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Another approach: Modeling the change in electric potential in one-dimensional space by using Green's functions

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Abstract. The change in electric potential as a result of lightning in a one-dimensional domain is evaluated. Green's function is used to evaluate the new potential. The change in the electric potential is a constant along the lightning channel, and it is the same as the pre-flash potential outside the channel. The governing equation for the electric potential is obtained from Maxwell's equations.

1. Introduction

Lightning is a process in which the electric charge buildup in two charge centers increase and eventually reach a breakdown threshold, leading to a lightning discharge. This forms a lightning channel, it is what we see as the lightning. During this process the conductivity along the lightning channel becomes very large, therefore causing a change in the electric potential as well. In this study, we develop a new approach to obtain a formula for the change in the electric potential immediately after lighting in a one-dimensional space by using Green's function, see [1], [3], [5]. To this end, we solve the following system:

$$\begin{aligned}
\dot{\phi}_{xx} &= -(\sigma\phi_x)_x \quad (x,t) \in [0,1]\times[0,\infty), && \text{(1a)}\\
\phi(0,t) &= 0 \quad t \in [0,\infty), && \text{(1b)}\\
\phi(1,t) &= 0 \quad t \in [0,\infty), && \text{(1c)}\\
\phi(x,0) &= \phi_0(x) \quad x \in [0,1]. && \text{(1d)}
\end{aligned}$$

The initial potential ϕ_0 lies in the space

$$\mathrm{L}^2_0([0,1]) = \{f \in \mathrm{L}^2([0,1]) | \int_0^1 f(x)\, dx = 0\},$$

where $\mathrm{L}^2([0,1])$ is the usual space of square integrable functions on $[0,1]$ and $\sigma > 0$ lies in the space $\mathrm{L}^\infty([0,1])$ of essentially bounded functions on $[0,1]$.

In the moments after a lightning discharge, the conductivity along the lightning channel becomes infinitely large. In our domain, we assume lightning channel $\mathcal{L}$ is centered around an interior point x_0,

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$\mathcal{L} = [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]$, with $\mathcal{L}^c = [0,1]\backslash[x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]$. Therefore, we write $\sigma = \sigma + \tau\Psi$, where Ψ is the characteristic function of $\mathcal{L}$, that is $\Psi = 0$ everywhere except on $\mathcal{L}$, and τ is a large scalar. If lightning occurs at time $t = 0$, then in the moments after lightning the electric potential is governed by

$$\dot{\phi}_{xx} = -((\sigma + \tau\Psi)\phi_x)_x \quad (x,t) \in [0,1] \times [0,\infty), \tag{2}$$

subject to the boundary conditions (1b) - (1d). If $\phi^\tau(x,t)$ is the solution to (2), then the potential after the lightning is given by

$$\phi^+(x) = \lim_{t \to 0^+} \lim_{\tau \to \infty} \phi^\tau(x,t) \tag{3}$$

assuming lightning is very fast.

In 2021, Aslan [2] obtained a formula that computes the change in electric potential due to lightning in one-dimensional space of $[0,1]$. They solved the problem (1a) - (1d) and obtained the following result for $\phi^+(x)$.

Theorem 1.1.

$$\phi^+(x) \;=\; \int_0^x \phi_s^+(s)\, ds,$$

where $\phi_x^+(x) = (1 - \Psi(x))(\phi_x(x,0) - a)$ *and*

$$a = \frac{1}{1-\Delta x} \int_{\mathcal{L}^c} \phi_s(s,0)\, ds.$$

Note that $1 - \Delta x$ is nonzero since Δx is the length of the lightning domain $\mathcal{L}$. In calculating the limit, they used functions $\{\kappa_i\}, i \geq 1$, where

$$\kappa_i(x) = \{\frac{1}{\sqrt{2}}, \cos(\pi x), \cos(2\pi x), \ldots\}$$

is a complete orthonormal basis for $L_0^2([-1,1])$. By using $\{\kappa_i\}, i \geq 1$, a new set of basis functions are obtained for $L_0^2([0,1])$. They used the orthonormality properties of these new basis functions as they computed the limit and prove Theorem 1.1. In addition, they calculated $\phi^+(x)$ explicitly and obtained the following formula.

$$\begin{aligned}
\phi^+(x) &= \lim_{\Delta t \to 0} \lim_{\tau \to \infty} \phi(x, \Delta t) \\
&= \int_0^x \phi_s^+(s)\, ds \\
&= \begin{cases} \phi_0(x) + x\frac{\bar{\phi}}{1-\Delta x} & \text{if } x \in [0, x_0 - \frac{\Delta x}{2}]\,, \\ \phi_0(x_0 - \frac{\Delta x}{2}) + (x_0 - \frac{\Delta x}{2})\frac{\bar{\phi}}{1-\Delta x} & \text{if } x \in [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}], \\ \phi_0(x) - (1-x)\frac{\bar{\phi}}{1-\Delta x} & \text{if } x \in [x_0 + \frac{\Delta x}{2}, 1], \end{cases}
\end{aligned} \tag{4}$$

where $\bar{\phi} = \phi_0(x_0 + \frac{\Delta x}{2}) - \phi_0(x_0 - \frac{\Delta x}{2})$.

In this paper, we consider the Green's function for the boundary value problem

$$\begin{aligned} u_{xx} &= -f(x), \quad 0 < x < 1, \\ u(0) &= u(1) \;=\; 0 \end{aligned}$$

to prove Theorem 1.1 and obtain the formula (4).

The paper is organized as follows: In Section 2, we explain how the governing equation for the electric potential is obtained from Maxwell's equations. In Section 3, we develop the problem in one-dimensional space of $[0,1]$. In Section 4, we evaluate the limit to prove Theorem 1.1. Conclusions are given in Section 5.

2. Maxwell equations

The governing equations for the potential are derived from the Maxwell's equations for linear materials. By Ampere's law, the curl of the magnetic field strength is given by

$$\nabla \times \mathbf{H} \;=\; \mathbf{J}_T + \varepsilon \frac{\partial \mathbf{E}}{\partial t},$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{J}_T$ is the total current density, $\mathbf{E}$ is the electric field, and ε is the permittivity of the air. The total current density $\mathbf{J}_T$ is partly due to the movement of ice and water in the cloud and partly due to the conductivity of the cloud. Therefore, we can write $\mathbf{J}_T = \mathbf{J}_p + \sigma_a \mathbf{E}$, where σ_a is the conductivity of the atmosphere. Replacing $\mathbf{J}_T$ and taking the divergence gives

$$\varepsilon \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = -\nabla \cdot (\sigma_a \mathbf{E}) - \nabla \cdot \mathbf{J}_p. \tag{5}$$

By Faraday's law of induction,

$$\nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

where $\mathbf{B}$ is the magnetic flux density. In our model, we assume the time derivative of $\mathbf{B}$ can be neglected, therefore obtaining $\nabla \cdot \mathbf{E} = 0$. This implies that $\mathbf{E} = \nabla\phi$ or $\mathbf{E} = -\nabla\phi$, where ϕ is the electric potential. It is common to assume $\mathbf{E} = -\nabla\phi$. Substituting this into (5), we obtain

$$\frac{\partial \nabla^2 \phi}{\partial t} = -\nabla \cdot (\sigma \nabla \phi) + \nabla \cdot \mathbf{J},$$

where $\sigma = \sigma_a/\varepsilon$ and $\mathbf{J} = \mathbf{J}_p/\varepsilon$.

3. The problem in a one-dimensional space

We consider the problem on the interval $[0,1]$. In addition, we assume $\mathbf{J}$ can be neglected. Therefore the following equations model our problem:

$$\begin{aligned}
\dot{\phi}_{xx} &= -(\sigma\phi_x)_x \quad (x,t) \in [0,1] \times [0,\infty), \\
\phi(0,t) &= 0 \quad t \in [0,\infty), \\
\phi(1,t) &= 0 \quad t \in [0,\infty), \\
\phi(x,0) &= \phi_0(x) \quad x \in [0,1],
\end{aligned} \tag{6}$$

Then we consider the Green's function for the boundary value problem

$$\begin{aligned}
u_{xx} &= -f(x), \quad 0 < x < 1, \\
u(0) &= u(1) \;=\; 0
\end{aligned} \tag{7}$$

to prove Theorem 1.1. By [6], Ch.9, we write the solution to (7) as

$$u(x) \;=\; \int_0^1 G(x,s) f(s)\, ds,$$

where

$$G(x,s) \;=\; \begin{cases} (1-s)x & \text{if } x < s\,, \\ (1-x)s & \text{if } x > s. \end{cases}$$

Now we multiply (6) by $G(x,s)$ and integrate over $[0,1]$ to obtain

$$\int_0^1 \dot{\phi}_{ss}(s)G(x,s)\,ds \;=\; -\int_0^1 (\sigma(s)\phi_s(s))_s G(x,s)\,ds \tag{8}$$

By using the Green's function formula for the solution to (6) in (8), we obtain $-\dot{\phi}(x,t)$ on the left hand side. Combining this with the integral of the right hand side gives

$$\dot{\phi}(x,t) = -\int_0^1 \sigma(s)\phi_s(s,t)G_s(x,s)\,ds. \tag{9}$$

Let M be a linear operator acting on ϕ defined by

$$\int_0^1 \sigma(s)\phi_s(s,t)G_s(x,s)\,ds. \tag{10}$$

Then we can rewrite (9) as

$$\dot{\phi}(x,t) = -M\phi(x,t). \tag{11}$$

Theorem 3.1. *The linear operator M given in (10) is positive definite and the equation given in (11) has a unique solution in the form of*

$$\phi(x,t) = e^{-Mt}\phi(x,0). \tag{12}$$

Proof. For non-zero $\phi \in \mathrm{L}^2([0,1])$, we have

$$\begin{aligned}
\langle M\phi,\phi\rangle &= \int_0^1 \left(\int_0^1 \sigma(s)\phi_s(s,t)G_s(x,s)\,ds\right)\phi(x,t)\,dx \\
&= \int_0^1 \phi(x,t)\phi(x,t)\,dx \\
&= \|\phi(x,t)\|_2^2 \\
&> 0.
\end{aligned}$$

Here we use integration by parts on the inside integral. By using the boundary conditions and the Green functions formula for the solution to (6), the inside integral reduces to $\phi(x,t)$ and the rest follows. Therefore by [4], Theorem 4.1, (9) has a unique solution and it is given by

$$\phi(x,t) = e^{-Mt}\phi(x,0). \tag{13}$$

□

Now, to simulate lightning, in (9) we let $\sigma \to \sigma + \tau\Psi$ on the lightning domain $\mathcal{L}$. Then we have

$$\begin{aligned}
\dot{\phi}(x,t) &= -\int_0^1 (\sigma(s)+\tau\Psi(s))\phi_s(s,t)G_s(x,s)\,ds \\
&= -\int_0^1 \sigma(s)\phi_s(s,t)G_s(x,s)\,ds - \tau\int_{\mathcal{L}} \phi_s(s,t)G_s(x,s)\,ds \\
&= -M\phi(x,t) - \tau\delta M\phi(x,t)
\end{aligned}$$

where δM is a linear operator acting on ϕ defined by

$$\delta M\phi(x,t) = \int_{\mathcal{L}} \phi_s(s,t)G_s(x,s)\,ds \;=\; \int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \phi_s(s,t)G_s(x,s)\,ds.$$

Therefore by using (13), the solution can be written as

$$\phi(x,\Delta t) = e^{-M\Delta t}e^{-\tau\delta M\Delta t}\phi_0(x), \tag{14}$$

where $\phi_0(x) = \phi(x,0)$.

4. Computation of $e^{-\tau\delta M\Delta t}\phi_0(x)$ and the limit

In this section we prove Theorem 1.1 by calculating $e^{-\tau\delta M\Delta t}\phi_0(x)$. By expanding this in Taylor series, we obtain

$$e^{-\tau\delta M\Delta t}\phi_0(x) = \sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(\delta M)^n\phi_0(x). \tag{15}$$

Since lightning domain $\mathcal{L}$ separates the domain $[0,1]$ into three subintervals, $[0, x_0 - \frac{\Delta x}{2}]$, $[x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]$, $[x_0 + \frac{\Delta x}{2}, 1]$, we compute $\delta M\phi_0(x)$ in each of these subintervals separately.

If $x \in [0, x_0 - \frac{\Delta x}{2}]$, then $x < s$ and $G_s(x,s) = -x$. Therefore,

$$\delta M\phi_0(x) \;=\; -x\int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}}\phi_s(s,0)\,ds \;=\; -x\bar{\phi}, \tag{16}$$

where $\bar{\phi} = \phi_0(x_0 + \frac{\Delta x}{2}) - \phi_0(x_0 - \frac{\Delta x}{2})$. Then we have

$$\begin{aligned}
(\delta M)^2\phi_0(x) &= -x[\delta M\phi_0(x_0 + \frac{\Delta x}{2}) - \delta M\phi_0(x_0 - \frac{\Delta x}{2})]\\
&= -x([1-(x_0+\frac{\Delta x}{2})]\Delta\phi - (x_0 - \frac{\Delta x}{2})\bar{\phi})\\
&= -x(1-\Delta x)\bar{\phi},
\end{aligned}$$

where

$$\begin{aligned}
\delta M\phi_0(x_0 + \frac{\Delta x}{2}) &= [1-(x_0+\frac{\Delta x}{2})]\bar{\phi},\\
\delta M\phi_0(x_0 - \frac{\Delta x}{2}) &= -(x_0 - \frac{\Delta x}{2})\bar{\phi}.
\end{aligned}$$

In general, we have

$$(\delta M)^n\phi_0(x) = -x(1-\Delta x)^{n-1}\bar{\phi}.$$

Therefore,

$$\begin{aligned}
e^{-\tau\delta M\Delta t}\phi_0(x) &= \sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(\delta M)^n\phi_0(x)\\
&= \phi_0(x) - \sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(1-\Delta x)^{n-1}x\phi(x)\\
&= \phi_0(x) - \frac{x\bar{\phi}}{1-\Delta x}\sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(1-\Delta x)^n\\
&= \phi_0(x) - \frac{x\bar{\phi}}{1-\Delta x}[e^{-\tau\Delta t(1-\Delta x)} - 1].
\end{aligned}$$

Since $\Delta t(1-\Delta x) > 0$, $e^{-\tau\Delta t(1-\Delta x)}$ approaches 0 as τ tends to infinity. Hence we have

$$\lim_{\tau\to\infty} e^{-\tau\delta M\Delta t}\phi(x) = \phi(x) + x\frac{\bar{\phi}}{1-\Delta x}. \tag{17}$$

If $x \in [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]$, then we have

$$\begin{aligned}
\delta M\phi_0(x) &= \int_{x_0-\frac{\Delta x}{2}}^{x} \phi_s(s,0)G_s(x,s)\,ds + \int_{x}^{x_0+\frac{\Delta x}{2}} \phi_s(s,0)G_s(x,s)\,ds \\
&= (1-x)\int_{x_0-\frac{\Delta x}{2}}^{x} \phi_s(s,0)\,ds - x\int_{x}^{x_0+\frac{\Delta x}{2}} \phi_s(s,0)\,ds \\
&= (1-x)(\phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2}) - x(\phi_0(x_0 + \frac{\Delta x}{2}) - \phi_0(x)) \\
&= \phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2}) - x\bar{\phi}. \qquad (18)
\end{aligned}$$

Then we have

$$\begin{aligned}
(\delta M)^2\phi_0(x) &= [\phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2}) - x\bar{\phi}] - \delta M\phi_0(x_0 - \frac{\Delta x}{2}) - x\delta M(\bar{\phi}) \\
&= \phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2}) + (x_0 - \frac{\Delta x}{2})\bar{\phi} - (1 + (1-\Delta x))x\bar{\phi},
\end{aligned}$$

where

$$\begin{aligned}
\delta M\phi_0(x_0 - \frac{\Delta x}{2}) &= (x_0 - \frac{\Delta x}{2})\bar{\phi}, \\
\delta M(\bar{\phi}) &= (1-\Delta x)\bar{\phi}.
\end{aligned}$$

In general, we have

$$\begin{aligned}
(\delta M)^n\phi_0(x) &= \phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2}) + [1 + (1-\Delta x) + \ldots + (1-\Delta x)^{n-2}](x_0 - \frac{\Delta x}{2})\bar{\phi} \\
&- [1 + (1-\Delta x) + \ldots + (1-\Delta x)^{n-1}]x\bar{\phi}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
e^{-\tau\delta M\Delta t}\phi_0(x) &= \sum_{n=0}^{\infty} \frac{(-\tau\Delta t)^n}{n!}(\delta M)^n\phi_0(x) \\
&= \phi_0(x) + \sum_{n=1}^{\infty} \frac{(-\tau\Delta t)^n}{n!}(\phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2})) \\
&\quad + \sum_{n=2}^{\infty} \frac{(-\tau\Delta t)^n}{n!}[1 + (1-\Delta x) + \ldots + (1-\Delta x)^{n-2}](x_0 - \frac{\Delta x}{2})\bar{\phi} \\
&\quad - \sum_{n=1}^{\infty} \frac{(-\tau\Delta t)^n}{n!}[1 + (1-\Delta x) + \ldots + (1-\Delta x)^{n-1}]x\bar{\phi} \\
&= \phi_0(x) + A + B - C, \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
A &= \Big(\phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2})\Big)\sum_{n=1}^{\infty} \frac{(-\tau\Delta t)^n}{n!} \\
&= \Big(\phi_0(x) - \phi_0(x_0 - \frac{\Delta x}{2})\Big)[e^{-\tau\Delta t} - 1],
\end{aligned}$$

$$\begin{aligned}
B &= (x_0-\frac{\Delta x}{2})\bar{\phi}\sum_{n=2}^{\infty}\frac{(-\tau\Delta t)^n}{n!}[1+(1-\Delta x)+\ldots+(1-\Delta x)^{n-2}]\\
&= (x_0-\frac{\Delta x}{2})\bar{\phi}\sum_{n=2}^{\infty}\frac{(-\tau\Delta t)^n}{n!}\frac{(1-(1-\Delta x)^{n-1})}{\Delta x}\\
&= (x_0-\frac{\Delta x}{2})\frac{\bar{\phi}}{\Delta x}\left[\sum_{n=2}^{\infty}\frac{(-\tau\Delta t)^n}{n!}-\frac{1}{1-\Delta x}\sum_{n=2}^{\infty}\frac{(-\tau\Delta t(1-\Delta x))^n}{n!}\right]\\
&= (x_0-\frac{\Delta x}{2})\frac{\bar{\phi}}{\Delta x}\left[e^{-\tau\Delta t}-(1-\tau\Delta t)-\frac{1}{1-\Delta x}[e^{-\tau\Delta t(1-\Delta x)}-(1-\tau\Delta t(1-\Delta x))]\right]\\
&= (x_0-\frac{\Delta x}{2})\frac{\bar{\phi}}{\Delta x}\left[e^{-\tau\Delta t}-\frac{1}{1-\Delta x}e^{-\tau\Delta t(1-\Delta x)}+\frac{\Delta x}{1-\Delta x}\right],
\end{aligned}$$

and

$$\begin{aligned}
C &= x\bar{\phi}\sum_{n=1}^{\infty}\frac{(-\tau\Delta t)^n}{n!}[1+(1-\Delta x)+\ldots+(1-\Delta x)^{n-1}]\\
&= x\bar{\phi}\sum_{n=1}^{\infty}\frac{(-\tau\Delta t)^n}{n!}\frac{1-(1-\Delta x)^n}{\Delta x}\\
&= x\frac{\bar{\phi}}{\Delta x}[e^{-\tau\Delta t}-1-(e^{-\tau\Delta t(1-\Delta x)}-1)]\\
&= x\frac{\bar{\phi}}{\Delta x}[e^{-\tau\Delta t}-e^{-\tau\Delta t(1-\Delta x)}.
\end{aligned}$$

Since $\Delta t > 0$, we have $\Delta t(1-\Delta x) > 0$. Therefore as τ tends to infinity, we obtain

$$\begin{aligned}
\lim_{\tau\to\infty} A &= -\phi_0(x)+\phi_0(x_0-\frac{\Delta x}{2}),\\
\lim_{\tau\to\infty} B &= (x_0-\frac{\Delta x}{2})\frac{\bar{\phi}}{1-\Delta x},\\
\lim_{\tau\to\infty} C &= 0.
\end{aligned}$$

Therefore by substituting these results into (19), we obtain

$$\begin{aligned}
\lim_{\tau\to\infty} e^{-\tau\delta M\Delta t}\phi_0(x) &= \lim_{\tau\to\infty}(\phi_0(x)+A+B-C)\\
&= \phi_0(x_0-\frac{\Delta x}{2})+(x_0-\frac{\Delta x}{2})\frac{\bar{\phi}}{1-\Delta x}. \qquad (20)
\end{aligned}$$

If $x\in[x_0+\frac{\Delta x}{2},1]$, then $x>s$ and $G_s(x,s)=1-x$. Therefore,

$$\delta M\phi_0(x) \;=\; (1-x)\int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}}\phi_s(s)\,ds \;=\; (1-x)\bar{\phi}. \qquad (21)$$

Then we have

$$\begin{aligned}
(\delta M)^2\phi_0(x) &= (1-x)[\delta M\phi(x_0+\frac{\Delta x}{2})-\delta M\phi(x_0-\frac{\Delta x}{2})]\\
&= (1-x)(1-\Delta x)\bar{\phi}.
\end{aligned}$$

In general, we have

$$(\delta M)^n\phi_0(x)=(1-x)(1-\Delta x)^{n-1}\bar{\phi}.$$

Therefore,

$$\begin{aligned} e^{-\tau\delta M\Delta t}\phi_0(x) &= \sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(\delta M)^n\phi_0(x) \\ &= \phi_0(x)+\sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(1-\Delta x)^{n-1}(1-x)\phi(x) \\ &= \phi_0(x)+(1-x)\frac{\bar{\phi}}{1-\Delta x}\sum_{n=0}^{\infty}\frac{(-\tau\Delta t)^n}{n!}(1-\Delta x)^n \\ &= \phi_0(x)+(1-x)\frac{\bar{\phi}}{1-\Delta x}[e^{-\tau\Delta t(1-\Delta x)}-1]. \end{aligned}$$

Since $\Delta t(1-\Delta x)>0$, as τ tends infinity we have

$$\lim_{\tau\to\infty} e^{-\tau\delta M\Delta t}\phi_0(x)=\phi_0(x)-(1-x)\frac{\bar{\phi}}{1-\Delta x}. \tag{22}$$

Note that

$$\lim_{\Delta t\to 0} e^{-M\Delta t}=1.$$

Combining this with (17), (20), and (22), we have

$$\begin{aligned} \phi^+(x) &= \lim_{\Delta t\to 0}\lim_{\tau\to\infty}\phi(x,\Delta t) \\ &= \lim_{\Delta t\to 0}\lim_{\tau\to\infty} e^{-M\Delta t}e^{-\tau\delta M\Delta t}\phi_0(x) \\ &= \begin{cases} \phi_0(x)+x\frac{\bar{\phi}}{1-\Delta x} & \text{if } x\in[0,x_0-\frac{\Delta x}{2}]\,, \\ \phi_0(x_0-\frac{\Delta x}{2})+(x_0-\frac{\Delta x}{2})\frac{\bar{\phi}}{1-\Delta x} & \text{if } x\in[x_0-\frac{\Delta x}{2},x_0+\frac{\Delta x}{2}], \\ \phi_0(x)-(1-x)\frac{\bar{\phi}}{1-\Delta x} & \text{if } x\in[x_0+\frac{\Delta x}{2},1], \end{cases} \end{aligned} \tag{23}$$

where $\bar{\phi}=\phi_0(x_0+\frac{\Delta x}{2})-\phi_0(x_0-\frac{\Delta x}{2})$.

5. Conclusions

We evaluated the new electric potential immediately after the lightning discharge. Theorem 1.1 shows that the electric potential is constant along the lightning domain and it is equal to the pre-flash potential elsewhere. The result was shown earlier in [2] by using eigenfunctions. Here, potential is evaluated by using the Green's function.

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References

[1] Alamgir S., Khan A.R., Pecaric J.E., *Linear inequalities via extension of Montgomery identity and weighted Hermite-Hadamard inequalities with and without green functions*, Filomat, **36 (11)**, (2022), 3593-3608.
[2] B. C. Aslan, *Modeling the change in electric potential due to lightning in a one-dimensional space*, Appl. Anal. **101** (2022), 6280–6296.
[3] Han X.L., Huang J.J., *The Generalized Green's function for boundary value problem of second order difference equation*, Journal of Function Spaces, **2015**, (2015), art.n.201946.
[4] S. G. Krein, *Linear Differential Equations in Banach Space*, American Mathematical Society, Rhode Island, 1971.
[5] Polidoro, S., Ragusa, M.A., *A green function and regularity results for an ultraparabolic equation with a singular potential*, Advances in Differential Equations, **7 (11)** , (2022) 1281–1314.
[6] H. Sagan, *Boundary and Eigenvalue Problems in Mathematical Physics*, Dover, New York, 1989.