (12) on the left by B and from right by CAC to gather with the condition $[A^*, B] = [A\ CAC] = 0$ we get

$$\sum_{0\leq j\leq m-1} \gamma_j A^{*n} B\Phi(B, A, C)^j(I)A^n = 0 \tag{13}$$

and

$$\sum_{0\leq j\leq m-1} \gamma_j A^{*n} \Phi(B, A, C)^j(I)A^n(CAC) = 0. \tag{14}$$

Subtracting two equations (13) and (14), we obtain

$$\sum_{0\leq j\leq m-1} \gamma_j A^{*n} \left(B\Phi(B, AC)^j(I) - \Phi(B, A, C)^j(I)CAC\right)A^n = \sum_{0\leq j\leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^{j+1}(I)A^n = 0. \tag{15}$$

The same procedure applied to equation (15) gives

$$\sum_{0\leq j\leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^{j+2}(I)A^n = 0$$

By continuing this process we obtain

$$\sum_{0\leq j\leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^{j+r}(I)A^n = 0\ for\ all\ r \in \mathbb{N}.$$

From Proposition 2.16, it is well known that if $A \in [nQ \cap \mathcal{HEL}_m(B, C)]$ then $A \in [nQ \cap \mathcal{HEL}_p(B, C)]$ for all $p \geqslant m$ and this means that the following implications hold.

For $r = m - 1$, $\sum_{0\leq j\leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^{j+m-1}(I)A^n = 0 \Longrightarrow \gamma_0 A^{*n}\Phi(B, A, C)^{m-1}(I)A^n = 0.$

So, $\gamma_0 = 0$ by the fact that $A^{*n}\Phi(B, A, C)^{m-1}(I)A^n \neq 0$.

For $r = m - 2$, $\sum_{0\leq j\leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^{j+m-2}(I)A^n = 0 \Longrightarrow \gamma_1 A^{*n}\Phi(B, A, C)^{m-1}(I)A^n = 0.$

So, $\gamma_1 = 0$.

Respiting this process for $r = m - 3, ..., r = 1$ and $r = 0$ we can found that all $\gamma_j = 0$ for $j = 2, ..., m - 1$.

Hence,

$$\sum_{0\leq j\leq m-1} \gamma_j A^{*n}\Phi(B, A, C)^j(I)A^n = 0 \Longrightarrow \gamma_0 = \gamma_1 = ... = \gamma_{m-1} = 0.$$

□

References

[1] A. Abeer AL-Dohiman, *Higher order quasi Hilton Class of Linear Transformations*.Jouf University Science and Engineering Journal (JUSEJ) (2021);**8(2)**: 01- 11.

[2] A. Abeer, O. A. Mahmoud Sid Ahmed and B. A. Farsin, *n-quasi-m-complex symmetric operators*, Symmetry (2023),**15**, 1662. https://doi.org/10.3390/ sym15091662.

[3] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space. I.* Integral Equations Operator Theory **21** (1995), no.**4**, 383-429.

[4] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space. II.* Integral Equations Operator Theory **23** (1995), no.**1**, 1-48.

[5] J. Agler, M. Stankus, *m-isometric transformations of Hilbert space. III.* Integral Equations Operator Theory **24** (1996), no.**4**, 379-421.

[6] P. Aiena , F. Burderi and S. Triolo, *Local Spectral Properties Under Conjugations*.Mediterr. J. Math. (2021) 18:89 https://doi.org/10.1007/s00009-021-01731-7

[7] T. Bermúdez, I. Marrero, A. Martinón, *On the orbit of an m-isometry*. Integral Equations Operator Theory **64** (2009), no. **4**, 487-494.

[8] M. Chō, E. Ko and J. E. Lee, On (m, C)-Isometric Operators, Complex Anal. Oper. Theory **10** (2016), 1679–1694.

[9] M. Chō, E. Ko and J. Lee, (∞, C)*-isometric operators*, Operators and Matrices, **11 3** (2017), 793-806.

[10] M. Chō, J. Lee and H. Motoyoshi, *On* $[m, C]$*-isometric operators*, Filomat **31:7** (2017), 2073-2080.

[11] B.P. Duggal, I.H. Kim, *Structure of elementary operators defining m-left invertible, m-selfadjoint and related classes of operators*, Journal of Mathematical Analysis and Applications. Volume **495**, Issue 1, 1 March (2021), 124718.

[12] B. P. Duggal and I. H. Kim, *Structure of n-quasi left-m-invertible and related classes of operators*, Demonstratio Mathematica 2020; **53**: 249–268.

[13] B. P. Duggal and V. Muller, *Tensor product of left n-invertible operators*, Studia Math. **215**(2013), no. **2**, 113–125.

[14] H. Ezzahraoui, On m-generalized invertible operators on Banach spaces. Ann. Funct. Anal. Volume **7**, Number **4** (2016), 609–621.

[15] SR. Garcia, Prodan E, Putinar M. Mathematical and physical aspects of complex symmetric operators. J Phys A. 2014;47:353001. (54pp).

[16] C. Gu, *Structures of left n-invertible operators and their applications*, Stud Math. (2015);**226(3)**: 189–211.

[17] C. Gu and M. Stankus, *Some results on higher order isometries and symmetries: Products and sums with a nilpotent operator*, Linear Algebra and its Applications, **469** (2015), 500-509

[18] J.W. Helton, *Infinite dimensional Jordan operators and Strum-Liouville conjugate point theory*, Trans. Amer. Math. Soc., **170** (1972), 305-331.

[19] Y. Kim, E. Ko and J.E. Lee, *On the Helton class of p-hyponormal operators*, Proc. Amer. Math. Soc., **135** (2007), 2113-2120.

[20] I. Kim, Y. Kim, E. Ko, and J. E. Lee, *Inherited proprieties through the Hilton class of an operator*, Bull. Korean Math. Soc. **48** (2011), No. **1**, pp. 183–195.

[21] Ji Eun Lee, The Helton class of operators and rank one perturbations of the unilateral shift, Ph. D. Thesis, Ewha Womans University, 2008.

[22] H. Lia, Y. Wanga, *Some Properties of* (m, C)*-Isometric Operators*, Filomat 33:3 (2019), 971–980 https://doi.org/10.2298/FIL1903971L

[23] O. A. Mahmoud Sid Ahmed, A. Saddi and K. Gherairi, *Some results on higher orders quasi-isometries*, Hacet. J. Math. Stat. Volume **49 (4)** (2020), 1315 – 1333. DOI : 10.15672/hujms.532964

[24] O. A. Mahmoud Sid Ahmed, *Some Properties of m-invertible operators in Banach Spaces*, Acta Mathematica Scientia (2012),**32** B **(2)**, 520–530.

[25] O. A. Mahmoud Sid Ahmed, *m-Isometric Operators on Banach Spaces*, Asian-European Journal of Mathematics Vol. **3**, No.**1** (2010) 1-19.

[26] O. A. Mahmoud Sid Ahmed, M. Chō and J. E. Lee, *On n-quasi-*(m, C)*-isomtric operators*, Linear and Multilinear Alebra (2020), Vol.**68**, NO. **5**, 1001–1020.

[27] S. Mecheri and O. A. Mahmoud Sid Ahmed, *Some classes of operators related to m-isometries*, Oper. and Matrices.Volume **14**, Number 1 (2020), 145-157.

[28] S. Mecheri and T. Prasad, On n-quasi-m-isometric operators, Asian-European Journal of Mathematics Vol.**9**, No. **4** (2016) 1650073 (8 pages).

[29] S. M. Patel, 2-isometry operators, Glasnik Matematicki, 37 (57) (2002) 143-147.

[30] R. Rabaoui, *On Skew* (A, m)*-Symmetric Operators in a Hilbert Space*, Filomat 36:10 (2022), 3261–3278.

[31] M. Salehi and K. Hedayatian, On higher order selfadjoint operators, Linear Algebra and its Applications. https://doi.org/10.1016/j.laa.2019.11.009.

[32] J. Shen, *On n-quasi-*$[m, C]$*-isometric operators*, Journal of Inequalities and Applications (2019) 2019:313. https://doi.org/10.1186/s13660-019-2268-3.

[33] M. Stankus, *m-Isometries, n-symmetries and other linear transformations which are hereditary roots*, Integral Equations Operator Theory, **75** (2013), 301-321.