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# On some 3-dimensional almost $\eta$-Ricci solitons with diagonal metrics

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**Abstract.** We study some properties of a 3-dimensional manifold with a diagonal Riemannian metric as an almost $\eta$-Ricci soliton from the following points of view: under certain assumptions, we determine the potential vector field if $\eta$ is given; we get constraints on the metric when the potential vector field has a particular expression; we compute the defining functions of the soliton when both the potential vector field and the 1-form are prescribed. Moreover, we find conditions for the manifold to be flat. Based on the theoretical results, we provide examples.

## 1. Introduction

The solitons' theory has been lately intensively investigated. The stationary solutions of the Ricci flow, namely, the *Ricci solitons*, still represent a very actual topic to be studied from various points of view. Problems like linear stability of compact Ricci solitons, curvature estimates, rigidity results for gradient Ricci solitons, etc. have been recently treated by Huai-Dong Cao *et. al.* in [7–9]. As a generalization of the notion of Ricci soliton given by Hamilton in [12], an *almost $\eta$-Ricci soliton* [4, 5, 10] is a Riemannian manifold $(M, \tilde{g})$ with a smooth vector field $V$ which satisfies the following equation

$$\frac{1}{2}\pounds_V \tilde{g} + \mathrm{Ric} + \lambda \tilde{g} + \mu \eta \otimes \eta = 0, \tag{1}$$

where $\lambda$ and $\mu$ are two smooth functions on $M$ with $\mu \neq 0$, Ric is the Ricci curvature tensor field, $\pounds_V \tilde{g}$ is the Lie derivative of the metric $\tilde{g}$ in the direction of $V$, and $\eta$ is a 1-form on $M$. If $V$ is a Killing vector field, i.e., $\pounds_V \tilde{g} = 0$, then the soliton is called *trivial*, and in this case, $(M, \tilde{g})$ is just a quasi-Einstein manifold. On the other hand, if $\eta = 0$, then $(M, \tilde{g}, V, \lambda)$ is called an *almost Ricci soliton* [13]. In this case, the soliton is said to be *shrinking*, *steady* or *expanding* according as $\lambda$ is negative, zero or positive, respectively [11]. In the above cases, if $\lambda$ and $\mu$ are real numbers, then we drop "almost".

The aim of the present paper is to describe a 3-dimensional manifold endowed with a diagonal Riemannian metric as an almost $\eta$-Ricci soliton. More precisely, under certain assumptions, we determine the potential vector field $V$ when $\eta$ is given, we find the conditions that must be satisfied by the Riemannian metric when the potential vector field has a particular expression, and we compute the defining functions $\lambda$ and $\mu$ when both the potential vector field and the 1-form are prescribed. Based on the theoretical results, we construct examples, among which the 3-dimensional $Sol_3$ and $\mathbb{H}^2 \times \mathbb{R}$ Lie groups. An analogous study

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for the canonical metric has been previously done by the present author in [6]. More recently, some particular Riemannian manifolds, namely, the 3-dimensional $Sol_3$ and $\mathbb{H}^2 \times \mathbb{R}$ Lie groups, have been described as Ricci solitons by Belarbi, Atashpeykara and Haji-Badali in [1–3] from a similar point of view.

## 2. The flatness condition

We consider $I = I_1 \times I_2 \times I_3 \subseteq \mathbb{R}^3$, where $I_i \subseteq \mathbb{R}$, $i \in \{1, 2, 3\}$, are open intervals, endowed with a diagonal Riemannian metric $\tilde{g}$ given by

$$\tilde{g} = \frac{1}{f_1^2} dx^1 \otimes dx^1 + \frac{1}{f_2^2} dx^2 \otimes dx^2 + dx^3 \otimes dx^3,$$

where $f_1$ and $f_2$ are two smooth functions nowhere zero on $I$, and $x^1, x^2, x^3$ are the standard coordinates in $\mathbb{R}^3$. Let

$$\left\{ E_1 := f_1 \frac{\partial}{\partial x^1},\ \ E_2 := f_2 \frac{\partial}{\partial x^2},\ \ E_3 := \frac{\partial}{\partial x^3} \right\}$$

be a local orthonormal frame, and, for the sake of simplicity, we will make the following notations:

$$\frac{f_2}{f_1} \cdot \frac{\partial f_1}{\partial x^2} =: a,\ \ \frac{1}{f_1} \cdot \frac{\partial f_1}{\partial x^3} =: b,\ \ \frac{f_1}{f_2} \cdot \frac{\partial f_2}{\partial x^1} =: c,\ \ \frac{1}{f_2} \cdot \frac{\partial f_2}{\partial x^3} =: d.$$

Computing the Lie brackets $[X, Y] := X \circ Y - Y \circ X$, we get

$$[E_1, E_2] = -aE_1 + cE_2,\ \ [E_1, E_3] = -bE_1,\ \ [E_2, E_3] = -dE_2.$$

The Levi-Civita connection $\nabla$ of $\tilde{g}$, deduced from the Koszul's formula

$$2\tilde{g}(\nabla_X Y, Z) = X(\tilde{g}(Y, Z)) + Y(\tilde{g}(Z, X)) - Z(\tilde{g}(X, Y)) - \tilde{g}(X, [Y, Z]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y]),$$

is given by

$$\nabla_{E_1} E_1 = aE_2 + bE_3,\ \ \nabla_{E_1} E_2 = -aE_1,\ \ \nabla_{E_1} E_3 = -bE_1,\ \ \nabla_{E_2} E_1 = -cE_2,$$
$$\nabla_{E_2} E_2 = cE_1 + dE_3,\ \ \nabla_{E_2} E_3 = -dE_2,\ \ \nabla_{E_3} E_1 = 0,\ \ \nabla_{E_3} E_2 = 0,\ \ \nabla_{E_3} E_3 = 0,$$

and the Riemann and Ricci curvature tensor fields

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\ \ \mathrm{Ric}(Y, Z) := \sum_{k=1}^{3} \tilde{g}(R(E_k, Y)Z, E_k)$$

are the following

$$\begin{aligned}
&R(E_1, E_2)E_2 = [E_1(c) + E_2(a) - a^2 - c^2 - bd]E_1 + [E_3(c) - cd]E_3,\\
&R(E_2, E_1)E_1 = [E_1(c) + E_2(a) - a^2 - c^2 - bd]E_2 + [E_3(a) - ab]E_3,\\
&R(E_1, E_3)E_3 = [E_3(b) - b^2]E_1,\\
&R(E_2, E_3)E_3 = [E_3(d) - d^2]E_2,\\
&R(E_3, E_1)E_1 = [E_3(a) - ab]E_2 + [E_3(b) - b^2]E_3,\\
&R(E_3, E_2)E_2 = [E_3(c) - cd]E_1 + [E_3(d) - d^2]E_3,\\
&R(E_1, E_2)E_3 = \tilde{g}(R(E_2, E_1)E_1, E_3)E_1 - \tilde{g}(R(E_1, E_2)E_2, E_3)E_2 = [E_3(a) - ab]E_1 - [E_3(c) - cd]E_2,\\
&R(E_2, E_3)E_1 = \tilde{g}(R(E_3, E_2)E_2, E_1)E_2 - \tilde{g}(R(E_2, E_3)E_3, E_1)E_3 = [E_3(c) - cd]E_2,\\
&R(E_3, E_1)E_2 = -\tilde{g}(R(E_3, E_1)E_1, E_2)E_1 + \tilde{g}(R(E_1, E_3)E_3, E_2)E_3 = -[E_3(a) - ab]E_1,\\
&\mathrm{Ric}(E_1, E_1) = E_1(c) + E_2(a) + E_3(b) - a^2 - b^2 - c^2 - bd,\\
&\mathrm{Ric}(E_2, E_2) = E_1(c) + E_2(a) + E_3(d) - a^2 - c^2 - d^2 - bd,\\
&\mathrm{Ric}(E_3, E_3) = E_3(b) + E_3(d) - b^2 - d^2,\\
&\mathrm{Ric}(E_1, E_2) = 0,\ \ \mathrm{Ric}(E_1, E_3) = E_3(c) - cd,\ \ \mathrm{Ric}(E_2, E_3) = E_3(a) - ab.
\end{aligned}$$

We shall determine the conditions that the two functions $f_1$ and $f_2$ must satisfy for the manifold to be flat, exploring the cases when each of the two functions depends on a single variable. In this paper, if a function $h$ on $\mathbb{R}^3$ depends only on some of its variables, then we will write in its argument only that variables in order to emphasize this fact, for example, $h(x^i)$, $h(x^i, x^j)$.

**Proposition 2.1.** *If $f_i = f_i(x^i)$ for $i \in \{1, 2\}$, then $(I, \tilde{g})$ is a flat Riemannian manifold.*

*Proof.* In this case, $a = b = c = d = 0$, and $R(E_i, E_j)E_k = 0$ for any $i, j, k \in \{1, 2, 3\}$. □

**Proposition 2.2.** *If $f_i = f_i(x^3)$ for $i \in \{1, 2\}$, then the following assertions are equivalent:*
*(1) $(I, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_1$ and $f_2$ are constant, or $f_i = k_i \in \mathbb{R} \setminus \{0\}$ and $f_j(x^3) = \dfrac{c_1}{x^3 - c_2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R} \setminus I_3$, and $i \neq j$.*

*Proof.* In this case, $a = 0$, $b = \dfrac{f_1'}{f_1}$, $c = 0$, $d = \dfrac{f_2'}{f_2}$, and

$$R(E_1, E_2)E_2 = -bdE_1, \quad R(E_2, E_1)E_1 = -bdE_2,$$

$$R(E_1, E_3)E_3 = (b' - b^2)E_1, \quad R(E_3, E_1)E_1 = (b' - b^2)E_3,$$
$$R(E_2, E_3)E_3 = (d' - d^2)E_2, \quad R(E_3, E_2)E_2 = (d' - d^2)E_3.$$

Then, $R = 0$ if and only if

$$\begin{cases} bd = 0 \\ b' = b^2 \\ d' = d^2 \end{cases},$$

that is,

$$\begin{cases} f_1' f_2' = 0 \\ \left(\dfrac{f_1'}{f_1}\right)' = \left(\dfrac{f_1'}{f_1}\right)^2 \\ \left(\dfrac{f_2'}{f_2}\right)' = \left(\dfrac{f_2'}{f_2}\right)^2 \end{cases}.$$

Therefore,
- $f_1' = 0$ and $f_2' = 0$, or
- $f_1' = 0$ and $-\dfrac{1}{\frac{f_2'(x^3)}{f_2(x^3)}} = x^3 + c_0$ $\left(\text{i.e., } f_2(x^3) = \dfrac{c_1}{x^3 - c_2}, \text{ where } c_1 \in \mathbb{R} \setminus \{0\}, c_2 \in \mathbb{R} \setminus I_3\right)$, or, similarly
- $f_1(x^3) = \dfrac{c_1}{x^3 - c_2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R} \setminus I_3$ and $f_2' = 0$. □

**Corollary 2.3.** *For $I = \mathbb{R}^3$, if $f_i = f_i(x^3)$ for $i \in \{1, 2\}$, then the following assertions are equivalent:*
*(1) $(\mathbb{R}^3, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_1$ and $f_2$ are constant.*

*Proof.* It follows immediately from Proposition 2.2. □

**Proposition 2.4.** *If $f_1 = f_1(x^1)$, $f_2 = f_2(x^3)$, then the following assertions are equivalent:*
*(1) $(I, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_2$ is constant, or $f_2(x^3) = \dfrac{c_1}{x^3 - c_2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R} \setminus I_3$.*

*Proof.* In this case, $a = b = c = 0$, $d = \dfrac{f_2'}{f_2}$, and

$$R(E_2, E_3)E_3 = (d' - d^2)E_2, \ \ R(E_3, E_2)E_2 = (d' - d^2)E_3,$$

$$R(E_1, E_2)E_2 = R(E_1, E_3)E_3 = R(E_2, E_1)E_1 = R(E_3, E_1)E_1 = 0.$$

Then, $R = 0$ if and only if

$$d' = d^2,$$

that is,

$$\left(\frac{f_2'}{f_2}\right)' = \left(\frac{f_2'}{f_2}\right)^2.$$

Therefore,
- $f_2' = 0$, or
- $-\dfrac{1}{\dfrac{f_2'(x^3)}{f_2(x^3)}} = x^3 + c_0$ $\left(\text{i.e., } f_2(x^3) = \dfrac{c_1}{x^3 - c_2}, \text{ where } c_1 \in \mathbb{R} \setminus \{0\}, c_2 \in \mathbb{R} \setminus I_3\right)$. □

**Corollary 2.5.** *For $I = \mathbb{R}^3$, if $f_1 = f_1(x^1)$, $f_2 = f_2(x^3)$, then the following assertions are equivalent:*
*(1) $(\mathbb{R}^3, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_2$ is constant.*

*Proof.* It follows immediately from Proposition 2.4. □

**Proposition 2.6.** *If $f_i = f_i(x^2)$ for $i \in \{1, 2\}$, then the following assertions are equivalent:*
*(1) $(I, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_1$ and $f_2$ satisfy the equation*

$$\frac{f_1'}{f_1}\frac{f_2'}{f_2} + \frac{f_1''}{f_1} = 2\left(\frac{f_1'}{f_1}\right)^2;$$

*(3) $f_1$ is constant, or $f_1'$ is nowhere zero and $f_2 = c_0\dfrac{f_1^2}{f_1'}$, where $c_0 \in \mathbb{R} \setminus \{0\}$.*

*Proof.* In this case, $a = f_2\dfrac{f_1'}{f_1}$, $b = c = d = 0$, and

$$R(E_1, E_2)E_2 = [E_2(a) - a^2]E_1, \ \ R(E_2, E_1)E_1 = [E_2(a) - a^2]E_2,$$

$$R(E_1, E_3)E_3 = R(E_3, E_1)E_1 = R(E_2, E_3)E_3 = R(E_3, E_2)E_2 = 0.$$

Then, $R = 0$ if and only if

$$f_2 a' = a^2,$$

that is,

$$f_2 f_2' \frac{f_1'}{f_1} + f_2^2 \frac{f_1'' f_1 - (f_1')^2}{f_1^2} = f_2^2\left(\frac{f_1'}{f_1}\right)^2,$$

which is equivalent to

$$\frac{f_1'}{f_1}\frac{f_2'}{f_2} + \frac{f_1''}{f_1} = 2\left(\frac{f_1'}{f_1}\right)^2.$$

If $f_1' = 0$, then $f_1$ is a constant function. If $f_1' \neq 0$ (hence, if it is nowhere zero), then the previous relation can be written as

$$\frac{f_2'}{f_2} = \frac{2\left(\frac{f_1'}{f_1}\right)^2 - \frac{f_1''}{f_1}}{\frac{f_1'}{f_1}}.$$

Let us notice that

$$\frac{f_2'}{f_2} = \frac{\left(\frac{f_1'}{f_1}\right)^2 - \left(\frac{f_1'}{f_1}\right)'}{\frac{f_1'}{f_1}} = \frac{f_1'}{f_1} - \frac{\left(\frac{f_1'}{f_1}\right)'}{\frac{f_1'}{f_1}},$$

which, by integration, gives

$$\ln|f_2| = \ln|f_1| - \ln\left|\frac{f_1'}{f_1}\right| + k = \ln\left(e^k \frac{f_1^2}{|f_1'|}\right),$$

where $k \in \mathbb{R}$; therefore, $f_2 = c_0 \frac{f_1^2}{f_1'}$, where $c_0 \in \mathbb{R} \setminus \{0\}$. □

**Corollary 2.7.** *For $I = \mathbb{R}^3$, if $f_1 = f_2 =: f(x^2)$, then the following assertions are equivalent:*
*(1) $(\mathbb{R}^3, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f(x^2) = c_1 e^{c_2 x^2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$.*

*Proof.* (1) is equivalent to $ff'' - (f')^2 = 0$, that is, $\left(\frac{f'}{f}\right)' = 0$, with the solution $f(x^2) = c_1 e^{c_2 x^2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$, hence we get the conclusion. □

**Proposition 2.8.** *If $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$, then the following assertions are equivalent:*
*(1) $(I, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_1$ and $f_2$ satisfy the equation*

$$\frac{f_1'' f_1 - 2(f_1')^2}{f_1^4} = -\frac{f_2'' f_2 - 2(f_2')^2}{f_2^4} = constant.$$

*Proof.* In this case, $a = f_2 \frac{f_1'}{f_1}$, $b = 0$, $c = f_1 \frac{f_2'}{f_2}$, $d = 0$, and

$$R(E_1, E_2)E_2 = [E_1(c) + E_2(a) - a^2 - c^2]E_1, \quad R(E_2, E_1)E_1 = [E_1(c) + E_2(a) - a^2 - c^2]E_2,$$

$$R(E_1, E_3)E_3 = R(E_3, E_1)E_1 = R(E_2, E_3)E_3 = R(E_3, E_2)E_2 = 0.$$

Then, $R = 0$ if and only if

$$f_1 \frac{\partial c}{\partial x^1} + f_2 \frac{\partial a}{\partial x^2} = a^2 + c^2,$$

that is,

$$f_1^2 \frac{f_2'' f_2 - (f_2')^2}{f_2^2} + f_2^2 \frac{f_1'' f_1 - (f_1')^2}{f_1^2} = f_2^2 \left(\frac{f_1'}{f_1}\right)^2 + f_1^2 \left(\frac{f_2'}{f_2}\right)^2,$$

which is equivalent to

$$\frac{f_1'' f_1 - 2(f_1')^2}{f_1^4} = -\frac{f_2'' f_2 - 2(f_2')^2}{f_2^4}.$$

Since $f_1$ depends only on $x^2$ and $f_2$ depends only on $x^1$, we deduce that the above ratio must be a constant. □

**Remark 2.9.** *We shall now look on the condition (2) from Proposition 2.8 satisfied by the two functions, namely*

$$\frac{f''f-2(f')^2}{f^4}=k\in\mathbb{R},$$

*when $f$ is a real function defined on a real interval. Let us notice that*

$$\frac{f''f-2(f')^2}{f^4}=-\left(\frac{1}{f}\right)''\frac{1}{f}.$$

*Denoting by $h:=\frac{1}{f}$, we have*

$$h''h=-k.$$

*Let $r\in\mathbb{R}$ and let $J\subseteq\mathbb{R}$ be an open interval such that $0\notin J$ and $-2k\ln|y|+r>0$ for any $y\in J$. Let $F$ be an antiderivative on $J$ of the function*

$$y\mapsto\frac{1}{\sqrt{-2k\ln|y|+r}}.$$

*Then, $F'(y)>0$ for any $y\in J$; therefore, $F$ is strictly increasing on $J$, hence, it is invertible onto its image. Let $\varepsilon\in\{\pm1\}$ and let $I_J:=\varepsilon\,(F(J)-c_0)$, where $c_0\in\mathbb{R}$. Then,*

$$h:I_J\to\mathbb{R},\ \ h(x):=F^{-1}(\varepsilon x+c_0)$$

*satisfies $h''(x)h(x)=-k$ for any $x\in I_J$, and we get*

$$f:I_J\to\mathbb{R},\ \ f(x)=\frac{1}{F^{-1}(\varepsilon x+c_0)}.$$

**Proposition 2.10.** *If $f_1=f_1(x^2)$, $f_2=f_2(x^3)$, then the following assertions are equivalent:*
*(1) $(I,\tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_1$ and $f_2$ are constant, or $f_1=k_1\in\mathbb{R}\setminus\{0\}$ and $f_2(x^3)=\frac{c_1}{x^3-c_2}$, where $c_1\in\mathbb{R}\setminus\{0\}$, $c_2\in\mathbb{R}\setminus I_3$, or $f_2=k_2\in\mathbb{R}\setminus\{0\}$ and $f_1(x^2)=\frac{c_1}{x^2-c_2}$, where $c_1\in\mathbb{R}\setminus\{0\}$, $c_2\in\mathbb{R}\setminus I_2$.*

*Proof.* In this case, $a=f_2\frac{f_1'}{f_1}$, $b=c=0$, $d=\frac{f_2'}{f_2}$, and

$$R(E_1,E_2)E_2=[E_2(a)-a^2]E_1,\ \ R(E_2,E_1)E_1=[E_2(a)-a^2]E_2+E_3(a)E_3,$$

$$R(E_2,E_3)E_3=[E_3(d)-d^2]E_2,\ \ R(E_3,E_2)E_2=[E_3(d)-d^2]E_3,$$

$$R(E_3,E_1)E_1=E_3(a)E_2,\ \ R(E_1,E_2)E_3=E_3(a)E_1,\ \ R(E_3,E_1)E_2=-E_3(a)E_1,$$

$$R(E_1,E_3)E_3=R(E_2,E_3)E_1=0.$$

Then, $R=0$ if and only if

$$\begin{cases} f_1'f_2'=0\\ \left(\frac{f_1'}{f_1}\right)'=\left(\frac{f_1'}{f_1}\right)^2\\ \left(\frac{f_2'}{f_2}\right)'=\left(\frac{f_2'}{f_2}\right)^2\end{cases}.$$

Therefore,

- $f_1' = 0$ and $f_2' = 0$, or
- $f_1' = 0$ and $-\dfrac{1}{\frac{f_2'(x^3)}{f_2(x^3)}} = x^3 + c_0$ $\left(\text{i.e., } f_2(x^3) = \dfrac{c_1}{x^3 - c_2}, \text{ where } c_1 \in \mathbb{R} \setminus \{0\}, c_2 \in \mathbb{R} \setminus I_3\right)$, or, similarly
- $f_1(x^2) = \dfrac{c_1}{x^2 - c_2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R} \setminus I_2$ and $f_2' = 0$. □

**Corollary 2.11.** *For $I = \mathbb{R}^3$, if $f_1 = f_1(x^2)$, $f_2 = f_2(x^3)$, then the following assertions are equivalent:*
*(1) $(\mathbb{R}^3, \tilde{g})$ is a flat Riemannian manifold;*
*(2) $f_1$ and $f_2$ are constant.*

*Proof.* It follows immediately from Proposition 2.10. □

## 3. $I$ as an almost $\eta$-Ricci soliton

Let $V = \sum_{k=1}^{3} V^k E_k$ and $\eta = \sum_{k=1}^{3} \eta^k e_k$, where $e_k$ is the dual 1-form of $E_k$ for $k \in \{1,2,3\}$. From (1) we get the equations that define an almost $\eta$-Ricci soliton $(I, \tilde{g}, V, \lambda, \mu)$:

$$\frac{1}{2}\Big\{E_i(V^j) + E_j(V^i) + \sum_{k=1}^{3} V^k[\tilde{g}(\nabla_{E_i}E_k, E_j) + \tilde{g}(E_i, \nabla_{E_j}E_k)]\Big\} + \mathrm{Ric}(E_i, E_j) + \lambda\delta^{ij} + \mu\eta^i\eta^j = 0$$

for any $(i, j) \in \{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$, which is equivalent to the following system

$$\begin{cases} E_1(V^1) - aV^2 - bV^3 + \mathrm{Ric}(E_1, E_1) + \lambda + \mu(\eta^1)^2 = 0 \\ E_2(V^2) - cV^1 - dV^3 + \mathrm{Ric}(E_2, E_2) + \lambda + \mu(\eta^2)^2 = 0 \\ E_3(V^3) + \mathrm{Ric}(E_3, E_3) + \lambda + \mu(\eta^3)^2 = 0 \\ \frac{1}{2}[E_1(V^2) + E_2(V^1) + aV^1 + cV^2] + \mu\eta^1\eta^2 = 0 \\ \frac{1}{2}[E_1(V^3) + E_3(V^1) + bV^1] + \mathrm{Ric}(E_1, E_3) + \mu\eta^1\eta^3 = 0 \\ \frac{1}{2}[E_2(V^3) + E_3(V^2) + dV^2] + \mathrm{Ric}(E_2, E_3) + \mu\eta^2\eta^3 = 0 \end{cases}. \tag{2}$$

We shall further consider the cases when the potential vector field of the almost $\eta$-Ricci soliton $(I, \tilde{g}, \lambda, \mu)$ is $V = \dfrac{\partial}{\partial x^3}$, or when $\eta = dx^3$.

### 3.1. *Almost $\eta$-Ricci solitons with $V = \dfrac{\partial}{\partial x^3}$*

If $V^1 = V^2 = 0$ and $V^3 = 1$, then the system (2) becomes

$$\begin{cases} -b + \mathrm{Ric}(E_1, E_1) + \lambda + \mu(\eta^1)^2 = 0 \\ -d + \mathrm{Ric}(E_2, E_2) + \lambda + \mu(\eta^2)^2 = 0 \\ \mathrm{Ric}(E_3, E_3) + \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = 0 \\ \mathrm{Ric}(E_1, E_3) + \mu\eta^1\eta^3 = 0 \\ \mathrm{Ric}(E_2, E_3) + \mu\eta^2\eta^3 = 0 \end{cases}. \tag{3}$$

**Proposition 3.1.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $V = \dfrac{\partial}{\partial x^3}$. If $f_i = f_i(x^i)$ for $i \in \{1,2\}$ and $\mu$ is nowhere zero on $I$, then $\eta = 0$ and $V$ is a Killing vector field.*

*Proof.* In this case, we have $a = b = c = d = 0$, and $\mathrm{Ric}(E_i, E_j) = 0$ for any $i, j \in \{1, 2, 3\}$, and (3) becomes

$$\begin{cases} \lambda + \mu(\eta^1)^2 = 0 \\ \lambda + \mu(\eta^2)^2 = 0 \\ \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = \mu\eta^1\eta^3 = \mu\eta^2\eta^3 = 0 \end{cases}. \tag{4}$$

Since $\mu$ is nowhere zero, we get $(\eta^1)^2 = (\eta^2)^2 = (\eta^3)^2$, hence, $\eta^i = 0$ for $i \in \{1, 2, 3\}$, and $\lambda = 0$ from (4); therefore, $\pounds_V \tilde{g} = 0$ by means of (1). □

**Proposition 3.2.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $V = \dfrac{\partial}{\partial x^3}$. If $f_i = f_i(x^3)$ for $i \in \{1, 2\}$, $\eta^i =: f$ for $i \in \{1, 2, 3\}$, and $\mu$ is nowhere zero on $I$, then $\eta = 0$. Moreover,*

*(i) if one of the functions $f_1$ and $f_2$ is constant, then the other one is constant, too; in this case, $(I, \tilde{g})$ is a flat Riemannian manifold, $\lambda = 0$, and $V$ is a Killing vector field;*

*(ii) if $\dfrac{f_1}{f_2}$ is constant, then $f_i(x^3) = c_i e^{c_0 e^{-x^3}}$ for $i \in \{1, 2\}$, where $c_0 \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R} \setminus \{0\}$.*

*Proof.* In this case, we have $a = 0$, $b = \dfrac{f_1'}{f_1}$, $c = 0$, $d = \dfrac{f_2'}{f_2}$, and

$$\mathrm{Ric}(E_1, E_1) = b' - b^2 - bd, \quad \mathrm{Ric}(E_2, E_2) = d' - d^2 - bd, \quad \mathrm{Ric}(E_3, E_3) = b' + d' - b^2 - d^2,$$

$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = \mathrm{Ric}(E_2, E_3) = 0,$$

and (3) becomes

$$\begin{cases} -b + b' - b^2 - bd + \lambda + \mu(\eta^1)^2 = 0 \\ -d + d' - d^2 - bd + \lambda + \mu(\eta^2)^2 = 0 \\ b' + d' - b^2 - d^2 + \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = \mu\eta^1\eta^3 = \mu\eta^2\eta^3 = 0 \end{cases}. \tag{5}$$

Since $\mu$ is nowhere zero, (5) is equivalent to

$$\begin{cases} b(d+1) = d^2 - d' \\ d(b+1) = b^2 - b' \\ \lambda = b^2 - b' + d^2 - d' \\ f^2 = 0 \end{cases}, \tag{6}$$

which implies $\eta = 0$.

(i) $f_1$ is constant if and only if $b = 0$, and from (6), we deduce that $b = 0$ if and only if $d = 0$, i.e., if and only if $f_2$ is constant. In this case, $R = 0$, $\lambda = 0$, and $\pounds_V \tilde{g} = 0$.

(ii) If $\dfrac{f_1}{f_2}$ is constant, then $b = d$, and from (6), we get

$$\begin{cases} b' = -b \\ \lambda = 2(b^2 - b') \end{cases}. \tag{7}$$

If $b = d = 0$, then $f_1$ and $f_2$ are constant and $\lambda = 0$. If $b = d \neq 0$ (hence, if they are nowhere zero), then

$$\begin{cases} \dfrac{b'}{b} = -1 \\ \lambda = 2(b^2 - b') \end{cases}, \tag{8}$$

from where we get

$$\frac{f_i'(x^3)}{f_i(x^3)} = b(x^3) = c_0 e^{-x^3}, \ \ c_0 \in \mathbb{R} \setminus \{0\}, i \in \{1,2\},$$

which, by integration, gives $f_i(x^3) = c_i e^{-c_0 e^{-x^3}}$ for $i \in \{1,2\}$, where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$. □

**Corollary 3.3.** *Under the hypotheses of Proposition 3.2, if* $f_1 = f_2 =: f(x^3)$, *then* $f(x^3) = c_1 e^{c_2 e^{-x^3}}$, *where* $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$.

*Proof.* It follows from Proposition 3.2 (ii). □

**Proposition 3.4.** *Let* $(I, \tilde{g}, V, \lambda, \mu)$ *be an almost* $\eta$*-Ricci soliton with* $V = \frac{\partial}{\partial x^3}$. *If* $f_1 = f_1(x^1)$, $f_2 = f_2(x^3)$, $\eta^i =: f$ *for* $i \in \{1,2,3\}$, *and* $\mu$ *is nowhere zero on* $I$, *then* $\eta = 0$ *and* $\lambda = 0$. *Moreover,* $f_2$ *is constant and* $V$ *is a Killing vector field.*

*Proof.* In this case, we have $a = b = c = 0$, $d = \frac{f_2'}{f_2}$, and

$$\mathrm{Ric}(E_2, E_2) = \mathrm{Ric}(E_3, E_3) = d' - d^2,$$
$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = \mathrm{Ric}(E_2, E_3) = 0,$$

and (3) becomes

$$\begin{cases} \lambda + \mu(\eta^1)^2 = 0 \\ -d + d' - d^2 + \lambda + \mu(\eta^2)^2 = 0 \\ d' - d^2 + \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = \mu\eta^1\eta^3 = \mu\eta^2\eta^3 = 0 \end{cases}. \tag{9}$$

Since $\mu$ is nowhere zero, (9) is equivalent to

$$\begin{cases} d = d' - d^2 = 0 \\ \lambda = -\mu f^2 \\ f^2 = 0 \end{cases}.$$

It follows that $\eta = 0$, $\lambda = 0$, and $\pounds_V \tilde{g} = 0$. □

**Proposition 3.5.** *Let* $(I, \tilde{g}, V, \lambda, \mu)$ *be an almost* $\eta$*-Ricci soliton with* $V = \frac{\partial}{\partial x^3}$. *If* $f_i = f_i(x^2)$ *for* $i \in \{1,2\}$, $\eta^i =: f$ *for* $i \in \{1,2,3\}$, *and* $\mu$ *is nowhere zero on* $I$, *then* $\eta = 0$ *and* $\lambda = 0$. *Moreover,* $f_1$ *is constant, or* $f_1'$ *is nowhere zero and* $f_2 = c_0 \frac{f_1^2}{f_1'}$, *where* $c_0 \in \mathbb{R} \setminus \{0\}$.

*Proof.* In this case, we have $a = f_2 \frac{f_1'}{f_1}$, $b = c = d = 0$, and

$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = E_2(a) - a^2,$$
$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = \mathrm{Ric}(E_2, E_3) = \mathrm{Ric}(E_3, E_3) = 0,$$

and (3) becomes

$$\begin{cases} f_2 f_2' \frac{f_1'}{f_1} + \frac{f_2^2}{f_1^2}[f_1'' f_1 - 2(f_1')^2] + \lambda + \mu(\eta^1)^2 = 0 \\ f_2 f_2' \frac{f_1'}{f_1} + \frac{f_2^2}{f_1^2}[f_1'' f_1 - 2(f_1')^2] + \lambda + \mu(\eta^2)^2 = 0 \\ \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = \mu\eta^1\eta^3 = \mu\eta^2\eta^3 = 0 \end{cases}. \tag{10}$$

Since $\mu$ is nowhere zero, (10) is equivalent to

$$\begin{cases} f_2 f_2' \dfrac{f_1'}{f_1} + \dfrac{f_2^2}{f_1^2}[f_1'' f_1 - 2(f_1')^2] + \lambda + \mu f^2 = 0 \\ \lambda = -\mu f^2 \\ f^2 = 0 \end{cases}.$$

It follows that $\eta = 0$, $\lambda = 0$, and, from the first equation of the previous system, we get

$$\frac{f_1'}{f_1}\frac{f_2'}{f_2} + \frac{f_1''}{f_1} = 2\left(\frac{f_1'}{f_1}\right)^2.$$

From Proposition 2.6, we deduce that $f_1$ is constant, or $f_1'$ is nowhere zero and $f_2 = c_0 \dfrac{f_1^2}{f_1'}$, where $c_0 \in \mathbb{R}\setminus\{0\}$. □

**Corollary 3.6.** *Under the hypotheses of Proposition 3.5, if $f_1 = f_2 =: f(x^2)$, then $f(x^2) = c_1 e^{c_2 x^2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$.*

*Proof.* It follows from Proposition 3.5 that $ff'' - (f')^2 = 0$, that is, $\left(\dfrac{f'}{f}\right)' = 0$, with the solution $f(x^2) = c_1 e^{c_2 x^2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$. □

**Proposition 3.7.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $V = \dfrac{\partial}{\partial x^3}$. If $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$, $\eta^i =: f$ for $i \in \{1, 2, 3\}$, and $\mu$ is nowhere zero on $I$, then $\eta = 0$ and $\lambda = 0$. Moreover,*

$$\frac{f_1'' f_1 - 2(f_1')^2}{f_1^4} = -\frac{f_2'' f_2 - 2(f_2')^2}{f_2^4} = \textit{constant}.$$

*Proof.* In this case, we have $a = f_2 \dfrac{f_1'}{f_1}$, $b = 0$, $c = f_1 \dfrac{f_2'}{f_2}$, $d = 0$, and

$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = E_1(c) + E_2(a) - a^2 - c^2,$$

$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = \mathrm{Ric}(E_2, E_3) = \mathrm{Ric}(E_3, E_3) = 0,$$

and (3) becomes

$$\begin{cases} f_1 \dfrac{\partial c}{\partial x^1} + f_2 \dfrac{\partial a}{\partial x^2} - a^2 - c^2 + \lambda + \mu(\eta^1)^2 = 0 \\ f_1 \dfrac{\partial c}{\partial x^1} + f_2 \dfrac{\partial a}{\partial x^2} - a^2 - c^2 + \lambda + \mu(\eta^2)^2 = 0 \\ \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = \mu\eta^1\eta^3 = \mu\eta^2\eta^3 = 0 \end{cases}. \tag{11}$$

Since $\mu$ is nowhere zero and $\eta^i = f$, (11) is equivalent to

$$\begin{cases} f_1 \dfrac{\partial c}{\partial x^1} + f_2 \dfrac{\partial a}{\partial x^2} - a^2 - c^2 + \lambda + \mu f^2 = 0 \\ \lambda = -\mu f^2 \\ f^2 = 0 \end{cases}.$$

It follows that $\eta = 0$, $\lambda = 0$, and, from the first equation of the previous system, we get the relation between $f_1$ and $f_2$. Since $f_1$ depends only on $x^2$ and $f_2$ depends only on $x^1$, we deduce that the obtained ratio must be a constant. □

**Proposition 3.8.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $V = \frac{\partial}{\partial x^3}$. If $f_1 = f_1(x^2)$, $f_2 = f_2(x^3)$, $\eta^i =: f$ for $i \in \{1, 2, 3\}$, and $\mu$ is nowhere zero on $I$, then $\eta = 0$ and $\lambda = 0$. Moreover, $V$ is a Killing vector field, $f_2$ is constant, and either $f_1$ is constant or $f_1(x^2) = \frac{c_1}{x^2 - c_2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R} \setminus I_2$.*

*Proof.* In this case, we have $a = f_2 \frac{f_1'}{f_1}$, $b = c = 0$, $d = \frac{f_2'}{f_2}$, and

$$\mathrm{Ric}(E_1, E_1) = E_2(a) - a^2, \;\; \mathrm{Ric}(E_2, E_2) = E_2(a) - a^2 + E_3(d) - d^2, \;\; \mathrm{Ric}(E_3, E_3) = E_3(d) - d^2,$$

$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = 0, \;\; \mathrm{Ric}(E_2, E_3) = E_3(a),$$

and (3) becomes

$$\begin{cases} f_2 \dfrac{\partial a}{\partial x^2} - a^2 + \lambda + \mu(\eta^1)^2 = 0 \\ -d + f_2 \dfrac{\partial a}{\partial x^2} - a^2 + d' - d^2 + \lambda + \mu(\eta^2)^2 = 0 \\ d' - d^2 + \lambda + \mu(\eta^3)^2 = 0 \\ \mu\eta^1\eta^2 = \mu\eta^1\eta^3 = 0 \\ \dfrac{\partial a}{\partial x^3} + \mu\eta^2\eta^3 = 0 \end{cases}. \tag{12}$$

Since $\mu$ is nowhere zero and $\eta^i = f$, (12) is equivalent to

$$\begin{cases} f_2 \dfrac{\partial a}{\partial x^2} - a^2 - d = 0 \\ d = d' - d^2 \\ \lambda = -d \\ f^2 = 0 \\ \dfrac{\partial a}{\partial x^3} = 0 \end{cases}.$$

It follows that $\eta = 0$ and $f_1' f_2' = 0$, which, together with the second equation of the system, implies that one of the functions $f_1$ and $f_2$ must be constant. If $f_1 = k_1 \in \mathbb{R} \setminus \{0\}$, then $a = 0$, $d = 0$ (hence, $f_2$ is constant), and $\lambda = 0$. If $f_2 = k_2 \in \mathbb{R} \setminus \{0\}$, then $d = 0$, $\lambda = 0$, and

$$\left(\frac{f_1'}{f_1}\right)' = \left(\frac{f_1'}{f_1}\right)^2.$$

Then, either $f_1$ is a constant, too, or $f_1' \neq 0$, in which case, by integration, we obtain $-\frac{1}{\frac{f_1'(x^2)}{f_1(x^2)}} = x^2 + c_0$, i.e., $f_1(x^2) = \frac{c_1}{x^2 - c_2}$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R} \setminus I_2$. Since $b = d = 0$, we deduce that $\pounds_V \tilde{g} = 0$. □

*3.2. Almost η-Ricci solitons with $\eta = dx^3$*

If $\eta^1 = \eta^2 = 0$ and $\eta^3 = 1$, then the system (2) becomes

$$\begin{cases} E_1(V^1) - aV^2 - bV^3 + \mathrm{Ric}(E_1, E_1) + \lambda = 0 \\ E_2(V^2) - cV^1 - dV^3 + \mathrm{Ric}(E_2, E_2) + \lambda = 0 \\ E_3(V^3) + \mathrm{Ric}(E_3, E_3) + \lambda + \mu = 0 \\ E_1(V^2) + E_2(V^1) + aV^1 + cV^2 = 0 \\ \frac{1}{2}[E_1(V^3) + E_3(V^1) + bV^1] + \mathrm{Ric}(E_1, E_3) = 0 \\ \frac{1}{2}[E_2(V^3) + E_3(V^2) + dV^2] + \mathrm{Ric}(E_2, E_3) = 0 \end{cases}. \tag{13}$$

**Theorem 3.9.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an η-Ricci soliton with $\eta = dx^3$. If $f_i = f_i(x^i)$ for $i \in \{1, 2\}$, then*

$$\begin{cases} V^1(x^1, x^2, x^3) = -\lambda F_1(x^1) + c_1 F_2(x^2) + c_2 x^3 + c_3 \\ V^2(x^1, x^2, x^3) = -c_1 F_1(x^1) - \lambda F_2(x^2) + c_4 x^3 + c_5 \\ V^3(x^1, x^2, x^3) = -c_2 F_1(x^1) - c_4 F_2(x^2) - (\lambda + \mu) x^3 + c_6 \end{cases}, \tag{14}$$

*where $F_i$ is an antiderivative of $\frac{1}{f_i}$ for $i \in \{1, 2\}$ and $c_i \in \mathbb{R}$ for $i \in \{1, 2, 3, 4, 5, 6\}$.*

*Proof.* In this case, we have $a = b = c = d = 0$, and $\mathrm{Ric}(E_i, E_j) = 0$ for any $i, j \in \{1, 2, 3\}$, and (13) becomes

$$\begin{cases} \frac{\partial V^1}{\partial x^1} = -\frac{\lambda}{f_1} \\ \frac{\partial V^2}{\partial x^2} = -\frac{\lambda}{f_2} \\ \frac{\partial V^3}{\partial x^3} = -(\lambda + \mu) \\ f_1 \frac{\partial V^2}{\partial x^1} = -f_2 \frac{\partial V^1}{\partial x^2} \\ f_1 \frac{\partial V^3}{\partial x^1} = -\frac{\partial V^1}{\partial x^3} \\ f_2 \frac{\partial V^3}{\partial x^2} = -\frac{\partial V^2}{\partial x^3} \end{cases}. \tag{15}$$

Since $f_i$ depends only on $x^i$, from the first three equations of (15), we find that

$$\begin{cases} V^1(x^1, x^2, x^3) = -\lambda F_1(x^1) + h_1(x^2, x^3) \\ V^2(x^1, x^2, x^3) = -\lambda F_2(x^2) + h_2(x^1, x^3) \\ V^3(x^1, x^2, x^3) = -(\lambda + \mu) x^3 + h_3(x^1, x^2) \end{cases},$$

where $F_i' = \frac{1}{f_i}$ for $i \in \{1, 2\}$. From the last three equations of (15), we infer:

$$f_1(x^1) \frac{\partial h_2}{\partial x^1}(x^1, x^3) = -f_2(x^2) \frac{\partial h_1}{\partial x^2}(x^2, x^3), \tag{16}$$

$$f_1(x^1) \frac{\partial h_3}{\partial x^1}(x^1, x^2) = -\frac{\partial h_1}{\partial x^3}(x^2, x^3), \tag{17}$$

$$f_2(x^2) \frac{\partial h_3}{\partial x^2}(x^1, x^2) = -\frac{\partial h_2}{\partial x^3}(x^1, x^3). \tag{18}$$

Denoting $f_2(x^2)\dfrac{\partial h_1}{\partial x^2}(x^2,x^3) =: \bar{h}_1(x^3)$, we have

$$\frac{\partial h_1}{\partial x^2}(x^2,x^3) = \frac{\bar{h}_1(x^3)}{f_2(x^2)},$$

which, by integration, implies

$$h_1(x^2,x^3) = \bar{h}_1(x^3)F_2(x^2) + \hat{h}_1(x^3),$$

and, by replacing it in (17), it gives

$$-\bar{h}_1'(x^3)F_2(x^2) - f_1(x^1)\frac{\partial h_3}{\partial x^1}(x^1,x^2) = \hat{h}_1'(x^3).$$

Differentiating the previous relation with respect to $x^2$, it implies

$$\bar{h}_1'(x^3) = -f_1(x^1)f_2(x^2)\frac{\partial^2 h_3}{\partial x^2 \partial x^1}(x^1,x^2),$$

which must be a constant, let's say $c_1$. Therefore,

$$\bar{h}_1(x^3) = c_1x^3 + c_2$$

and

$$\frac{\partial^2 h_3}{\partial x^2 \partial x^1}(x^1,x^2) = -\frac{c_1}{f_1(x^1)f_2(x^2)},$$

which, by double integration, implies

$$h_3(x^1,x^2) = -c_1F_1(x^1)F_2(x^2) + l_1(x^1) + l_2(x^2).$$

Also, from (16), we have $\dfrac{\partial h_2}{\partial x^1}(x^1,x^3) = -\dfrac{\bar{h}_1(x^3)}{f_1(x^1)}$, which gives

$$h_2(x^1,x^3) = -\bar{h}_1(x^3)F_1(x^1) + \hat{h}_2(x^3).$$

From (17), we find

$$\hat{h}_1'(x^3) = -f_1(x^1)l_1'(x^1),$$

which must be a constant, let's say, $c_3$, which gives

$$\hat{h}_1(x^3) = c_3x^3 + c_4.$$

Since $f_1(x^1)l_1'(x^1) = -c_3$, we get

$$l_1(x^1) = -c_3F_1(x^1) + c_5.$$

From (18), we find

$$\hat{h}_2'(x^3) = 2c_1F_1(x^1) - f_2(x^2)l_2'(x^2),$$

which must be a constant, let's say, $c_6$, which gives

$$\hat{h}_2(x^3) = c_6x^3 + c_7.$$

Since $2c_1F_1(x^1) = f_2(x^2)l_2'(x^2) + c_6$ and $F_1$ is not constant, we get

$$c_1 = 0, \;\; l_2(x^2) = -c_6F_2(x^2) + c_8.$$

We have obtained

$$\begin{cases} V^1(x^1,x^2,x^3) = -\lambda F_1(x^1) + c_2F_2(x^2) + c_3x^3 + c_4 \\ V^2(x^1,x^2,x^3) = -\lambda F_2(x^2) - c_2F_1(x^1) + c_6x^3 + c_7 \\ V^3(x^1,x^2,x^3) = -(\lambda+\mu)x^3 - c_3F_1(x^1) + c_5 - c_6F_2(x^2) + c_8 \end{cases},$$

which, by changing the indices of some constants, gives (14). □

**Example 3.10.** *For $f_i = k_i \in \mathbb{R} \setminus \{0\}$, $i \in \{1,2\}$, the vector field $V$ with $V^1$, $V^2$ and $V^3$ given by:*

$$\begin{cases} V^1(x^1,x^2,x^3) = -\frac{\lambda}{k_1}x^1 + \frac{c_1}{k_2}x^2 + c_2x^3 + c_3 \\ V^2(x^1,x^2,x^3) = -\frac{c_1}{k_1}x^1 - \frac{\lambda}{k_2}x^2 + c_4x^3 + c_5 \\ V^3(x^1,x^2,x^3) = -\frac{c_2}{k_1}x^1 - \frac{c_4}{k_2}x^2 - (\lambda+\mu)x^3 + c_6 \end{cases},$$

*where $c_i \in \mathbb{R}$ for $i \in \{1,2,3,4,5,6\}$, is the potential vector field of the $\eta$-Ricci soliton $(I, \tilde{g}, \lambda, \mu)$ for $\eta = dx^3$.*

**Remark 3.11.** *If $f_i = f_i(x^i)$ for $i \in \{1,2\}$, then $V$ with $V^1$, $V^2$ and $V^3$ given by:*

$$\begin{cases} V^1(x^2,x^3) = c_1F_2(x^2) + c_2x^3 + c_3 \\ V^2(x^1,x^3) = -c_1F_1(x^1) + c_4x^3 + c_5 \\ V^3(x^1,x^2) = -c_2F_1(x^1) - c_4F_2(x^2) + c_6 \end{cases}$$

*is a Killing vector field on $(I, \tilde{g})$, where $F_i$ is an antiderivative of $\frac{1}{f_i}$ for $i \in \{1,2\}$ and $c_i \in \mathbb{R}$ for $i \in \{1,2,3,4,5,6\}$.*

**Theorem 3.12.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $\eta = dx^3$. If $f_i = f_i(x^3)$ for $i \in \{1,2\}$, and $V^i = V^i(x^3)$ for $i \in \{1,2,3\}$, then:*

$$\begin{cases} V^1 = \frac{c_1}{f_1} \\ V^2 = \frac{c_2}{f_2} \\ (b-d)V^3 = (b-d)' - (b^2-d^2) \\ (b-d)\lambda = b'd - bd' \\ (b-d)^2\mu = -(b-d)[(b-d)'' + b'd - bd'] + [(b-d)']^2 + (b-d)^2(b^2+d^2) \end{cases},$$

*where $c_1$, $c_2 \in \mathbb{R}$.*

*We have the following cases.*

*(i) On any open interval $J_3 \subseteq I_3$ on which $\left(\frac{f_1}{f_2}\right)' \neq 0$ everywhere (equivalent to $b \neq d$ on $J_3$), we have:*

$$V^1 = \frac{c_1}{f_1}, \ V^2 = \frac{c_2}{f_2}, \ V^3 = \frac{(b-d)'}{b-d} - (b+d),$$

$$\lambda = \frac{b'd - bd'}{b-d}, \ \mu = -\frac{(b-d)'' + b'd - bd'}{b-d} + \left(\frac{(b-d)'}{b-d}\right)^2 + b^2 + d^2,$$

*where $c_1$, $c_2 \in \mathbb{R}$.*

*In particular:*

*(1) if $f_1$ is constant on an open interval $J \subseteq J_3$ (from which $f_2' \neq 0$ everywhere on $J$), we have on $J$:*

$$V^1 = c_1, \ V^2 = \frac{c_2}{f_2}, \ V^3 = \frac{d' - d^2}{d},$$

$$\lambda = 0, \ \mu = -\frac{d''}{d} + \left(\frac{d'}{d}\right)^2 + d^2,$$

*where $c_1$, $c_2 \in \mathbb{R}$;*

*(2) if $f_2$ is constant on an open interval $J \subseteq J_3$ (from which $f_1' \neq 0$ everywhere on $J$), we have on $J$:*

$$V^1 = \frac{c_1}{f_1}, \ \ V^2 = c_2, \ \ V^3 = \frac{b' - b^2}{b},$$

$$\lambda = 0, \ \ \mu = -\frac{b''}{b} + \left(\frac{b'}{b}\right)^2 + b^2,$$

*where $c_1, c_2 \in \mathbb{R}$.*

*(ii) On any open interval $J_3 \subseteq I_3$ on which $\frac{f_1}{f_2}$ is constant (equivalent to $b = d$ on $J_3$), we have:*

$$V^1 = \frac{c_1}{f_1}, \ \ V^2 = \frac{c_2}{f_2}, \ \ bV^3 = b' - 2b^2 + \lambda, \ \ (V^3)' = -2(b' - b^2) - (\lambda + \mu),$$

*where $c_1, c_2 \in \mathbb{R}$.*

*In addition:*

*(1) if $J \subseteq J_3$ is an open interval with $f_1' \neq 0$ (and $f_2' \neq 0$) everywhere on $J$, we have on $J$:*

$$V^1 = \frac{c_1}{f_1}, \ \ V^2 = \frac{c_2}{f_2}, \ \ V^3 = \frac{b' - 2b^2 + \lambda}{b},$$

$$\lambda = -\left(\frac{b' + \lambda}{b}\right)' + 2b^2 - \mu,$$

*where $c_1, c_2 \in \mathbb{R}$;*

*(2) if $f_1$ and $f_2$ are constant on an open interval $J \subseteq J_3$, we have on $J$:*

$$V^1 = c_1, \ \ V^2 = c_2, \ \ V^3 = F,$$

$$\lambda = 0,$$

*where $F' = -\mu$ and $c_1, c_2 \in \mathbb{R}$.*

*Proof.* In this case, we have $a = 0$, $b = \frac{f_1'}{f_1}$, $c = 0$, $d = \frac{f_2'}{f_2}$, and

$$\mathrm{Ric}(E_1, E_1) = b' - b^2 - bd, \ \ \mathrm{Ric}(E_2, E_2) = d' - d^2 - bd, \ \ \mathrm{Ric}(E_3, E_3) = b' + d' - b^2 - d^2,$$

$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = \mathrm{Ric}(E_2, E_3) = 0,$$

and (13) becomes

$$\begin{cases} f_1 \dfrac{\partial V^1}{\partial x^1} - bV^3 + b' - b^2 - bd + \lambda = 0 \\ f_2 \dfrac{\partial V^2}{\partial x^2} - dV^3 + d' - d^2 - bd + \lambda = 0 \\ \dfrac{\partial V^3}{\partial x^3} + b' - b^2 + d' - d^2 + \lambda + \mu = 0 \\ f_1 \dfrac{\partial V^2}{\partial x^1} + f_2 \dfrac{\partial V^1}{\partial x^2} = 0 \\ f_1 \dfrac{\partial V^3}{\partial x^1} + \dfrac{\partial V^1}{\partial x^3} + bV^1 = 0 \\ f_2 \dfrac{\partial V^3}{\partial x^2} + \dfrac{\partial V^2}{\partial x^3} + dV^2 = 0 \end{cases},$$

which is equivalent to

$$\begin{cases}(V^1)' = -bV^1\\ (V^2)' = -dV^2\\ (V^3)' = -(b' - b^2 + d' - d^2) - (\lambda + \mu)\\ bV^3 = b' - b^2 - bd + \lambda\\ dV^3 = d' - d^2 - bd + \lambda\end{cases}. \tag{19}$$

From the first two equations, we get either $V^i = 0$ or $\frac{(V^i)'}{V^i} = -\frac{f_i'}{f_i}$, i.e., $V^i = \frac{c_i}{f_i}$, where $c_i \in \mathbb{R} \setminus \{0\}$ for $i \in \{1, 2\}$. From the last two equations, we obtain

$$(b - d)V^3 = (b - d)' - (b^2 - d^2)$$

and

$$(b - d)\lambda = b'd - bd'.$$

Differentiating the second-last equality, multiplying the result with $b-d$ and substituting in it the expressions of $(b - d)V^3$ and $(V^3)'$, we get

$$(b - d)^2\mu = -(b - d)[(b - d)'' + b'd - bd'] + [(b - d)']^2 + (b - d)^2(b^2 + d^2).$$

(i) For $x^3 \in I_3$, we notice that $\left(\frac{f_1}{f_2}\right)'(x^3) \neq 0$ if and only if $b(x^3) \neq d(x^3)$. From the last three equalities from above, on any open interval $J_3 \subseteq I_3$ such that $\left(\frac{f_1}{f_2}\right)'(x^3) \neq 0$ for any $x^3 \in J_3$ (equivalent to $b \neq d$ everywhere on $J_3$), we get the expressions of $V^3$, $\lambda$, and $\mu$.

(ii) We notice that $\frac{f_1}{f_2}$ is constant on $J_3$ if and only if $\left(\frac{f_1}{f_2}\right)' = 0$ on $J_3$ (equivalent to $b = d$ on $J_3$). In this case, (19) becomes

$$\begin{cases}(V^1)' = -bV^1\\ (V^2)' = -bV^2\\ (V^3)' = -2(b' - b^2) - (\lambda + \mu)\\ bV^3 = b' - 2b^2 + \lambda\end{cases}.$$

On any open interval $J \subseteq J_3$ where $f_1' \neq 0$ everywhere, we have $b \neq 0$ and

$$V^3 = \frac{b' - 2b^2 + \lambda}{b},$$

which, by differentiation and using the third relation, gives the expression of $\mu$.

If $f_1$ and $f_2$ are constant on $J \subseteq J_3$, then $b = d = 0$ on $J$ and (19) becomes

$$\begin{cases}\lambda = 0\\ (V^1)' = 0\\ (V^2)' = 0\\ (V^3)' = -\mu\end{cases}. \qquad \square$$

**Example 3.13.** *The Riemannian $Sol_3$ Lie group $(\mathbb{R}^3, \tilde{g})$, where $f_1(x^3) = e^{-x^3}$ and $f_2(x^3) = e^{x^3}$, is an $\eta$-Ricci soliton for $\eta = dx^3$, with*

$$V = c_1\frac{\partial}{\partial x^1} + c_2\frac{\partial}{\partial x^2}\ \ (c_1, c_2 \in \mathbb{R}),\ \ \lambda = 0,\ \ \mu = 2.$$

**Remark 3.14.** *If $f_i = f_i(x^3)$ for $i \in \{1, 2\}$, then $V$ with $V^1$, $V^2$ and $V^3$ given by:*

$$\begin{cases} V^1(x^3) = \dfrac{c_1}{f_1(x^3)} \\ V^2(x^3) = \dfrac{c_2}{f_2(x^3)} \\ V^3 = 0 \end{cases}$$

*is a Killing vector field on $(I, \tilde{g})$, where $c_i \in \mathbb{R}$ for $i \in \{1, 2\}$.*

**Theorem 3.15.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $\eta = dx^3$. If $f_1 = f_1(x^1)$, $f_2 = f_2(x^3)$, and $V^i = V^i(x^3)$ for $i \in \{1, 2, 3\}$, then:*

$$\begin{cases} V^1 = c_1 \\ V^2 = \dfrac{c_2}{f_2} \\ dV^3 = d' - d^2 \\ \lambda = 0 \\ d^2\mu = (d')^2 + d^4 - dd'' \end{cases},$$

*where $c_1, c_2 \in \mathbb{R}$.*

*We have the following cases.*

*(i) On any open interval $J_3 \subseteq I_3$ on which $f_2$ is constant, we have:*

$$V^1 = c_1, \;\; V^2 = c_2, \;\; V^3 = F,$$

$$\lambda = 0,$$

*where $F' = -\mu$ and $c_1, c_2 \in \mathbb{R}$.*

*(ii) On any open interval $J_3 \subseteq I_3$ on which $f_2' \neq 0$ everywhere (equivalent to $d \neq 0$ on $J_3$), we have:*

$$V^1 = c_1, \;\; V^2 = \frac{c_2}{f_2}, \;\; V^3 = \frac{d' - d^2}{d},$$

$$\lambda = 0, \;\; \mu = -\frac{d''}{d} + \left(\frac{d'}{d}\right)^2 + d^2,$$

*where $c_1, c_2 \in \mathbb{R}$.*

*Proof.* In this case, we have $a = b = c = 0$, $d = \dfrac{f_2'}{f_2}$, and

$$\text{Ric}(E_2, E_2) = \text{Ric}(E_3, E_3) = d' - d^2,$$

$$\text{Ric}(E_1, E_1) = \text{Ric}(E_1, E_2) = \text{Ric}(E_1, E_3) = \text{Ric}(E_2, E_3) = 0,$$

and (13) becomes

$$\begin{cases} f_1\dfrac{\partial V^1}{\partial x^1}+\lambda=0\\ f_2\dfrac{\partial V^2}{\partial x^2}-dV^3+d'-d^2+\lambda=0\\ \dfrac{\partial V^3}{\partial x^3}+d'-d^2+\lambda+\mu=0\\ f_1\dfrac{\partial V^2}{\partial x^1}+f_2\dfrac{\partial V^1}{\partial x^2}=0\\ f_1\dfrac{\partial V^3}{\partial x^1}+\dfrac{\partial V^1}{\partial x^3}=0\\ f_2\dfrac{\partial V^3}{\partial x^2}+\dfrac{\partial V^2}{\partial x^3}+dV^2=0 \end{cases},$$

equivalent to

$$\begin{cases} \lambda=0\\ (V^1)'=0\\ (V^2)'=-dV^2\\ (V^3)'=-d'+d^2-\mu\\ dV^3=d'-d^2 \end{cases}, \tag{20}$$

from which we get

$$V^1=c_1,\;\; V^2=\frac{c_2}{f_2},$$

where $c_1, c_2 \in \mathbb{R}$. Differentiating the last equality of the previous system, multiplying the result with $d$ and substituting in it the expressions of $dV^3$ and $(V^3)'$, we get

$$d^2\mu=(d')^2+d^4-dd''.$$

(i) We notice that $f_2$ is constant on $J_3$ if and only if $d=0$ on $J_3$, and (20) becomes

$$\begin{cases} \lambda=0\\ (V^1)'=0\\ (V^2)'=0\\ (V^3)'=-\mu \end{cases}.$$

(ii) We notice that $f_2'\neq 0$ on $J_3$ if and only if $d\neq 0$ on $J_3$. Dividing the relations for $dV^3$ and $d^2\mu$ by $d$ and $d^2$, respectively, we obtain the expressions of $V^3$ and $\mu$. □

**Example 3.16.** *For $f_1(x^1)=e^{x^1}$, $f_2(x^3)=e^{x^3}$, the vector field*

$$V=c_1e^{x^1}\frac{\partial}{\partial x^1}+c_2\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}\;\;(c_1,c_2\in\mathbb{R})$$

*is the potential vector field of the $\eta$-Ricci soliton $(\mathbb{R}^3,\tilde{g},0,1)$ for $\eta=dx^3$.*

**Theorem 3.17.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $\eta = dx^3$. If $f_i = f_i(x^2)$ for $i \in \{1,2\}$ and $V^i = V^i(x^3)$ for $i \in \{1,2,3\}$, then:*

$$\begin{cases} V^1 = c_1 \\ V^2 = c_2 \\ V^3 = F \\ c_1 f_1' = 0 = c_2 f_1' \\ \lambda = 2f_2^2 \left(\frac{f_1'}{f_1}\right)^2 - f_2 f_2' \frac{f_1'}{f_1} - f_2^2 \frac{f_1''}{f_1} \end{cases},$$

*where $c_1, c_2 \in \mathbb{R}$, $F' = G$, $\mu(x^2, x^3) = -\lambda(x^2) - G(x^3)$, with $G = G(x^3)$ a smooth function on $I_3$.*

*We have the following cases.*

*(i) For $J_2 \subseteq I_2$ a nontrivial interval on which $f_1$ is constant, we have*

$$\lambda(x^2) = 0, \quad F'(x^3) = -\mu(x^2, x^3) \ \text{for } x^2 \in J_2, x^3 \in I_3.$$

*(ii) For $f_1$ constant, we have*

$$\lambda = 0, \quad \mu = \mu(x^3), \quad F'(x^3) = -\mu(x^3) \ \text{for } x^3 \in I_3.$$

*(iii) For $f_1$ not constant, we have*

$$V^1 = V^2 = 0.$$

*Proof.* In this case, we have $a = f_2 \frac{f_1'}{f_1}$, $b = c = d = 0$, and

$$\text{Ric}(E_1, E_1) = \text{Ric}(E_2, E_2) = E_2(a) - a^2,$$

$$\text{Ric}(E_1, E_2) = \text{Ric}(E_1, E_3) = \text{Ric}(E_2, E_3) = \text{Ric}(E_3, E_3) = 0,$$

and (13) becomes

$$\begin{cases} f_1 \frac{\partial V^1}{\partial x^1} - aV^2 + f_2 a' - a^2 + \lambda = 0 \\ f_2 \frac{\partial V^2}{\partial x^2} + f_2 a' - a^2 + \lambda = 0 \\ \frac{\partial V^3}{\partial x^3} + \lambda + \mu = 0 \\ f_1 \frac{\partial V^2}{\partial x^1} + f_2 \frac{\partial V^1}{\partial x^2} + aV^1 = 0 \\ f_1 \frac{\partial V^3}{\partial x^1} + \frac{\partial V^1}{\partial x^3} = 0 \\ f_2 \frac{\partial V^3}{\partial x^2} + \frac{\partial V^2}{\partial x^3} = 0 \end{cases},$$

that is,

$$\begin{cases} aV^1 = 0 \\ aV^2 = 0 \\ (V^1)' = 0 \\ (V^2)' = 0 \\ (V^3)' = -(\lambda + \mu) \\ f_2 a' - a^2 + \lambda = 0 \end{cases}.$$

From the second-last equation, we deduce that $\lambda + \mu$ depends only on $x^3$; therefore,

$$0 = \frac{\partial(\lambda + \mu)}{\partial x^2} = \lambda' + \frac{\partial \mu}{\partial x^2} \, ,$$

which, by integration, gives

$$\mu(x^2, x^3) = -\lambda(x^2) - G(x^3)$$

with $G = G(x^3)$ a smooth function on $I_3$. We obtain

$$\begin{cases} V^1 = c_1 \\ V^2 = c_2 \\ V^3 = F \\ aV^1 = 0 \\ aV^2 = 0 \\ \lambda = -f_2 a' + a^2 \end{cases} , \tag{21}$$

where $F' = G$ and $c_1, c_2 \in \mathbb{R}$.

(i) We notice that $f_1$ is constant on $J_2$ if and only if $a = 0$ on $J_2$; therefore, $a' = 0$ and $\lambda = 0$, hence $F' = -\mu$ on $J_2$.

(ii) It follows from (i) for $J_2 = I_2$.

(iii) For $f_1$ not constant, there exists $x_0^2 \in I_2$ such that $f_1'(x_0^2) \neq 0$, i.e., $a(x_0^2) \neq 0$, and from the second- and third-last equations of (21), it follows that $V_1 = V_2 = 0$. □

**Example 3.18.** *The manifold* $\mathbb{H}^2 \times \mathbb{R} := \{(x^1, x^2, x^3) \in \mathbb{R}^3 \ : \ x^2 > 0\}$ *with*

$$\tilde{g} = \frac{1}{f^2} dx^1 \otimes dx^1 + \frac{1}{f^2} dx^2 \otimes dx^2 + dx^3 \otimes dx^3,$$

*where* $f(x^2) = x^2$, *is an* $\eta$*-Ricci soliton for* $\eta = dx^3$, *with*

$$V = \frac{\partial}{\partial x^3}, \ \ \lambda = 1, \ \ \mu = -1.$$

**Theorem 3.19.** *Let* $(I, \tilde{g}, V, \lambda, \mu)$ *be an almost* $\eta$*-Ricci soliton with* $\eta = dx^3$. *If* $f_1 = f_1(x^2)$, $f_2 = f_2(x^1)$, *and* $V^i = V^i(x^3)$ *for* $i \in \{1, 2, 3\}$, *then:*

$$\begin{cases} V^1 = c_1 \\ V^2 = c_2 \\ V^3 = F \\ c_1 f_2 \dfrac{f_1'}{f_1} = -c_2 f_1 \dfrac{f_2'}{f_2} \\ c_2 f_2 \dfrac{f_1'}{f_1} = c_1 f_1 \dfrac{f_2'}{f_2} \\ \lambda = c_1 f_1 \dfrac{f_2'}{f_2} + f_1^2 \left[ \left( \dfrac{f_2'}{f_2} \right)^2 - \left( \dfrac{f_2'}{f_2} \right)' \right] + f_2^2 \left[ \left( \dfrac{f_1'}{f_1} \right)^2 - \left( \dfrac{f_1'}{f_1} \right)' \right] \end{cases} ,$$

*where* $c_1, c_2 \in \mathbb{R}$ *and* $F' = -(\lambda + \mu)$.

*For* $c_1 \neq 0$ *or* $c_2 \neq 0$, *we get* $f_1$ *and* $f_2$ *constant,* $\lambda = 0$, *and* $F' = -\mu$.

*Proof.* In this case, we have $a = f_2\frac{f_1'}{f_1}$, $b = 0$, $c = f_1\frac{f_2'}{f_2}$, $d = 0$, and

$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = E_1(c) + E_2(a) - a^2 - c^2,$$

$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = \mathrm{Ric}(E_2, E_3) = \mathrm{Ric}(E_3, E_3) = 0,$$

and (13) becomes

$$\begin{cases} E_1(V^1) - aV^2 + E_1(c) + E_2(a) - a^2 - c^2 + \lambda = 0 \\ E_2(V^2) - cV^1 + E_1(c) + E_2(a) - a^2 - c^2 + \lambda = 0 \\ E_3(V^3) + \lambda + \mu = 0 \\ E_1(V^2) + E_2(V^1) + aV^1 + cV^2 = 0 \\ E_1(V^3) + E_3(V^1) = 0 \\ E_2(V^3) + E_3(V^2) = 0 \end{cases}. \tag{22}$$

Since $V^i$ for $i \in \{1, 2, 3\}$ depends only on $x^3$, (22) becomes

$$\begin{cases} aV^2 = f_1\frac{\partial c}{\partial x^1} + f_2\frac{\partial a}{\partial x^2} - a^2 - c^2 + \lambda = cV^1 \\ aV^1 = -cV^2 \\ (V^1)' = 0 \\ (V^2)' = 0 \\ (V^3)' = -(\lambda + \mu) \end{cases}.$$

From the last three equations, we infer $V^1 = c_1$, $V^2 = c_2$, where $c_1, c_2 \in \mathbb{R}$, and $V^3 = F$, where $F' = -(\lambda + \mu)$, and the previous system is equivalent to

$$\begin{cases} V^1 = c_1 \\ V^2 = c_2 \\ \lambda = c_1 f_1\frac{f_2'}{f_2} + f_1^2\left[\left(\frac{f_2'}{f_2}\right)^2 - \left(\frac{f_2'}{f_2}\right)'\right] + f_2^2\left[\left(\frac{f_1'}{f_1}\right)^2 - \left(\frac{f_1'}{f_1}\right)'\right] \\ c_2 a = c_1 c \\ c_1 a = -c_2 c \\ V^3 = F \end{cases}.$$

We get $(c_1^2 + c_2^2)a = 0$. If $c_1 \neq 0$ or $c_2 \neq 0$, then $a = 0$, i.e., $f_1$ is constant, and further, $c = 0$, i.e., $f_2$ is constant. In this case,

$$\lambda = 0, \quad V^3 = F,$$

where $F' = -\mu$. So, we get the conclusion. □

**Example 3.20.** *For $f_1(x^2) = e^{x^2}$ and $f_2(x^1) = e^{x^1}$, the vector field*

$$V = \frac{1}{2}e^{2x^3}\frac{\partial}{\partial x^3}$$

*is the potential vector field of the almost $\eta$-Ricci soliton*

$$\left(\mathbb{R}^3, \tilde{g}, \lambda(x^1, x^2) = e^{2x^1} + e^{2x^2}, \mu(x^1, x^2, x^3) = -(e^{2x^1} + e^{2x^2} + e^{2x^3})\right)$$

*for $\eta = dx^3$.*

**Theorem 3.21.** *Let $(I, \tilde{g}, V, \lambda, \mu)$ be an almost $\eta$-Ricci soliton with $\eta = dx^3$. If $f_1 = f_1(x^2)$, $f_2 = f_2(x^3)$, and $V^i = V^i(x^3)$ for $i \in \{1, 2, 3\}$, then:*

$$\begin{cases} V^1 = c_1 \\ V^2 = c_2 f_2 + \dfrac{c_3}{f_2} \\ aV^1 = 0 \\ dV^3 = aV^2 + d' - d^2 \\ a = -c_2 f_2 + F \\ \lambda = aV^2 - f_2 F' + a^2 \\ (V^3)' = -(d' - d^2) - (\lambda + \mu) \end{cases},$$

*where $c_1, c_2, c_3 \in \mathbb{R}$, and $F$ is a smooth real function on $I_2$. Moreover,*

*(i) if $f_2$ is constant, then $V^2$ is constant;*

*(ii) if $f_2$ is not constant, then $f_1(x^2) = c_4 e^{-c_2 x^2}$, where $c_4 \in \mathbb{R} \setminus \{0\}$, and $a = -c_2 f_2$;*

*(iii) if $V^1 \neq 0$, then $a = 0$, $dV^3 = d' - d^2$, $\lambda = -f_2 F'$, and $f_1$ is constant. If, in addition, $f_2$ is not constant, then $V^2 = \dfrac{c_3}{f_2}$, $F = 0$, $\lambda = 0$, and $\mu = -(V^3)' - (d' - d^2)$.*

*Proof.* In this case, we have $a = f_2 \dfrac{f_1'}{f_1}$, $b = c = 0$, $d = \dfrac{f_2'}{f_2}$, and

$$\mathrm{Ric}(E_1, E_1) = E_2(a) - a^2, \ \ \mathrm{Ric}(E_2, E_2) = E_2(a) - a^2 + E_3(d) - d^2, \ \ \mathrm{Ric}(E_3, E_3) = E_3(d) - d^2,$$
$$\mathrm{Ric}(E_1, E_2) = \mathrm{Ric}(E_1, E_3) = 0, \ \ \mathrm{Ric}(E_2, E_3) = E_3(a),$$

and (13) becomes

$$\begin{cases} f_1 \dfrac{\partial V^1}{\partial x^1} - aV^2 + f_2 \dfrac{\partial a}{\partial x^2} - a^2 + \lambda = 0 \\ f_2 \dfrac{\partial V^2}{\partial x^2} - dV^3 + d' - d^2 + f_2 \dfrac{\partial a}{\partial x^2} - a^2 + \lambda = 0 \\ \dfrac{\partial V^3}{\partial x^3} + d' - d^2 + \lambda + \mu = 0 \\ f_1 \dfrac{\partial V^2}{\partial x^1} + f_2 \dfrac{\partial V^1}{\partial x^2} + aV^1 = 0 \\ f_1 \dfrac{\partial V^3}{\partial x^1} + \dfrac{\partial V^1}{\partial x^3} = 0 \\ \dfrac{1}{2}\left[ f_2 \dfrac{\partial V^3}{\partial x^2} + \dfrac{\partial V^2}{\partial x^3} + dV^2 \right] + \dfrac{\partial a}{\partial x^3} = 0 \end{cases},$$

which is equivalent to

$$\begin{cases} (V^1)' = 0 \\ aV^1 = 0 \\ (V^2)' + dV^2 = -2 \dfrac{\partial a}{\partial x^3} \\ aV^2 = f_2 \dfrac{\partial a}{\partial x^2} - a^2 + \lambda \\ (V^3)' = -(d' - d^2) - (\lambda + \mu) \\ dV^3 = f_2 \dfrac{\partial a}{\partial x^2} - a^2 + d' - d^2 + \lambda \end{cases}. \tag{23}$$

From the first equation, it follows that $V^1$ is constant. From the third equation, we deduce that $\frac{\partial a}{\partial x^3}$ depends only on $x^3$, while $a$ depends on $x^2$ and $x^3$. On the other hand, we have $\frac{\partial a}{\partial x^3} = f_2' \frac{f_1'}{f_1}$. Since $\frac{f_1'}{f_1}$ depends only on $x^2$, if there exists $x_0^3 \in I_3$ such that $f_2'(x_0^3) \neq 0$, then $\frac{f_1'}{f_1}$ is constant on $I_2$, let's say $\frac{f_1'}{f_1} = c_1$ with $c_1 \in \mathbb{R}$. So, in the case $f_2$ not constant, we infer that $f_1(x^2) = c_2 e^{c_1 x^2}$, where $c_2 \in \mathbb{R} \setminus \{0\}$, and $a = c_1 f_2$. Hence, $\frac{\partial a}{\partial x^3} = c_1 f_2'$, and we obtain the equation

$$(V^2)' + \frac{f_2'}{f_2} V^2 = -2c_1 f_2',$$

with the solution $V^2 = -c_1 f_2 + \frac{c_3}{f_2}$, where $c_3 \in \mathbb{R}$. If $f_2$ is constant on $I_3$, we obtain $\frac{\partial a}{\partial x^3} = 0$ and $d = 0$, and the third equation of the system becomes $(V^2)' = 0$, so $V^2 = c_4$, where $c_4 \in \mathbb{R}$. We notice that we always have

$$\frac{\partial a}{\partial x^3} = c_1 f_2' \ \text{ and } \ V^2 = -c_1 f_2 + \frac{c_3}{f_2},$$

where $c_1, c_3 \in \mathbb{R}$. We infer that $a(x^2, x^3) = c_1 f_2 + F(x^2)$, where $F$ is a smooth real function on $I_2$. Hence, $\frac{\partial a}{\partial x^2} = F'$, and, from the fourth equation of the system, we get

$$\lambda = aV^2 - f_2 F' + a^2.$$

The fifth equation of (23) gives

$$\mu = -(V^3)' - (d' - d^2) - \lambda,$$

while, from the last equation of the system and the expression of $\lambda$, we get

$$dV^3 = aV^2 + d' - d^2,$$

that is,

$$dV^3 = -c_1 f_2^2 \frac{f_1'}{f_1} + c_3 \frac{f_1'}{f_1} + d' - d^2. \qquad \square$$

**Example 3.22.** *For $f_1(x^2) = e^{x^2}$ and $f_2(x^3) = e^{x^3}$, the vector field*

$$V = (1 - e^{2x^3}) \frac{\partial}{\partial x^2} - e^{2x^3} \frac{\partial}{\partial x^3}$$

*is a potential vector field of the almost $\eta$-Ricci soliton*

$$\left(\mathbb{R}^3, \tilde{g}, \lambda = 1, \mu(x^3) = 2e^{2x^3}\right)$$

*for $\eta = dx^3$.*

### *3.3. Almost $\eta$-Ricci solitons with $\eta = dx^3$ and $V = \frac{\partial}{\partial x^3}$*

If $V^1 = V^2 = \eta^1 = \eta^2 = 0$ and $V^3 = \eta^3 = 1$, then the system (2) becomes

$$\begin{cases} -b + \mathrm{Ric}(E_1, E_1) + \lambda = 0 \\ -d + \mathrm{Ric}(E_2, E_2) + \lambda = 0 \\ \mathrm{Ric}(E_3, E_3) + \lambda + \mu = 0 \\ \mathrm{Ric}(E_1, E_3) = 0 \\ \mathrm{Ric}(E_2, E_3) = 0 \end{cases}. \tag{24}$$

**Theorem 3.23.** *Let $V = \frac{\partial}{\partial x^3}$ and $\eta = dx^3$. If $f_i = f_i(x^3)$ for $i \in \{1,2\}$, then $(I, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$\lambda = \frac{f_1'}{f_1}\left(\frac{f_2'}{f_2}+1\right) - \frac{f_1''}{f_1} + 2\left(\frac{f_1'}{f_1}\right)^2 \ \text{and} \ \mu = -\frac{f_1'}{f_1}\left(\frac{f_2'}{f_2}+1\right) - \frac{f_2''}{f_2} + 2\left(\frac{f_2'}{f_2}\right)^2,$$

*and $f_1$ and $f_2$ satisfy*

$$\frac{f_1'' - f_1'}{f_1} - 2\left(\frac{f_1'}{f_1}\right)^2 = \frac{f_2'' - f_2'}{f_2} - 2\left(\frac{f_2'}{f_2}\right)^2. \tag{25}$$

*Proof.* In this case, we have $a = 0$, $b = \frac{f_1'}{f_1}$, $c = 0$, $d = \frac{f_2'}{f_2}$, and (24) becomes

$$\begin{cases} -b + \mathrm{Ric}(E_1, E_1) + \lambda = 0 \\ -d + \mathrm{Ric}(E_2, E_2) + \lambda = 0 \,, \\ \mathrm{Ric}(E_3, E_3) + \lambda + \mu = 0 \end{cases} \tag{26}$$

with

$$\mathrm{Ric}(E_1, E_1) = \frac{f_1''}{f_1} - 2\left(\frac{f_1'}{f_1}\right)^2 - \frac{f_1'}{f_1}\frac{f_2'}{f_2},$$
$$\mathrm{Ric}(E_2, E_2) = \frac{f_2''}{f_2} - 2\left(\frac{f_2'}{f_2}\right)^2 - \frac{f_1'}{f_1}\frac{f_2'}{f_2},$$
$$\mathrm{Ric}(E_3, E_3) = \frac{f_1''}{f_1} - 2\left(\frac{f_1'}{f_1}\right)^2 + \frac{f_2''}{f_2} - 2\left(\frac{f_2'}{f_2}\right)^2.$$

The two functions $f_1$ and $f_2$ must satisfy (25) due to the first two equations of (26), and we obtain the expressions of $\lambda$ and $\mu$ from the first and the last equation of the same system. □

**Example 3.24.** *For $f_i(x^3) = k_i e^{kx^3}$, $i \in \{1,2\}$, $k, k_1, k_2 \in \mathbb{R} \setminus \{0\}$, $V = \frac{\partial}{\partial x^3}$, and $\eta = dx^3$, $(\mathbb{R}^3, \tilde{g}, V, \lambda, \mu)$ is an $\eta$-Ricci soliton if and only if*

$$\lambda = 2k^2 + k \ \text{and} \ \mu = -k.$$

**Corollary 3.25.** *Under the hypotheses of Theorem 3.23, if $f_1 = f_2 =: f(x^3)$, then $(I, \tilde{g}, V)$ is an almost $\eta$-Ricci soliton with $\lambda$ and $\mu$ as scalar functions if and only if they are given by*

$$\lambda = 3\left(\frac{f'}{f}\right)^2 + \frac{f' - f''}{f} \ \text{and} \ \mu = \left(\frac{f'}{f}\right)^2 - \frac{f' + f''}{f}.$$

*Proof.* It follows immediately from Theorem 3.23. □

**Theorem 3.26.** *Let $V = \frac{\partial}{\partial x^3}$ and $\eta = dx^3$. If $f_1 = f_1(x^1)$, $f_2 = f_2(x^3)$, then $(I, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$f_2(x^3) = c_0 e^{-x^3}, \ \lambda = 0, \ \mu = 1,$$

*where $c_0 \in \mathbb{R} \setminus \{0\}$, or*

$$f_2(x^3) = \frac{c_1}{c_2 e^{x^3} + 1}, \ \lambda = 0, \ \mu(x^3) = \frac{c_2 e^{x^3}}{c_2 e^{x^3} + 1},$$

*where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ such that $c_2 \neq -e^{-x^3}$ for all $x^3 \in I_3$.*

*In particular, for $I_3 = \mathbb{R}$, we additionally have the condition $c_2 > 0$.*

*Proof.* In this case, we have $a = b = c = 0$, $d = \frac{f_2'}{f_2}$, and (24) becomes

$$\begin{cases} \lambda = 0 \\ -d + d' - d^2 = 0 \\ d' - d^2 + \mu = 0 \end{cases}.$$

If $d = 0$, then $f_2$ is constant, and $\lambda = \mu = 0$ (which contradicts $\mu \neq 0$). If $d = -1$, then $\ln|f_2(x^3)| = -x^3 + k_0$, where $k_0 \in \mathbb{R}$, and we get $f_2(x^3) = \pm e^{-x^3+k_0}$ and $\mu = 1$. If $d \neq -1$ and $d \neq 0$, from the second equation of the above system, we get

$$1 = \frac{d'}{d^2+d} = \frac{d'}{d} - \frac{d'}{d+1},$$

which, by integration, gives

$$\ln\left(\left|\frac{d(x^3)}{d(x^3)+1}\right|\right) = x^3 + k_1,$$

where $k_1 \in \mathbb{R}$, from which

$$(1 - k_2 e^{x^3})d(x^3) = k_2 e^{x^3},$$

where $k_2 \in \mathbb{R} \setminus \{0\}$ and for all $x^3 \in I_3$. This infers

$$\frac{f_2'(x^3)}{f_2(x^3)} = d(x^3) = \frac{k_2 e^{x^3}}{1 - k_2 e^{x^3}},$$

where $k_2 \neq e^{-x^3}$ for all $x^3 \in I_3$, and, by integrating, we get

$$f_2(x^3) = \frac{k_3}{1 - k_2 e^{x^3}},$$

where $k_3 \in \mathbb{R} \setminus \{0\}$, and

$$\mu(x^3) = d^2(x^3) - d'(x^3) = -d(x^3) = \frac{-k_2 e^{x^3}}{1 - k_2 e^{x^3}}. \qquad \square$$

**Theorem 3.27.** *Let $V = \frac{\partial}{\partial x^3}$ and $\eta = dx^3$. If $f_i = f_i(x^2)$ for $i \in \{1, 2\}$, then $(I, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$\lambda = -f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2 + \frac{f_1'}{f_1}\frac{f_2'}{f_2}\right] \text{ and } \mu = f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2 + \frac{f_1'}{f_1}\frac{f_2'}{f_2}\right].$$

*Proof.* In this case, we have $a = f_2\frac{f_1'}{f_1}$, $b = c = d = 0$, and (24) becomes

$$\begin{cases} \mathrm{Ric}(E_1, E_1) + \lambda = 0 \\ \mathrm{Ric}(E_2, E_2) + \lambda = 0 \\ \lambda + \mu = 0 \end{cases},$$

with

$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2 + \frac{f_1'}{f_1}\frac{f_2'}{f_2}\right]. \qquad \square$$

**Example 3.28.** *For $f_1(x^2) = k_1 e^{-x^2}$, $f_2(x^2) = k_2 e^{x^2}$, $k_1, k_2 \in \mathbb{R} \setminus \{0\}$, $V = \frac{\partial}{\partial x^3}$, and $\eta = dx^3$, $(\mathbb{R}^3, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$\lambda(x^2) = 2k_2^2 e^{2x^2} \text{ and } \mu(x^2) = -2k_2^2 e^{2x^2}.$$

**Corollary 3.29.** *Under the hypotheses of Theorem 3.27, if $f_1 = f_2 =: f(x^2)$, then $(I, \tilde{g}, V)$ is an almost $\eta$-Ricci soliton with $\lambda$ and $\mu$ as scalar functions if and only if they are given by*

$$\lambda = (f')^2 - f''f \text{ and } \mu = -(f')^2 + f''f.$$

*Proof.* It follows immediately from Theorem 3.27. □

**Theorem 3.30.** *Let $V = \frac{\partial}{\partial x^3}$ and $\eta = dx^3$. If $f_1 = f_1(x^2)$ and $f_2 = f_2(x^1)$, then $(I, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$\lambda = -f_1^2\left[\left(\frac{f_2'}{f_2}\right)' - \left(\frac{f_2'}{f_2}\right)^2\right] - f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right] \text{ and } \mu = f_1^2\left[\left(\frac{f_2'}{f_2}\right)' - \left(\frac{f_2'}{f_2}\right)^2\right] + f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right].$$

*Proof.* In this case, we have $a = f_2\frac{f_1'}{f_1}$, $b = 0$, $c = f_1\frac{f_2'}{f_2}$, $d = 0$, and (24) becomes

$$\begin{cases} \mathrm{Ric}(E_1, E_1) + \lambda = 0 \\ \mathrm{Ric}(E_2, E_2) + \lambda = 0 \\ \lambda + \mu = 0 \end{cases},$$

with

$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = f_1^2\left[\left(\frac{f_2'}{f_2}\right)' - \left(\frac{f_2'}{f_2}\right)^2\right] + f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right].$$ □

**Example 3.31.** *For $f_1(x^2) = e^{x^2}$, $f_2(x^1) = e^{x^1}$, $V = \frac{\partial}{\partial x^3}$, and $\eta = dx^3$, $(\mathbb{R}^3, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$\lambda(x^1, x^2) = e^{2x^1} + e^{2x^2} \text{ and } \mu(x^1, x^2) = -e^{2x^1} - e^{2x^2}.$$

**Theorem 3.32.** *Let $V = \frac{\partial}{\partial x^3}$ and $\eta = dx^3$. If $f_1 = f_1(x^2)$ and $f_2 = f_2(x^3)$, then $(I, \tilde{g}, V, \lambda, \mu)$ is an almost $\eta$-Ricci soliton if and only if*

$$d' = d(d+1), \quad f_1'f_2' = 0,$$

$$\lambda = -f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right] \text{ and } \mu = f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right] - \left[\left(\frac{f_2'}{f_2}\right)' - \left(\frac{f_2'}{f_2}\right)^2\right].$$

*Moreover, one of the following cases is satisfied:*
*(i) $f_2 = k_2 \in \mathbb{R} \setminus \{0\}$ and*

$$\lambda = -f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right], \quad \mu = f_2^2\left[\left(\frac{f_1'}{f_1}\right)' - \left(\frac{f_1'}{f_1}\right)^2\right];$$

*(ii) $f_1 = k_1 \in \mathbb{R} \setminus \{0\}$, $f_2(x^3) = \frac{c_1}{c_2e^{x^3} + 1}$, where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ such that $c_2 \neq -e^{-x^3}$ for all $x^3 \in I_3$, and*

$$\lambda = 0, \quad \mu = \frac{c_2e^{x^3}}{c_2e^{x^3} + 1};$$

*(iii) $f_1 = k_1 \in \mathbb{R} \setminus \{0\}$, $f_2(x^3) = c_0e^{-x^3}$, where $c_0 \in \mathbb{R} \setminus \{0\}$, and*

$$\lambda = 0, \quad \mu = 1.$$

*Proof.* In this case, we have $a = f_2 \frac{f_1'}{f_1}$, $b = c = 0$, $d = \frac{f_2'}{f_2}$, and (24) becomes

$$\begin{cases} \mathrm{Ric}(E_1, E_1) + \lambda = 0 \\ -d + \mathrm{Ric}(E_2, E_2) + \lambda = 0 \\ \mathrm{Ric}(E_3, E_3) + \lambda + \mu = 0 \\ \mathrm{Ric}(E_2, E_3) = 0 \end{cases}, \tag{27}$$

with

$$\mathrm{Ric}(E_1, E_1) = f_2^2 \left[ \left( \frac{f_1'}{f_1} \right)' - \left( \frac{f_1'}{f_1} \right)^2 \right],$$
$$\mathrm{Ric}(E_2, E_2) = f_2^2 \left[ \left( \frac{f_1'}{f_1} \right)' - \left( \frac{f_1'}{f_1} \right)^2 \right] + \left( \frac{f_2'}{f_2} \right)' - \left( \frac{f_2'}{f_2} \right)^2,$$
$$\mathrm{Ric}(E_3, E_3) = \left( \frac{f_2'}{f_2} \right)' - \left( \frac{f_2'}{f_2} \right)^2,$$
$$\mathrm{Ric}(E_2, E_3) = \frac{f_1' f_2'}{f_1}.$$

$f_2$ must satisfy $d' = d(d + 1)$ due to the first two equations of (27), and we obtain the cases:
(i) $f_2$ constant;
(ii) $f_1$ constant and $f_2(x^3) = \frac{c_1}{c_2 e^{x^3} + 1}$ on $I_3$, where $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ and $e^{x^3} \neq -\frac{1}{c_2}$ on $I_3$;
(iii) $f_1$ constant and $f_2(x^3) = c_0 e^{-x^3}$, where $c_0 \in \mathbb{R} \setminus \{0\}$. □

**Example 3.33.** *For $f_1 = 1$, $f_2(x^3) = e^{-x^3}$, $V = \frac{\partial}{\partial x^3}$, and $\eta = dx^3$, $(\mathbb{R}^3, \tilde{g}, V, \lambda, \mu)$ is an $\eta$-Ricci soliton if and only if*

$$\lambda = 0 \ \text{ and } \ \mu = 1.$$

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