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A class of the cyclic anti-periodic and nonlocal boundary value problem to the self-adjoint tripled fractional Langevin differential systems

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Abstract. This paper aims to study a class of the cyclic anti-periodic and nonlocal boundary value problem to the self-adjoint tripled fractional Langevin differential systems. Moreover, some sufficient conditions for the existence and uniqueness of solutions to the problem have been presented. Furthemore, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the problem have been acquired. Our main results enrich and expand some previous results. And some examples are given to validate our main results.

1. Introduction

In this paper, we are concerned with the following cyclic anti-periodic and integral boundary conditions of the self-adjoint tripled fractional Langevin differential system.

$$\begin{cases} {}^CD^{\beta}_{0+}\left({}^CD^{\alpha}_{0+}+\lambda\right)x_i\left(t\right)=f_i\left(t,x_1\left(t\right),x_2\left(t\right),x_3\left(t\right)\right),t\in(0,1),\ i=1,2,3,\\ x_1(0)+x_2(0)=-(x_2(1)+x_3(1)),\int_{\eta_1}^{\xi_1}(x_2(s)+x_3(s))ds=A_1,\\ x_2(0)+x_3(0)=-(x_3(1)+x_1(1)),\int_{\eta_2}^{\xi_2}(x_3(s)+x_1(s))ds=A_2,\\ x_3(0)+x_1(0)=-(x_1(1)+x_2(1)),\int_{\eta_3}^{\xi_3}(x_1(s)+x_2(s))ds=A_3,\end{cases} \tag{1.1}$$

where ${}^CD^{\alpha}_{0+}$ and ${}^CD^{\beta}_{0+}$ mean the Caputo fractional derivatives of order α and β with $0<\alpha<1,\ 0<\beta<1,\ 1<\alpha+\beta<2,\ 0<\eta_i<\xi_i<1,\ \lambda>0,\ A_i\in\mathbb{R},\ f_i:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ stand for continuous functions, $i=1,2,3$.

Fractional calculus is established on the basis of integer calculus. Subsequently, fractional differential equations also emerged as the times require, which has extensive applications in many disciplines, such as physics, geology, biology (see [17,18,20,23]). The qualitative theory of fractional differential equations is a core component of the fundamental theory of fractional differential equations. It has attracted a lot of discussions from many scholars (see [1,3,4,8-10,13-15] and references therein). For example, Ahmad et al. [3] studied the existence and uniqueness of solutions to a class of Caputo fractional coupled anti-periodic and nonlocal integral boundary value problems by some fixed point theorems as follows.

$$\begin{cases} {}^CD^{\beta}x(t)=f\left(t,x\left(t\right),y\left(t\right)\right),t\in[0,T],\\ {}^CD^{\alpha}x(t)=g\left(t,x\left(t\right),y\left(t\right)\right),t\in[0,T],\\ (x+y)(0)=-(x+y)(T),\int_{\eta}^{\xi}(x(s)+y(s))ds=A,\end{cases} \tag{1.2}$$

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where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo fractional derivatives of order α and β with $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $0 < \eta < \xi < T, A \in \mathbb{R}$, $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

In the recent years, the topics on boundary value problems of fractional Langevin equation have become more popular. The main reason is that the fractional Langevin equation is a generalization of integer form and can be used to present the fractional Brownian motion in anomalous diffusion of the memory property (see [11]). Therefore, it has a practical physic background. Fazli and Nieto [12] was concerned with the existence and uniqueness of solutions to the anti-periodic boundary value problems for a coupled system of factional Langevin equation via some fixed point theorems of mixed monotone mappings established in partially ordered metric spaces as follows.

$$\begin{cases} D^{\beta}\left(D^{\alpha}+\lambda\right)x\left(t\right)=f\left(t,x\left(t\right)\right), t\in(0,1), \\ x(0)+x(1)=0,\ D^{\alpha}x(0)+D^{\alpha}x(1)=0,\ D^{2\alpha}x(0)+D^{2\alpha}x(1)=0, \end{cases} \tag{1.3}$$

where D^{α} and D^{β} are the Caputo fractional derivatives of order α and β with $\alpha \in (0, 1], \beta \in (1, 2]$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\lambda \in \mathbb{R}$. After that, Salem [19] considered the existence and uniqueness of solutions for the following anti-periodic and nonlocal integral boundary value problem to fractional Langevin equation.

$$\begin{cases} {}^{C}D^{\beta}\left({}^{C}D^{\alpha}+\lambda\right)x\left(t\right)=f(t,x(t),{}^{C}D^{\alpha}x(t)),\ t\in[0,1], \\ x(0)+x(1)=0,\ x'(0)=0,\ {}^{C}D^{\alpha}x(1)=\frac{\mu}{\Gamma(\gamma)}\int_0^{\eta}(\eta-s)^{\gamma-1}x(s)ds, \end{cases} \tag{1.4}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo fractional derivatives of order β and α with $\alpha \in (0, 1]$, $\beta \in (1, 2]$, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\eta \in (0, 1)$, $\gamma > 0$, $\mu \in \mathbb{R}$. And, for the topics on boundary value problems of fractional Langevin equation, the readers can see [5,6,7,17,21] and references therein.

On the other hand, fractional coupling equations with the cyclic boundary value problems have become popular. Ahmad et al. [2] discussed a new kind of cyclic anti-periodic boundary conditions of the self-adjoint nonlinear tripled system of second order Sturm-Liouville differential equations. Subsequently, Zhang and Ni [22] expanded the results of [2] to case of fractional Langevin equations as follows.

$$\begin{cases} {}^{C}D^{\beta}_{0+}\left({}^{C}D^{\alpha}_{0+}+\lambda\right)x_i\left(t\right)=f_i\left(t,x_1\left(t\right),x_2\left(t\right),x_3\left(t\right)\right), t\in(0,1), i=1,2,3, \\ x_1(0)+x_2(1)=0,\quad {}^{C}D^{\alpha}_{0+}x_1(0)+{}^{C}D^{\alpha}_{0+}x_2(1)=0, \\ x_2(0)+x_3(1)=0,\quad {}^{C}D^{\alpha}_{0+}x_2(0)+{}^{C}D^{\alpha}_{0+}x_3(1)=0, \\ x_3(0)+x_2(1)=0,\quad {}^{C}D^{\alpha}_{0+}x_3(0)+{}^{C}D^{\alpha}_{0+}x_1(1)=0, \end{cases} \tag{1.5}$$

where ${}^{C}D^{\alpha}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ represent the Caputo fractional derivatives of order α and β, $0 < \alpha < 1$, $0 < \beta < 1$, $1 < \alpha + \beta < 2$, $f_i : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}, i = 1, 2, 3$ are continuous, $\lambda \in (0, +\infty)$. The existence and uniqueness of solutions to (1.5) are established. And the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of (1.5) are also presented. It should be mentioned that the cyclic boundary value problems of the self-adjoint nonlinear tripled system are more general and complex than boundary value problems of the normal coupled systems.

Motivated by the works mentioned above, we are devoted to discussing a class of the cyclic anti-periodic and nonlocal boundary value problem to the self-adjoint nonlinear tripled fractional Langevin differential system (1.1). Moreover, from Krasnoselskii fixed point theorem and Banach contraction mapping theorem, some sufficient conditions for the existence and uniqueness of solutions to the problem have been presented. Furthemore, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the problem have been acquired. Our main results enrich some previous results. Let's present the contributions of this paper: Firstly, our model are more general than (1.2). Secondly, there few papers investigating the cyclic nonlocal boundary value problems to the self-adjoint nonlinear tripled fractional Langevin differential system. Our main results expand some existing results. Thirdly, the cyclic boundary value problems of the self-adjoint nonlinear tripled system are more difficult and challenging than boundary value problems of the normal coupled systems.

2. Preliminaries

For the convenience of the reader, we recall some necessary definitions of the Riemann-Liouville fractional integral, the Caputo fractional derivative and some of their properties. In the meanwhile, we state two theorems, which will be used to prove the main results of the paper.

Definition 2.1 [16] The left-sided Riemann-Liouville fractional integral of order $\zeta > 0$ of a function $l : (0, \infty) \to \mathbb{R}$ is gived by

$$(I_{0+}^{\zeta} l)(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} l(s) ds, \ t > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 [16] Let $l \in AC^n[0, \infty)$. The left-side Caputo fractional derivative of order $\zeta > 0$ for a function l is defined by

$${}^{C}D_{0+}^{\zeta} l(t) = \frac{1}{\Gamma(n-\zeta)} \int_0^t (t-s)^{n-\zeta-1} l^{(n)}(s) ds, \ t > 0,$$

where $n = [\zeta] + 1$.

Lemma 2.3 [16] If $\alpha, \ \mu > 0, \ l \in C(0, 1)$, then

$$(I_{0+}^{\alpha} I_{0+}^{\mu} l)(t) = (I_{0+}^{\alpha+\mu} l)(t), \ {}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} l(t) = l(t),$$

$$I_{0+}^{\alpha} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\alpha+\mu)} t^{\alpha+\mu-1}, \ {}^{C}D_{0+}^{\alpha} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)} t^{\mu-\alpha-1}.$$

Lemma 2.4 [16] Assuming that $l \in AC^n[0, 1]$, $\alpha > 0$, then

$$I_{0+}^{\alpha} \, {}^{C}D_{0+}^{\alpha} l(t) = l(t) + \sum_{i=0}^{n-1} c_i t^i, \ 0 < t < 1,$$

where $c_0, \ c_1, \ \cdots, \ c_{n-1} \in \mathbb{R}, \ n = [\alpha] + 1$.

Lemma 2.5 [23] (Krasnoselskii fixed point theorem) Let D be a closed, convex, bounded and nonempty subset of a Banach space $\mathbb{X}$. Let A and B be two operators such that
(i) $Ax + By \in D$ for all $x, y \in D$;
(ii) A is a completely continuous operator;
(iii) B is a contraction mapping.
Then there exists $z \in D$ such that $z = Az + Bz$.

Lemma 2.6 [23] (Banach contraction mapping theorem) Let $\mathbb{X}$ be a Banach space, $D \subset \mathbb{X}$ closed and $T : D \to D$ a strict contraction, that is, $\|Tx - Ty\| \leq \mu \|x - y\|$ for some $\mu \in (0, 1)$ and all $x, y \in D$. Then T has a fixed point in D.

3. Existence and Uniqueness

We consider the Banach space $\mathbb{X} = C[0, 1]$ endowed with the norm $\|x\|_\infty = \max_{t \in [0,1]} |x(t)|$. Define the space $X = \mathbb{X} \times \mathbb{X} \times \mathbb{X}$ with the norm.

$$\|(x_1, x_2, x_3)\|_X = \|x_1\|_\infty + \|x_2\|_\infty + \|x_3\|_\infty, \ (x_1, x_2, x_3) \in X.$$

Then $(X, \| \cdot \|_X)$ is also a Banach space.

Lemma 3.1 Let $h_i \in AC([a,b],\mathbb{R}),\ i = 1,2,3$. Then $x = (x_1, x_2, x_3) \in X$ is a solution of the system of linear fractional Langevin equations

$$^{C}D^{\beta}_{0+}(^{C}D^{\alpha}_{0+} + \lambda)x_i(t) = h_i(t),\ t \in (0,1),\ i = 1,2,3, \tag{3.1}$$

supplemented with the boundary conditions

$$x_1(0) + x_2(0) = -(x_2(1) + x_3(1)), \int_{\eta_1}^{\xi_1}(x_2(s) + x_3(s))ds = A_1, \tag{3.2}$$

$$x_2(0) + x_3(0) = -(x_3(1) + x_1(1)), \int_{\eta_2}^{\xi_2}(x_3(s) + x_1(s))ds = A_2, \tag{3.3}$$

$$x_3(0) + x_1(0) = -(x_1(1) + x_2(1)), \int_{\eta_3}^{\xi_3}(x_1(s) + x_2(s))ds = A_3, \tag{3.4}$$

$$0 < \eta_i < \xi_i < 1,\ A_i \in \mathbb{R},\ i = 1,2,3,$$

is given by

$$\begin{aligned} x_i(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h_i(s)ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_i(s)ds \\ &+[t^{\alpha}(\alpha+1)e_{i,1} + e_{i,2}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(h_2(s)+h_3(s))ds \\ &+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_2(s)+x_3(s))ds] \\ &+[t^{\alpha}(\alpha+1)e_{i,3} + e_{i,4}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(h_3(s)+h_1(s))ds \\ &+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_3(s)+x_1(s))ds] \\ &+[t^{\alpha}(\alpha+1)e_{i,5} + e_{i,6}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(h_1(s)+h_2(s))ds \\ &+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_1(s)+x_2(s))ds] \\ &+[t^{\alpha}(\alpha+1)e_{i,7} + e_{i,8}]\{A_1 - \int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(h_2(\theta)+h_3(\theta))d\theta \\ &-\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(x_2(\theta)+x_3(\theta))d\theta]ds\} \\ &+[t^{\alpha}(\alpha+1)e_{i,9} + e_{i,10}]\{A_2 - \int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(h_3(\theta)+h_1(\theta))d\theta \\ &-\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(x_3(\theta)+x_1(\theta))d\theta]ds\} \\ &+[t^{\alpha}(\alpha+1)e_{i,11} + e_{i,12}]\{A_3 - \int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(h_1(\theta)+h_2(\theta))d\theta \\ &-\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(x_1(\theta)+x_2(\theta))d\theta]ds\},\ i = 1,2,3,\ t \in [0,1], \end{aligned} \tag{3.5}$$

where $e_{i,j}$ stand for different constants for the convenience of calculations.

Proof. we know Lemma 2.4 and apply the operator I^{β}_{0+} on the sides of (3.1),

$$ {}^{C}D^{\alpha}_{0+}x_i(t) = I^{\beta}_{0+}h_i(t) - \lambda x_i(t) + c_{i,1},\ c_{i,1} \in \mathbb{R},\ i = 1,2,3. \tag{3.6}$$

Applying the operator I^{α}_{0+} on the sides of (3.6) and using Lemma 2.3 and Lemma 2.4, we get

$$x_i(t) = I^{\alpha+\beta}_{0+}h_i(t) - \lambda I^{\alpha}_{0+}x_i(t) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}c_{i,1} + c_{i,2},\ c_{i,1},\ c_{i,2} \in \mathbb{R},\ i = 1,2,3. \tag{3.7}$$

By (3.6) and (3.7) ,we can get

$$x_i(0) = c_{i,2},\ \ x_i(1) = I^{\alpha+\beta}_{0+}h_i(1) - \lambda I^{\alpha}_{0+}x_i(1) + \frac{c_{i,1}}{\Gamma(\alpha+1)} + c_{i,2},\ i = 1,2,3. \tag{3.8}$$

Then, bring (3.8) into (3.2)-(3.4), we have

$$\begin{aligned}
c_{12} + c_{22} &= -(I^{\alpha+\beta}_{0+}(h_2(1) + h_3(1)) - \lambda I^{\alpha}_{0+}(x_2(1) + x_3(1)) + \frac{c_{21} + c_{31}}{\Gamma(\alpha+1)} + c_{22} + c_{32}),\\
c_{22} + c_{32} &= -(I^{\alpha+\beta}_{0+}(h_3(1) + h_1(1)) - \lambda I^{\alpha}_{0+}(x_3(1) + x_1(1)) + \frac{c_{31} + c_{11}}{\Gamma(\alpha+1)} + c_{32} + c_{12}),\\
c_{32} + c_{12} &= -(I^{\alpha+\beta}_{0+}(h_1(1) + h_2(1)) - \lambda I^{\alpha}_{0+}(x_1(1) + x_2(1)) + \frac{c_{11} + c_{21}}{\Gamma(\alpha+1)} + c_{12} + c_{22}),\\
\int_{\eta_1}^{\xi_1} & [I^{\alpha+\beta}_{0+}(h_2(s) + h_3(s)) - \lambda I^{\alpha}_{0+}(x_2(s) + x_3(s)) + \frac{c_{21} + c_{31}}{\Gamma(\alpha+1)} \cdot s^{\alpha} + c_{22} + c_{32}]ds = A_1,\\
\int_{\eta_2}^{\xi_2} & [I^{\alpha+\beta}_{0+}(h_3(s) + h_1(s)) - \lambda I^{\alpha}_{0+}(x_3(s) + x_1(s)) + \frac{c_{31} + c_{11}}{\Gamma(\alpha+1)} \cdot s^{\alpha} + c_{32} + c_{12}]ds = A_2,\\
\int_{\eta_3}^{\xi_3} & [I^{\alpha+\beta}_{0+}(h_1(s) + h_2(s)) - \lambda I^{\alpha}_{0+}(x_1(s) + x_2(s)) + \frac{c_{11} + c_{21}}{\Gamma(\alpha+1)} \cdot s^{\alpha} + c_{12} + c_{22}]ds = A_3.
\end{aligned}$$

We solve the following system of linear equations to determine the values of $c_{i,1}, c_{i,2},\ i = 1,2,3.$

$$\begin{pmatrix}
0 & 1 & \frac{1}{\Gamma(\alpha+1)} & 2 & \frac{1}{\Gamma(\alpha+1)} & 1\\
\frac{1}{\Gamma(\alpha+1)} & 1 & 0 & 1 & \frac{1}{\Gamma(\alpha+1)} & 2\\
\frac{1}{\Gamma(\alpha+1)} & 2 & \frac{1}{\Gamma(\alpha+1)} & 1 & 0 & 1\\
0 & 0 & \frac{b_1}{\Gamma(\alpha+2)} & a_1 & \frac{b_1}{\Gamma(\alpha+2)} & a_1\\
\frac{b_2}{\Gamma(\alpha+2)} & a_2 & 0 & 0 & \frac{b_2}{\Gamma(\alpha+2)} & a_2\\
\frac{b_3}{\Gamma(\alpha+2)} & a_3 & \frac{b_3}{\Gamma(\alpha+2)} & a_3 & 0 & 0
\end{pmatrix}
\begin{pmatrix} c_{11}\\ c_{12}\\ c_{21}\\ c_{22}\\ c_{31}\\ c_{32}\end{pmatrix} =$$

$$\begin{pmatrix}
-I^{\alpha+\beta}_{0+}(h_2(1) + h_3(1)) + \lambda I^{\alpha}_{0+}(x_2(1) + x_3(1))\\
-I^{\alpha+\beta}_{0+}(h_3(1) + h_1(1)) + \lambda I^{\alpha}_{0+}(x_3(1) + x_1(1))\\
-I^{\alpha+\beta}_{0+}(h_1(1) + h_2(1)) + \lambda I^{\alpha}_{0+}(x_1(1) + x_2(1))\\
A_1 - \int_{\eta_1}^{\xi_1}[I^{\alpha+\beta}_{0+}(h_2(s) + h_3(s)) - \lambda I^{\alpha}_{0+}(x_2(s) + x_3(s))]ds\\
A_2 - \int_{\eta_2}^{\xi_2}[I^{\alpha+\beta}_{0+}(h_3(s) + h_1(s)) - \lambda I^{\alpha}_{0+}(x_3(s) + x_1(s))]ds\\
A_3 - \int_{\eta_3}^{\xi_3}[I^{\alpha+\beta}_{0+}(h_1(s) + h_2(s)) - \lambda I^{\alpha}_{0+}(x_1(s) + x_2(s))]ds
\end{pmatrix} \tag{3.9}$$

After verification, the coefficient determinant of the system of linear equations (3.9) is not equal to zero. So, (3.9) has the unique solution

$$\begin{aligned}
c_{11} &= \Gamma(\alpha+2)(e_{1,1}D_1 + e_{1,3}D_2 + e_{1,5}D_3 + e_{1,7}D_4 + e_{1,9}D_5 + e_{1,11}D_6),\\
c_{12} &= e_{1,2}D_1 + e_{1,4}D_2 + e_{1,6}D_3 + e_{1,8}D_4 + e_{1,10}D_5 + e_{1,12}D_6,\\
c_{21} &= \Gamma(\alpha+2)(e_{2,1}D_1 + e_{2,3}D_2 + e_{2,5}D_3 + e_{2,7}D_4 + e_{2,9}D_5 + e_{2,11}D_6),\\
c_{22} &= e_{2,2}D_1 + e_{2,4}D_2 + e_{2,6}D_3 + e_{2,8}D_4 + e_{2,10}D_5 + e_{2,12}D_6,\\
c_{31} &= \Gamma(\alpha+2)(e_{3,1}D_1 + e_{3,3}D_2 + e_{3,5}D_3 + e_{3,7}D_4 + e_{3,9}D_5 + e_{3,11}D_6),\\
c_{32} &= e_{3,2}D_1 + e_{3,4}D_2 + e_{3,6}D_3 + e_{3,8}D_4 + e_{3,10}D_5 + e_{3,12}D_6,
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= -I_{0+}^{\alpha+\beta}(h_2(1)+h_3(1)) + \lambda I_{0+}^{\alpha}(x_2(1)+x_3(1)),\\
D_2 &= -I_{0+}^{\alpha+\beta}(h_3(1)+h_1(1)) + \lambda I_{0+}^{\alpha}(x_3(1)+x_1(1)),\\
D_3 &= -I_{0+}^{\alpha+\beta}(h_1(1)+h_2(1)) + \lambda I_{0+}^{\alpha}(x_1(1)+x_2(1)),\\
D_4 &= A_1 - \int_{\eta_1}^{\xi_1}[I_{0+}^{\alpha+\beta}(h_2(s)+h_3(s)) - \lambda I_{0+}^{\alpha}(x_2(s)+x_3(s))]ds,\\
D_5 &= A_2 - \int_{\eta_2}^{\xi_2}[I_{0+}^{\alpha+\beta}(h_3(s)+h_1(s)) - \lambda I_{0+}^{\alpha}(x_3(s)+x_1(s))]ds,\\
D_6 &= A_3 - \int_{\eta_3}^{\xi_3}[I_{0+}^{\alpha+\beta}(h_1(s)+h_2(s)) - \lambda I_{0+}^{\alpha}(x_1(s)+x_2(s))]ds.
\end{aligned}$$

We are getting the desired solution (3.5) when we substitute the values of $c_{i,1},\ c_{i,2},\ i=1,2,3$, in (3.7). Conversely, it is not difficult to verify that $(x_1,x_2,x_3)\in X$ given by (3.5) satisfies the system (3.1) and the boundary conditions (3.2)-(3.4). The proof is completed. □

In view of Lemma 2.5, we define the operator $T: X\to X$ by

$$\begin{aligned}
(Tx)(t) &:= ((T_1x)(t),\ (T_2x)(t),\ (T_3x)(t))\\
&= (T_1(x_1,x_2,x_3)(t),\ T_2(x_1,x_2,x_3)(t),\ T_3(x_1,x_2,x_3)(t)),
\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}
(T_ix)(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\varphi_i(s)ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_i(s)ds\\
&\quad+[t^{\alpha}(\alpha+1)e_{i,1}+e_{i,2}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(\varphi_2(s)+\varphi_3(s))ds\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(x_2(s)+x_3(s))ds]\\
&\quad+[t^{\alpha}(\alpha+1)e_{i,3}+e_{i,4}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(\varphi_3(s)+\varphi_1(s))ds\\
&\quad+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(x_3(s)+x_1(s))ds]
\end{aligned}$$

$$+[t^{\alpha}(\alpha+1)e_{i,5}+e_{i,6}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(\varphi_1(s)+\varphi_2(s))ds$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(x_1(s)+x_2(s))ds]$$
$$+[t^{\alpha}(\alpha+1)e_{i,7}+e_{i,8}]\{A_1-\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s(s-\theta)^{\alpha+\beta-1}(\varphi_2(\theta)+\varphi_3(\theta))d\theta$$
$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\theta)^{\alpha-1}(x_2(\theta)+x_3(\theta))d\theta]ds\}$$
$$+[t^{\alpha}(\alpha+1)e_{i,9}+e_{i,10}]\{A_2-\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s(s-\theta)^{\alpha+\beta-1}(\varphi_3(\theta)+\varphi_1(\theta))d\theta$$
$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\theta)^{\alpha-1}(x_3(\theta)+x_1(\theta))d\theta]ds\}$$
$$+[t^{\alpha}(\alpha+1)e_{i,11}+e_{i,12}]\{A_3-\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s(s-\theta)^{\alpha+\beta-1}(\varphi_1(\theta)+\varphi_2(\theta))d\theta$$
$$-\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\theta)^{\alpha-1}(x_1(\theta)+x_2(\theta))d\theta]ds\},\quad i=1,2,3,t\in[0,1],$$

and

$$\varphi_i(s)=f_i(s,x_1(s),x_2(s),x_3(s)),\quad i=1,2,3.$$

So, the function $x=(x_1,x_2,x_3)$ is a solution of problem (1.1), if and only if x is a fixed point of operator T. In the following, we will calculate our main result. For the convenience of calculation, we have import the following functions:

$$\begin{aligned}
m_{i,j}(t) &= t^{\alpha}(\alpha+1)e_{i,2j-1}+e_{i,2j},\\
M_i(t) &= (t^{\alpha}(\alpha+1)e_{i,7}+e_{i,8})A_1+(t^{\alpha}(\alpha+1)e_{i,9}+e_{i,10})A_2+(t^{\alpha}(\alpha+1)e_{i,11}+e_{i,12})A_3,\\
M &= \max\{\|M_1\|_\infty,\|M_2\|_\infty,\|M_3\|_\infty\},\\
\tau_i(t) &= \frac{1}{\Gamma(\alpha+\beta+1)}(1+m_{i,1}(t)+m_{i,2}(t)+m_{i,3}(t)+m_{i,4}(t)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}\\
&\quad +m_{i,5}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}+m_{i,6}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}),\\
\omega_i(t) &= \frac{1}{\Gamma(\alpha+\beta+1)}(m_{i,1}(t)+m_{i,2}(t)+m_{i,3}(t)+m_{i,4}(t)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}\\
&\quad +m_{i,5}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}+m_{i,6}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}),\\
\gamma_i(t) &= \frac{1}{\Gamma(\alpha+\beta+1)}(m_{i,1}(t)+m_{i,2}(t)+m_{i,3}(t)+m_{i,4}(t)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}\\
&\quad +m_{i,5}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}+m_{i,6}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}),
\end{aligned}$$

$$
\begin{aligned}
\tau &= \max\{\|\tau_1\|_\infty, \|\tau_2\|_\infty, \|\tau_3\|_\infty\},\\
\omega &= \max\{\|\omega_1\|_\infty, \|\omega_2\|_\infty, \|\omega_3\|_\infty\},\\
\gamma &= \max\{\|\gamma_1\|_\infty, \|\gamma_2\|_\infty, \|\gamma_3\|_\infty\},\\
N_i(t) &= m_{i,1}(t) + m_{i,2}(t) + m_{i,3}(t) + m_{i,4}(t)\frac{\xi_1^{\alpha+1} - \eta_1^{\alpha+1}}{\alpha+1} + m_{i,5}(t)\frac{\xi_2^{\alpha+1} - \eta_2^{\alpha+1}}{\alpha+1}\\
&\quad + m_{i,6}(t)\frac{\xi_3^{\alpha+1} - \eta_3^{\alpha+1}}{\alpha+1},\\
N &= \max\{\|N_1\|_\infty, \|N_2\|_\infty, \|N_3\|_\infty\},\ i = 1,2,3,\ j = 1,2,3,4,5,6.
\end{aligned}
$$

With aid of the Krasnoselskii's fixed point theorem, we now put forward the existence result for the system (1.1).

Theorem 3.2 Assume that the following conditions hold.
(H_1) The functions $f_i : [0,1] \times \mathbb{R}^3 \to \mathbb{R}, i = 1,2,3$ are continuous.
(H_2) There exist nonnegative functions $p_i,\ q_i,\ r_i,\ k_i \in C[0,1]$, such that, for all $(t, \varrho, \sigma, \varpi) \in [0,1] \times \mathbb{R}^3$, $(i = 1,2,3)$,

$$|f_i(t, \varrho, \sigma, \varpi)| \le k_i(t) + p_i(t)|\varrho| + q_i(t)|\sigma| + r_i(t)|\varpi|,$$

hold. Then the system (1.1) has at least one solution, provided that

$$\Gamma(\alpha+1) > \lambda(1 + 3N) + \Gamma(\alpha+1)(\tau + \omega + \gamma)\sum_{i=1}^{3} \ell_i, \tag{3.11}$$

where

$$p_i = \max_{t\in[0,1]} |p_i(t)|,\ q_i = \max_{t\in[0,1]} |q_i(t)|,\ r_i = \max_{t\in[0,1]} |r_i(t)|,\ k_i = \max_{t\in[0,1]} |k_i(t)|,\ \ell_i = p_i + q_i + r_i,\ i = 1,2,3.$$

Proof. Let us fix $\delta > 0$ such that

$$\delta \ge \frac{\Gamma(\alpha+1)(\tau + \omega + \gamma)\sum_{i=1}^{3} k_i + 3M\Gamma(\alpha+1)}{\Gamma(\alpha+1) - (1 + 3N)\lambda - \Gamma(\alpha+1)(\tau + \omega + \gamma)\sum_{i=1}^{3} \ell_i},$$

and consider the set

$$B_\delta = \{x = (x_1, x_2, x_3) \in \mathbb{X}^3 : \|x\|_X \le \delta\}.$$

We define now the operators $F,\ G : B_\delta \to X$ by

$$
\begin{aligned}
(Fx)(t) &= ((F_1x)(t), (F_2x)(t), (F_3x)(t))\\
&= (F_1(x_1, x_2, x_3)(t), F_2(x_1, x_2, x_3)(t), F_3(x_1, x_2, x_3)(t)),\\
(Gx)(t) &= ((G_1x)(t), (G_2x)(t), (G_3x)(t))\\
&= (G_1(x_1, x_2, x_3)(t), G_2(x_1, x_2, x_3)(t), G_3(x_1, x_2, x_3)(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
(F_ix)(t) = & -\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_i(s)ds \\
& +[t^\alpha(\alpha+1)e_{i,1}+e_{i,2}]\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_2(s)+x_3(s))ds \\
& +[t^\alpha(\alpha+1)e_{i,3}+e_{i,4}]\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_3(s)+x_1(s))ds \\
& +[t^\alpha(\alpha+1)e_{i,5}+e_{i,6}]\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_1(s)+x_2(s))ds \\
& +[t^\alpha(\alpha+1)e_{i,7}+e_{i,8}]\int_{\eta_1}^{\xi_1}[\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(x_2(\theta)+x_3(\theta))d\theta]ds \\
& +[t^\alpha(\alpha+1)e_{i,9}+e_{i,10}]\int_{\eta_2}^{\xi_2}[\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(x_3(\theta)+x_1(\theta))d\theta]ds \\
& +[t^\alpha(\alpha+1)e_{i,11}+e_{i,12}]\int_{\eta_3}^{\xi_3}[\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(x_1(\theta)+x_2(\theta))d\theta]ds, i=1,2,3.
\end{aligned}
$$

$$
\begin{aligned}
(G_ix)(t) = & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\varphi_i(s)ds \\
& -[t^\alpha(\alpha+1)e_{i,1}+e_{i,2}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\varphi_2(s)+\varphi_3(s))ds \\
& -[t^\alpha(\alpha+1)e_{i,3}+e_{i,4}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\varphi_3(s)+\varphi_1(s))ds \\
& -[t^\alpha(\alpha+1)e_{i,5}+e_{i,6}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\varphi_1(s)+\varphi_2(s))ds \\
& -[t^\alpha(\alpha+1)e_{i,7}+e_{i,8}]\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\varphi_2(\theta)+\varphi_3(\theta))d\theta]ds \\
& -[t^\alpha(\alpha+1)e_{i,9}+e_{i,10}]\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\varphi_3(\theta)+\varphi_1(\theta))d\theta]ds \\
& -[t^\alpha(\alpha+1)e_{i,11}+e_{i,12}]\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\varphi_1(\theta)+\varphi_2(\theta))d\theta]ds \\
& +[t^\alpha(\alpha+1)e_{i,7}+e_{i,8}]A_1+[t^\alpha(\alpha+1)e_{i,9}+e_{i,10}]A_2+[t^\alpha(\alpha+1)e_{i,11}+e_{i,12}]A_3, \\
& i=1,2,3.
\end{aligned}
$$

By using Krasnoselskii's fixed point theorem, we divide our proof into three steps.

(i) Firstly, we need to verify the following property.

$$Gx+Fy\in B_\delta,$$

for any $x=(x_1,x_2,x_3),\ y=(y_1,y_2,y_3)\in B_\delta$. In reality, it follows from $x,\ y\in B_\delta$ that $\|x\|_X\le\delta,\ \|y\|_X\le\delta$. Then, by using condition (H_2), we get

$$\begin{aligned}
|(G_1x)(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|\varphi_1(s)|ds \\
&+[t^\alpha(\alpha+1)e_{1,1}+e_{1,2}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\varphi_2(s)|+|\varphi_3(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,3}+e_{1,4}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\varphi_3(s)|+|\varphi_1(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,5}+e_{1,6}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\varphi_1(s)|+|\varphi_2(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,7}+e_{1,8}]\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varphi_2(\theta)|+|\varphi_3(\theta)|)d\theta]ds \\
&+[t^\alpha(\alpha+1)e_{1,9}+e_{1,10}]\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varphi_3(\theta)|+|\varphi_1(\theta)|)d\theta]ds \\
&+[t^\alpha(\alpha+1)e_{1,11}+e_{1,12}]\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varphi_1(\theta)|+|\varphi_2(\theta)|)d\theta]ds \\
&+\|M_1\|_\infty \\
&\leq \frac{k_1+\ell_1\|x\|_X}{\Gamma(\alpha+\beta+1)}+\frac{m_{11}(t)\sum_{i=1}^3(k_i+\ell_i\|x\|_X)}{\Gamma(\alpha+\beta+1)}+\frac{m_{12}(t)\sum_{i=1}^3(k_i+\ell_i\|x\|_X)}{\Gamma(\alpha+\beta+1)} \\
&+\frac{m_{13}(t)\sum_{i=1}^3(k_i+\ell_i\|x\|_X)}{\Gamma(\alpha+\beta+1)}+\frac{m_{14}(t)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)}{\Gamma(\alpha+\beta+1)} \\
&+\frac{m_{15}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)}{\Gamma(\alpha+\beta+1)} \\
&+\frac{m_{16}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)}{\Gamma(\alpha+\beta+1)}+\|M_1\|_\infty \\
&\leq \|\tau_1\|_\infty k_1+\|\omega_1\|_\infty k_2+\|\gamma_1\|_\infty k_3+(\|\tau_1\|_\infty\ell_1+\|\omega_1\|_\infty\ell_2+\|\gamma_1\|_\infty\ell_3)\delta+\|M_1\|_\infty.
\end{aligned}$$

In the same way, we also find

$$|(G_2x)(t)| \leq \|\gamma_2\|_\infty k_1+\|\tau_2\|_\infty k_2+\|\omega_2\|_\infty k_3+(\|\gamma_2\|_\infty\ell_1+\|\tau_2\|_\infty\ell_2+\|\omega_2\|_\infty\ell_3)\delta+\|M_2\|_\infty,$$
$$|(G_3x)(t)| \leq \|\omega_3\|_\infty k_1+\|\gamma_3\|_\infty k_2+\|\tau_3\|_\infty k_3+(\|\omega_3\|_\infty\ell_1+\|\gamma_3\|_\infty\ell_2+\|\tau_3\|_\infty\ell_3)\delta+\|M_3\|_\infty.$$

Besides, for all $t \in [0,1]$, we obtain the following inequality.

$$\begin{aligned}
|(F_iy)(t)| &\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y_i(s)|ds \\
&+[t^\alpha(\alpha+1)e_{i,1}+e_{i,2}]\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(|y_2(s)|+|y_3(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{i,3}+e_{i,4}]\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(|y_3(s)|+|y_1(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{i,5}+e_{i,6}]\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(|y_1(s)|+|y_2(s)|)ds
\end{aligned}$$

$$
\begin{aligned}
&+[t^{\alpha}(\alpha+1)e_{i,7}+e_{i,8}]\int_{\eta_1}^{\xi_1}[\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\theta)^{\alpha-1}(|y_2(\theta)|+|y_3(\theta)|)d\theta]ds\\
&+[t^{\alpha}(\alpha+1)e_{i,9}+e_{i,10}]\int_{\eta_2}^{\xi_2}[\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\theta)^{\alpha-1}(|y_3(\theta)|+|y_1(\theta)|)d\theta]ds\\
&+[t^{\alpha}(\alpha+1)e_{i,11}+e_{i,12}]\int_{\eta_3}^{\xi_3}[\frac{\lambda}{\Gamma(\alpha)}\int_0^s(s-\theta)^{\alpha-1}(|y_1(\theta)|+|y_2(\theta)|)d\theta]ds\\
\leq\;& \frac{\lambda\|y_i\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda m_{i,1}(t)\|y\|_X}{\Gamma(\alpha+1)}+\frac{\lambda m_{i,2}(t)\|y\|_X}{\Gamma(\alpha+1)}+\frac{\lambda m_{i,3}(t)\|y\|_X}{\Gamma(\alpha+1)}\\
&+\frac{\lambda m_{i,4}(t)\frac{\xi_1^{\alpha+1}-\eta_1^{\alpha+1}}{\alpha+1}\|y\|_X}{\Gamma(\alpha+1)}+\frac{\lambda m_{i,5}(t)\frac{\xi_2^{\alpha+1}-\eta_2^{\alpha+1}}{\alpha+1}\|y\|_X}{\Gamma(\alpha+1)}+\frac{\lambda m_{i,6}(t)\frac{\xi_3^{\alpha+1}-\eta_3^{\alpha+1}}{\alpha+1}\|y\|_X}{\Gamma(\alpha+1)}\\
=\;& \frac{\lambda\|y_i\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_i\|_\infty\|y\|_X}{\Gamma(\alpha+1)}\\
\leq\;& \frac{\lambda\|y_i\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_i\|_\infty\delta}{\Gamma(\alpha+1)},\quad i=1,2,3.
\end{aligned}
$$

According to above, we can obtain the following estimates immediately,

$$
\begin{aligned}
|(G_1x)(t)+(F_1y)(t)| \leq\;& \|\tau_1\|_\infty k_1+\|\omega_1\|_\infty k_2+\|\gamma_1\|_\infty k_3+(\|\tau_1\|_\infty\ell_1+\|\omega_1\|_\infty\ell_2+\|\gamma_1\|_\infty\ell_3)\delta+\|M_1\|_\infty\\
&+\frac{\lambda\|y_1\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_1\|_\infty\delta}{\Gamma(\alpha+1)},\\
|(G_2x)(t)+(F_2y)(t)| \leq\;& \|\gamma_2\|_\infty k_1+\|\tau_2\|_\infty k_2+\|\omega_2\|_\infty k_3+(\|\gamma_2\|_\infty\ell_1+\|\tau_2\|_\infty\ell_2+\|\omega_2\|_\infty\ell_3)\delta+\|M_2\|_\infty\\
&+\frac{\lambda\|y_2\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_2\|_\infty\delta}{\Gamma(\alpha+1)},\\
|(G_3x)(t)+(F_3y)(t)| \leq\;& \|\omega_3\|_\infty k_1+\|\gamma_3\|_\infty k_2+\|\tau_3\|_\infty k_3+(\|\omega_3\|_\infty\ell_1+\|\gamma_3\|_\infty\ell_2+\|\tau_3\|_\infty\ell_3)\delta+\|M_3\|_\infty\\
&+\frac{\lambda\|y_3\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_3\|_\infty\delta}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Taking the norm for $Gx+Fy$ on X, we finally obtain

$$
\begin{aligned}
\|Gx+Fy\|_X =\;& \|G_1x+F_1y\|_\infty+\|G_2x+F_2y\|_\infty+\|G_3x+F_3y\|_\infty\\
\leq\;& (\tau+\omega+\gamma)\sum_{i=1}^{3}(k_i+\ell_i\delta)+3M+\frac{(1+3N)\lambda\delta}{\Gamma(\alpha+1)}\leq\delta.
\end{aligned}
$$

Therefore, $Gx+Fy\in B_\delta$, for all $x,y\in B_\delta$.

(ii) We claim that F is a contraction on B_δ. In fact, for any $x=(x_1,x_2,x_3),\ y=(y_1,y_2,y_3)\in B_\delta$, we can get

$$
\begin{aligned}
|(F_ix)(t)-(F_iy)(t)| =\;& \frac{\lambda\|x_i-y_i\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_i\|_\infty}{\Gamma(\alpha+1)}(\|x_1-y_1\|_\infty+\|x_2-y_2\|_\infty+\|x_3-y_3\|_\infty),\\
\leq\;& \frac{\lambda\|x_i-y_i\|_\infty}{\Gamma(\alpha+1)}+\frac{\lambda\|N_i\|_\infty}{\Gamma(\alpha+1)}\|x-y\|_X.
\end{aligned}
$$

Using the norm $Fx-Fy$, we conclude

$$
\|Fx-Fy\|_X \leq \frac{\lambda(1+3N)}{\Gamma(\alpha+1)}\|x-y\|_X.
$$

In view of the condition (3.11), we infer that F is a contraction.

(iii) We have to show that G is equicontinuous on B_δ. As a matter of fact, since the functions f_1,f_2,f_3 are continuous, which implies that operator G is continuous on B_δ. Therefore, it remains to prove that G is compact on B_δ. Firstly, for any $x(t)\in B_\delta,\ t\in[0,1]$, by using (i), we obtain G is uniformly bounded on B_δ. Next, for $x=(x_1,x_2,x_3)\in B_\delta$ and $t_1,t_2\in[0,1]$ with $0\le t_1\le t_2\le 1$, we get

$$\begin{aligned}
|(G_1x)(t_2)-(G_1x)(t_1)| \le\ & \frac{k_1+(p_1+q_1+r_1)\|x\|_X}{\Gamma(\alpha+\beta)}\{\int_0^{t_1}[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}]ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds\}+\frac{3e_1(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)\\
&+\frac{e_{1,7}(\alpha+1)(t_2^\alpha-t_1^\alpha)(\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1})}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)\\
&+\frac{e_{1,9}(\alpha+1)(t_2^\alpha-t_1^\alpha)(\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1})}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)\\
&+\frac{e_{1,11}(\alpha+1)(t_2^\alpha-t_1^\alpha)(\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1})}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)\\
&+(t_2^\alpha-t_1^\alpha)(e_{1,7}A_1+e_{1,9}A_2+e_{1,11}A_3)\\
\le\ & \frac{k_1+(p_1+q_1+r_1)\|x\|_X}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+\frac{3e_1(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)\\
&+\frac{3e_{1'}(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\|x\|_X)+(t_2^\alpha-t_1^\alpha)(e_{1,7}A_1+e_{1,9}A_2+e_{1,11}A_3)\\
\le\ & \frac{k_1+(p_1+q_1+r_1)\delta}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+\frac{3e_1+3e_{1'}}{\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\delta)(t_2^\alpha-t_1^\alpha)\\
&+(t_2^\alpha-t_1^\alpha)(e_{1,7}A_1+e_{1,9}A_2+e_{1,11}A_3).
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
&|(G_2x)(t_2)-(G_2x)(t_1)|\\
\le\ & \frac{k_2+(p_2+q_2+r_2)\delta}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+\frac{3e_2+3e_{2'}}{(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\delta)(t_2^\alpha-t_1^\alpha)\\
&+(e_{2,7}A_1+e_{2,9}A_2+e_{2,11}A_3)(t_2^\alpha-t_1^\alpha),
\end{aligned}$$

$$\begin{aligned}
&|(G_3x)(t_2)-(G_3x)(t_1)|\\
\le\ & \frac{k_3+(p_3+q_3+r_3)\delta}{\Gamma(\alpha+\beta+1)}(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+\frac{3e_3+3e_{3'}}{\Gamma(\alpha+\beta+1)}\sum_{i=1}^3(k_i+\ell_i\delta)(t_2^\alpha-t_1^\alpha)\\
&+(e_{3,7}A_1+e_{3,9}A_2+e_{3,11}A_3)(t_2^\alpha-t_1^\alpha).
\end{aligned}$$

For the convenience of calculation, we have import the following constants.

$$e_i=\max\{e_{i,1}(\alpha+1),\ e_{i,3}(\alpha+1),\ e_{i,5}(\alpha+1)\},$$

$$e_{i'}=\max\{e_{i,7}(\alpha+1)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1},\ e_{i,9}(\alpha+1)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1},\ e_{i,11}(\alpha+1)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}\},$$

where $i=1,2,3$. Since $t^{\alpha+\beta}$ and t^α are uniformly continuous on $[0,1]$, we can get

$$|(G_ix)(t_2)-(G_ix)(t_1)|\to 0,\ as\ t_2\to t_1\ independent\ of\ x,\ i=1,2,3.$$

Therefore, the operator G is equicontinuous on B_δ, and so, through using the Arzelá-Ascoli theorem, we obtain that G is a compact on B_δ. Hence, all the assumptions of Lemma 2.5 are satisfied, and then by Lemma 2.5 we deduce that there exists a fixed point of operator $G + F$, which is a solution of the system (1.1). This completes the proof. □

In the following result, we prove the uniqueness of solution to the system (1.1) with the Banach's contraction mapping theorem.

Theorem 3.3 Suppose that the (H_1) and the following condition hold.
(H_3) There exist constants $L_i > 0$ $(i = 1, 2, 3)$ such that for any $t \in [0, 1]$, x_i, $y_i \in \mathbb{R}(i = 1, 2, 3)$,

$$|f_i(t, x_1, x_2, x_3) - f_i(t, y_1, y_2, y_3)| \leq L_i(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),\ i = 1, 2, 3,$$

Then system (1.1) has a unique solution, provided that

$$\lambda(3 + 3N) + \Gamma(\alpha + 1)(\Delta_1 + \Delta_2 + \Delta_3) < \Gamma(\alpha + 1), \tag{3.12}$$

where

$$\Delta_1 = \tau L_1 + \omega L_2 + \gamma L_3,\ \ \Delta_2 = \gamma L_1 + \tau L_2 + \omega L_3,\ \ \Delta_3 = \omega L_1 + \gamma L_2 + \tau L_3.$$

Proof. Let us fix $\rho > 0$ such that

$$\rho \geq \frac{\Gamma(\alpha + 1)(\tau + \omega + \gamma)(\vartheta_1 + \vartheta_2 + \vartheta_3) + 3M\Gamma(\alpha + 1)}{\Gamma(\alpha + 1) - 3\lambda - 3\lambda N - \Gamma(\alpha + 1)(\tau + \omega + \gamma)(L_1 + L_2 + L_3)},$$

where

$$\vartheta_i = \max_{t\in[0,1]} |f_i(t, 0, 0, 0)|,\ i = 1, 2, 3.$$

We first consider the set

$$B_\rho = \{(x_1, x_2, x_3) \in X : \|x\|_X \leq \rho\},$$

and show that $TB_\rho \subset B_\rho$. In fact, for any $x = (x_1, x_2, x_3) \in B_\rho, t \in [0, 1]$, from (H_3), it follows

$$\begin{aligned}
|f_i(t, x_1, x_2, x_3)| &\leq |f_i(t, x_1, x_2, x_3) - f_i(t, 0, 0, 0)| + |f_i(t, 0, 0, 0)| \\
&\leq L_i(\|x_1\|_\infty + \|x_2\|_\infty + \|x_3\|_\infty) + \vartheta_i \\
&= L_i\|x\|_X + \vartheta_i \leq L_i\rho + \vartheta_i,\ \ i = 1, 2, 3.
\end{aligned}$$

Next, we have

$$\begin{aligned}
&|(T_1x)(t)| \\
&\leq \frac{\lambda\rho}{\Gamma(\alpha + 1)} + \frac{\lambda\|N_1\|_\infty\rho}{\Gamma(\alpha + 1)} + \frac{L_1\rho + \vartheta_1}{\Gamma(\alpha + \beta + 1)} \\
&\quad + \frac{m_{11}(t)(L_1\rho + \vartheta_1 + L_2\rho + \vartheta_2 + L_3\rho + \vartheta_3)}{\Gamma(\alpha + \beta + 1)} + \frac{m_{12}(t)(L_1\rho + \vartheta_1 + L_2\rho + \vartheta_2 + L_3\rho + \vartheta_3)}{\Gamma(\alpha + \beta + 1)} \\
&\quad + \frac{m_{13}(t)(L_1\rho + \vartheta_1 + L_2\rho + \vartheta_2 + L_3\rho + \vartheta_3)}{\Gamma(\alpha + \beta + 1)} \\
&\quad + \frac{m_{14}(t)\frac{\xi_1^{\alpha+\beta+1} - \eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}(L_1\rho + \vartheta_1 + L_2\rho + \vartheta_2 + L_3\rho + \vartheta_3)}{\Gamma(\alpha + \beta + 1)}
\end{aligned}$$

$$+\frac{m_{15}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}(L_1\rho+\vartheta_1+L_2\rho+\vartheta_2+L_3\rho+\vartheta_3)}{\Gamma(\alpha+\beta+1)}$$
$$+\frac{m_{16}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}(L_1\rho+\vartheta_1+L_2\rho+\vartheta_2+L_3\rho+\vartheta_3)}{\Gamma(\alpha+\beta+1)}+\|M_1\|_\infty$$
$$\le \frac{\lambda\rho}{\Gamma(\alpha+1)}+\frac{\lambda\|N_1\|_\infty\rho}{\Gamma(\alpha+1)}+\|\tau_1\|_\infty(L_1\rho+\vartheta_1)+\|\omega_1\|_\infty(L_2\rho+\vartheta_2)+\|\gamma_1\|_\infty(L_3\rho+\vartheta_3)+\|M_1\|_\infty.$$

In similar manner, we have

$$\begin{aligned}
|(T_2x)(t)| &\le \frac{\lambda\rho}{\Gamma(\alpha+1)}+\frac{\lambda\|N_2\|_\infty\rho}{\Gamma(\alpha+1)}+\|\gamma_2\|_\infty(L_1\rho+\vartheta_1)\\
&\quad+\|\tau_2\|_\infty(L_2\rho+\vartheta_2)+\|\omega_2\|_\infty(L_3\rho+\vartheta_3)+\|M_2\|_\infty,\\
|(T_3x)(t)| &\le \frac{\lambda\rho}{\Gamma(\alpha+1)}+\frac{\lambda\|N_3\|_\infty\rho}{\Gamma(\alpha+1)}+\|\omega_3\|_\infty(L_1\rho+\vartheta_1)\\
&\quad+\|\gamma_3\|_\infty(L_2\rho+\vartheta_2)+\|\tau_3\|_\infty(L_3\rho+\vartheta_3)+\|M_3\|_\infty,
\end{aligned}$$

which, on taking the norm for $t\in[0,1]$, yields

$$\|Tx\|_X \le \frac{3\lambda\rho}{\Gamma(\alpha+1)}+\frac{3\lambda N\rho}{\Gamma(\alpha+1)}+(\tau+\omega+\gamma)(L_1\rho+\vartheta_1+L_2\rho+\vartheta_2+L_3\rho+\vartheta_3)+3M\le\rho.$$

This implies, $TB_\rho\subset B_\rho$. We now show that T is a contraction mapping on B_ρ. In fact, for any $x=(x_1,x_2,x_3),\ y=(y_1,y_2,y_3)\in X$, let

$$\varphi_{ix}(s)=f_i(s,x_1(s),x_2(s),x_3(s)),\ \varphi_{iy}(s)=f_i(s,y_1(s),y_2(s),y_3(s)),\ i=1,2,3.$$

Then, we have

$$\begin{aligned}
&|(T_ix)(t)-(T_iy)(t)|\\
\le\ &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}|\varphi_{ix}(s)-\varphi_{iy}(s)|ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|x_i(s)-y_i(s)|ds\\
&+[t^\alpha(\alpha+1)e_{i,1}+e_{i,2}][\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(|\varphi_{2x}(s)-\varphi_{2y}(s)|+|\varphi_{3x}(s)-\varphi_{3y}(s)|)ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(|x_2(s)-y_2(s)|+|x_3(s)-y_3(s)|)ds]\\
&+[t^\alpha(\alpha+1)e_{i,3}+e_{i,4}][\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(|\varphi_{3x}(s)-\varphi_{3y}(s)|+|\varphi_{1x}(s)-\varphi_{1y}(s)|)ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(|x_3(s)-y_3(s)|+|x_1(s)-y_1(s)|)ds]\\
&+[t^\alpha(\alpha+1)e_{i,5}+e_{i,6}][\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(|\varphi_{1x}(s)-\varphi_{1y}(s)|+|\varphi_{2x}(s)-\varphi_{2y}(s)|)ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(|x_1(s)-y_1(s)|+|x_2(s)-y_2(s)|)ds]\\
&+[t^\alpha(\alpha+1)e_{i,7}+e_{i,8}]\\
&\times\{\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s(s-\theta)^{\alpha+\beta-1}(|\varphi_{2x}(\theta)-\varphi_{2y}(\theta)|+|\varphi_{3x}(\theta)-\varphi_{3y}(\theta)|)d\theta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(|x_2(\theta)-y_2(\theta)|+|x_3(\theta)-y_3(\theta)|)d\theta]ds\}\\
&+[t^{\alpha}(\alpha+1)e_{i,9}+e_{i,10}]\\
&\times\{\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varphi_{3x}(\theta)-\varphi_{3y}(\theta)|+|\varphi_{1x}(\theta)-\varphi_{1y}(\theta)|)d\theta\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(|x_3(\theta)-y_3(\theta)|+|x_1(\theta)-y_1(\theta)|)d\theta]ds\}\\
&+[t^{\alpha}(\alpha+1)e_{i,11}+e_{i,12}]\\
&\times\{\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varphi_{1x}(\theta)-\varphi_{1y}(\theta)|+|\varphi_{2x}(\theta)-\varphi_{2y}(\theta)|)d\theta\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(|x_1(\theta)-y_1(\theta)|+|x_2(\theta)-y_2(\theta)|)d\theta]ds\}\\
\leq\ &\frac{\lambda}{\Gamma(\alpha+1)}\|x_i-y_i\|_\infty+\frac{\lambda m_{i,1}(t)}{\Gamma(\alpha+1)}\|x-y\|_X+\frac{\lambda m_{i,2}(t)}{\Gamma(\alpha+1)}\|x-y\|_X\\
&+\frac{\lambda m_{i,3}(t)}{\Gamma(\alpha+1)}\|x-y\|_X+\frac{\lambda m_{i,4}(t)\frac{\xi_1^{\alpha+1}-\eta_1^{\alpha+1}}{\alpha+1}}{\Gamma(\alpha+1)}\|x-y\|_X\\
&+\frac{\lambda m_{i,5}(t)\frac{\xi_2^{\alpha+1}-\eta_2^{\alpha+1}}{\alpha+1}}{\Gamma(\alpha+1)}\|x-y\|_X+\frac{\lambda m_{i,6}(t)\frac{\xi_3^{\alpha+1}-\eta_3^{\alpha+1}}{\alpha+1}}{\Gamma(\alpha+1)}\|x-y\|_X\\
&+\frac{L_i\|x-y\|_X}{\Gamma(\alpha+\beta+1)}+\frac{m_{i,1}(t)(L_1+L_2+L_3)\|x-y\|_X}{\Gamma(\alpha+\beta+1)}+\frac{m_{i,2}(t)(L_1+L_2+L_3)\|x-y\|_X}{\Gamma(\alpha+\beta+1)}\\
&+\frac{m_{i,3}(t)(L_1+L_2+L_3)\|x-y\|_X}{\Gamma(\alpha+\beta+1)}+\frac{m_{i,4}(t)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}(L_1+L_2+L_3)\|x-y\|_X}{\Gamma(\alpha+\beta+1)}\\
&+\frac{m_{i,5}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}(L_1+L_2+L_3)\|x-y\|_X}{\Gamma(\alpha+\beta+1)}+\frac{m_{i,6}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}(L_1+L_2+L_3)\|x-y\|_X}{\Gamma(\alpha+\beta+1)}\\
\leq\ &\Delta_i\|x-y\|_X+\frac{\lambda}{\Gamma(\alpha+1)}\|x_i-y_i\|_\infty+\frac{\lambda\|N_i\|\infty}{\Gamma(\alpha+1)}\|x-y\|_X,\quad i=1,2,3.
\end{aligned}
$$

According to the above inequalities, we can get

$$
\|Tx-Ty\|_X \leq (\Delta_1+\Delta_2+\Delta_3+\frac{\lambda}{\Gamma(\alpha+1)}+\frac{3\lambda N}{\Gamma(\alpha+1)})\|x-y\|_X. \tag{3.13}
$$

In view of the condition (3.12), we deduce that T is a contraction. Then, we obtain by the Lemma 2.6 that the operator T has a unique fixed point $x \in B_\rho$, which is the unique solution of the system (1.1). The theorem is proved. □

4. Ulam Stability Analysis

In this section, we prove the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the system (1.1). For this purpose, we first provide the related stability concepts of our problem. Let $\varepsilon_i > 0,\ f_i : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ be continuous functions and $\psi_i(t) : [0,1] \to \mathbb{R}^+$ $(i = 1,2,3)$ are non-increasing continuous functions. Consider the following inequalities:

$$
\left|{}^C D_{0+}^{\beta}\left({}^C D_{0+}^{\alpha}+\lambda\right)x_i(t)-f_i(t,x_1(t),x_2(t),x_3(t))\right| \leq \varepsilon_i,\ t\in[0,1],\ i=1,2,3. \tag{4.1}
$$

$$
\left|{}^C D_{0+}^{\beta}\left({}^C D_{0+}^{\alpha}+\lambda\right)x_i(t)-f_i(t,x_1(t),x_2(t),x_3(t))\right| \leq \psi_i(t)\,\varepsilon_i,\ t\in[0,1],\ i=1,2,3. \tag{4.2}
$$

Definition 4.1 If there is a constant $c_{f_1,f_2,f_3} > 0$ such that for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \varepsilon_3) > 0$ and for each $v = (v_1, v_2, v_3) \in X$ be such that the inequalities (4.1) and conditions (3.2)-(3.4), there exists a solution $u = (u_1, u_2, u_3) \in X$ of (1.1) with

$$\| u - v\|_X \leq c_{f_1,f_2,f_3}\varepsilon,$$

the system (1.1) is called Ulam-Hyers stable.

Remark 4.2 If there exist functions $\chi_i \in C([0,1],\mathbb{R})\,(i = 1,2,3)$, which depend on $v_i\,(i = 1,2,3)$, separately such that
(i) $|\chi_i(t)| \leq \varepsilon_i,\ t \in [0,1],\ (i = 1,2,\cdots,n)$;
(ii) ${}^{C}D^{\beta}_{0+}\left({}^{C}D^{\alpha}_{0+} + \lambda\right)v_i(t) = f_i(t, v_1(t), v_2(t), v_3(t)) + \chi_i(t),\ t \in [0,1],\ i = 1,2,3,$
we say that $v = (v_1, v_2, v_3) \in X$ be a solution of (4.1).

Definition 4.3 If there exists a constant $c_{f_1,f_2,f_3,\psi} > 0$ such that for every $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \varepsilon_3) > 0$ and for each $v = (v_1, v_2, v_3) \in X$ be such that inequalities (4.2) and conditions (3.2)-(3.4), there exists a solution $u = (u_1, u_2, u_3) \in X$ of (1.1) with

$$\| u - v\|_X \leq c_{f_1,f_2,f_3,\psi}\varepsilon\psi(t), \quad t \in [0,1],$$

the system (1.1) is called Ulam-Hyers-Rassias stable with respect to $\psi = \psi(\psi_1, \psi_2, \psi_3) \in C([0,1],\mathbb{R}^+)$.

Remark 4.4 If there exist functions $\phi_i \in C([0,1],\mathbb{R})\,(i = 1,2,3)$, which depend on $v_i\,(i = 1,2,3)$, separately such that
(i) $\left|\phi_i(t)\right| \leq \psi_i(t)\,\varepsilon_i,\ t \in [0,1],\ (i = 1,2,\cdots,n)$;
(ii) ${}^{C}D^{\beta}_{0+}\left({}^{C}D^{\alpha}_{0+} + \lambda\right)v_i(t) = f_i(t, v_1(t), v_2(t), v_3(t)) + \phi_i(t),\ t \in [0,1],\ i = 1,2,3,$
we say that $v = (v_1, v_2, v_3) \in X$ be a solution of (4.2).

In the next two theorems, we present sufficient conditions on which the system (1.1) is Ulam-Hyers stable and Ulam-Hyers-Rassias stable.

Theorem 4.5 Assume that (H_1), (H_3) and (3.12) are satisfied. Let $u = (u_1, u_2, u_3) \in X$ be the solution of the system (1.1) and $v = (v_1, v_2, v_3) \in X$ is the solution of the inequality problem (4.1) and (3.2)-(3.4). Then, there exists a constant $c_{f_1,f_2,f_3} > 0$ such that for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \varepsilon_3) > 0$,

$$\| u - v\|_X \leq c_{f_1,f_2,f_3}\varepsilon$$

that is, system (1.1) is Ulam-Hyers stable.

Proof. Since v is the solution of (4.1) and (3.2)-(3.4), then by Remark 4.2, v_i is the solution of the following system.

$$\begin{cases} {}^{C}D^{\beta}_{0+}\left({}^{C}D^{\alpha}_{0+} + \lambda\right)v_i(t) = f_i(t, v_1(t), v_2(t), v_3(t)) + \chi_i(t), t \in (0,1), i = 1,2,3, \\ v_1(0) + v_2(0) = -(v_2(1) + v_3(1)), \int_{\eta_1}^{\xi_1}(v_2(s) + v_3(s))ds = A_1, \\ v_2(0) + v_3(0) = -(v_3(1) + v_1(1)), \int_{\eta_2}^{\xi_2}(v_3(s) + v_1(s))ds = A_2, \\ v_3(0) + v_1(0) = -(v_1(1) + v_2(1)), \int_{\eta_3}^{\xi_3}(v_1(s) + v_2(s))ds = A_3. \end{cases} \tag{4.3}$$

In view of Lemma 3.1, the solution $v = (v_1, v_2, v_3) \in X$ of system (4.3) is given by

$$\begin{aligned}
v_i(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(\tilde{\varphi}_i(s)+\chi_i(s))ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_i(s)ds \\
&+[t^\alpha(\alpha+1)e_{i,1}+e_{i,2}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\tilde{\varphi}_2(s)+\tilde{\varphi}_3(s)+\chi_2(s)+\chi_3(s))ds \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(v_2(s)+v_3(s))ds] \\
&+[t^\alpha(\alpha+1)e_{i,3}+e_{i,4}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\tilde{\varphi}_3(s)+\tilde{\varphi}_1(s)+\chi_3(s)+\chi_1(s))ds \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(v_3(s)+v_1(s))ds] \\
&+[t^\alpha(\alpha+1)e_{i,5}+e_{i,6}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\tilde{\varphi}_1(s)+\tilde{\varphi}_2(s)+\chi_1(s)+\chi_2(s))ds \\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(v_1(s)+v_2(s))ds] \\
&+[t^\alpha(\alpha+1)e_{i,7}+e_{i,8}]\{A_1-\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\tilde{\varphi}_2(\theta)+\tilde{\varphi}_3(\theta)+\chi_2(\theta)+\chi_3(\theta))d\theta \\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(v_2(\theta)+v_3(\theta))d\theta]ds\} \\
&+[t^\alpha(\alpha+1)e_{i,9}+e_{i,10}]\{A_2-\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\tilde{\varphi}_3(\theta)+\tilde{\varphi}_1(\theta)+\chi_3(\theta)+\chi_1(\theta))d\theta \\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(v_3(\theta)+v_1(\theta))d\theta]ds\} \\
&+[t^\alpha(\alpha+1)e_{i,11}+e_{i,12}]\{A_3-\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\tilde{\varphi}_1(\theta)+\tilde{\varphi}_2(\theta)+\chi_1(\theta)+\chi_2(\theta))d\theta \\
&-\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(v_1(\theta)+v_2(\theta))d\theta]ds\}, \qquad i=1,2,3, \ \ t\in[0,1].
\end{aligned}$$

where

$$\tilde{\varphi}_i(s) = f_i(s, v_1(s), v_2(s), v_3(s)), i = 1, 2, 3.$$

Recalling the operator T , defined in (3.10), under current conditions, T is a contraction, and hence (1.1) has a unique solution $u = (u_1, u_2, u_3) \in X$, which is the fixed point of T. From (3.13) we also know that

$$\| Tu - Tv\|_X = \| u - Tv\|_X \le (\Delta_1+\Delta_2+\Delta_3+\frac{\lambda}{\Gamma(\alpha+1)}+\frac{3\lambda N}{\Gamma(\alpha+1)}) \| u-v\|_X$$

This implies,

$$\| u - v\|_X \le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)[1-(\Delta_1+\Delta_2+\Delta_3)]-\lambda-3\lambda N}\|Tv-v\|_X. \tag{4.4}$$

On the other hand, we have the following estimate

$$
\begin{aligned}
& |(T_1v)(t) - v_1(t)| \\
\leq \; & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|\chi_1(s)|ds \\
& +[t^{\alpha}(\alpha+1)e_{1,1}+e_{1,2}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\chi_2(s)+\chi_3(s)|)ds \\
& +[t^{\alpha}(\alpha+1)e_{1,3}+e_{1,4}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\chi_3(s)+\chi_1(s)|)ds \\
& +[t^{\alpha}(\alpha+1)e_{1,5}+e_{1,6}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\chi_1(s)+\chi_2(s)|)ds \\
& +[t^{\alpha}(\alpha+1)e_{1,7}+e_{1,8}]\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\chi_2(\theta)+\chi_3(\theta)|)d\theta]ds \\
& +[t^{\alpha}(\alpha+1)e_{1,9}+e_{1,10}]\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\chi_3(\theta)+\chi_1(\theta)|)d\theta]ds \\
& +[t^{\alpha}(\alpha+1)e_{1,11}+e_{1,12}]\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\chi_1(\theta)+\chi_2(\theta)|)d\theta]ds \\
\leq \; & \frac{\varepsilon_1}{\Gamma(\alpha+\beta+1)}+\frac{m_{11}(t)\Sigma_{i=1}^3\varepsilon_i}{\Gamma(\alpha+\beta+1)}+\frac{m_{12}(t)\Sigma_{i=1}^3\varepsilon_i}{\Gamma(\alpha+\beta+1)}+\frac{m_{13}(t)\Sigma_{i=1}^3\varepsilon_i}{\Gamma(\alpha+\beta+1)} \\
& +\frac{m_{14}(t)\frac{\xi_1^{\alpha+\beta+1}-\eta_1^{\alpha+\beta+1}}{\alpha+\beta+1}\Sigma_{i=1}^3\varepsilon_i}{\Gamma(\alpha+\beta+1)}+\frac{m_{15}(t)\frac{\xi_2^{\alpha+\beta+1}-\eta_2^{\alpha+\beta+1}}{\alpha+\beta+1}\Sigma_{i=1}^3\varepsilon_i}{\Gamma(\alpha+\beta+1)}+\frac{m_{16}(t)\frac{\xi_3^{\alpha+\beta+1}-\eta_3^{\alpha+\beta+1}}{\alpha+\beta+1}\Sigma_{i=1}^3\varepsilon_i}{\Gamma(\alpha+\beta+1)}, \\
\leq \; & \|\tau_1\|_\infty\varepsilon_1+\|\omega_1\|_\infty\varepsilon_2+\|\gamma_1\|_\infty\varepsilon_3.
\end{aligned}
$$

In a similar manner,we get

$$
\begin{aligned}
|(T_2v)(t)-v_2(t)| &\leq \|\gamma_2\|_\infty\varepsilon_1+\|\tau_2\|_\infty\varepsilon_2+\|\omega_2\|_\infty\varepsilon_3, \\
|(T_3v)(t)-v_3(t)| &\leq \|\omega_3\|_\infty\varepsilon_1+\|\gamma_3\|_\infty\varepsilon_2+\|\tau_3\|_\infty\varepsilon_3.
\end{aligned}
$$

We deduce by the above inequalities that

$$
\begin{aligned}
\| Tv-v\|_X &= \|T_1v-v_1\|_\infty+\|T_2v-v_2\|_\infty+\|T_3v-v_3\|_\infty \\
&\leq (\tau+\omega+\gamma)\Sigma_{i=1}^3\varepsilon_i.
\end{aligned}
$$

Let $\varepsilon=\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$, which is combined with (4.4), we obtain

$$
\| u-v\|_X \leq \frac{3\Gamma(\alpha+1)}{\Gamma(\alpha+1)[1-(\Delta_1+\Delta_2+\Delta_3)]-\lambda-3\lambda N}(\tau+\omega+\gamma)\varepsilon.
$$

Hence, we conclude that the system (1.1) is Ulam-Hyers stable. Thus, Theorem 4.5 is proved. □

Theorem 4.6 Assume that (H_1), (H_3), and (3.12) are satisfied. Let $u=(u_1,u_2,u_3)\in X$ be the solution of the system (1.1), $v=(v_1,v_2,v_3)\in X$ is the solution of the inequality problem (4.2) and (3.2)-(3.4), $\psi_i\in C([0,1],\mathbb{R}^+), i=1,2,3$ and there exist $\rho_{\psi_i}>0$, such that for each $t\in[0,1]$,

$$
I_{0+}^{\alpha+\beta}\psi_i(t)\leq\rho_{\psi_i}\psi_i(t),\ \ i=1,2,3.
$$

Then, there exist a constant $c_{f_1,f_2,f_3,\psi}>0$ and $\psi\in C([0,1],\mathbb{R}^+)$ such that for each $\varepsilon=\varepsilon(\varepsilon_1,\varepsilon_2,\varepsilon_3)>0$,

$$
\|u-v\|_X\leq c_{f_1,f_2,f_3,\psi}\varepsilon\psi(t),\ \ t\in[0,1],
$$

that is, system (1.1) is Ulam-Hyers-Rassias stable.

Proof. Let $v = (v_1, v_2, v_3) \in X$ is a solution of (4.2) and (3.2)-(3.4), by using Remark 4.4, Lemma 3.1 and proceeding as in the proof of Theorem 4.5, v_1, v_2, v_3 can be given by

$$\begin{aligned}
v_i(t) = & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(\tilde{\varphi}_i(s)+\phi_i(s))ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_i(s)ds \\
& +[t^\alpha(\alpha+1)e_{i,1}+e_{i,2}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\tilde{\varphi}_2(s)+\tilde{\varphi}_3(s)+\phi_2(s)+\phi_3(s))ds \\
& +\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(v_2(s)+v_3(s))ds] \\
& +[t^\alpha(\alpha+1)e_{i,3}+e_{i,4}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\tilde{\varphi}_3(s)+\tilde{\varphi}_1(s)+\phi_3(s)+\phi_1(s))ds \\
& +\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(v_3(s)+v_1(s))ds] \\
& +[t^\alpha(\alpha+1)e_{i,5}+e_{i,6}][-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(\tilde{\varphi}_1(s)+\tilde{\varphi}_2(s)+\phi_1(s)+\phi_2(s))ds \\
& +\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(v_1(s)+v_2(s))ds] \\
& +[t^\alpha(\alpha+1)e_{i,7}+e_{i,8}]\{A_1 - \int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\tilde{\varphi}_2(\theta)+\tilde{\varphi}_3(\theta)+\phi_2(\theta)+\phi_3(\theta))d\theta \\
& -\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(v_2(\theta)+v_3(\theta))d\theta]ds\} \\
& +[t^\alpha(\alpha+1)e_{i,9}+e_{i,10}]\{A_2 - \int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\tilde{\varphi}_3(\theta)+\tilde{\varphi}_1(\theta)+\phi_3(\theta)+\phi_1(\theta))d\theta \\
& -\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(v_3(\theta)+v_1(\theta))d\theta]ds\} \\
& +[t^\alpha(\alpha+1)e_{i,11}+e_{i,12}]\{A_3 - \int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(\tilde{\varphi}_1(\theta)+\tilde{\varphi}_2(\theta)+\phi_1(\theta)+\phi_2(\theta))d\theta \\
& -\frac{\lambda}{\Gamma(\alpha)}\int_0^s (s-\theta)^{\alpha-1}(v_1(\theta)+v_2(\theta))d\theta]ds\}, \qquad i=1,2,3, \ \ t\in[0,1].
\end{aligned}$$

Note that $u = (u_1, u_2, u_3) \in X$ is the solution of the system (1.1). An argument similar to the one used in the proof of Theorem 4.5 shows that

$$\| Tu - Tv\|_X = \| u - Tv\|_X \le (\Delta_1+\Delta_2+\Delta_3+\frac{\lambda}{\Gamma(\alpha+1)}+\frac{3\lambda N}{\Gamma(\alpha+1)}) \| u-v\|_X,$$

we deduce that

$$\| u - v\|_X \le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)[1-(\Delta_1+\Delta_2+\Delta_3)]-\lambda-3\lambda N}\|Tv-v\|_X. \tag{4.5}$$

On the other hand, we have the following estimate

$$
\begin{aligned}
&|(T_1v)(t) - v_1(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|\phi_i(s)|ds \\
&+[t^\alpha(\alpha+1)e_{1,1}+e_{1,2}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\phi_2(s)+\phi_3(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,3}+e_{1,4}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\phi_3(s)+\phi_1(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,5}+e_{1,6}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\phi_1(s)+\phi_2(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,7}+e_{1,8}]\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\phi_2(\theta)+\phi_3(\theta)|)d\theta]ds \\
&+[t^\alpha(\alpha+1)e_{1,9}+e_{1,10}]\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\phi_3(\theta)+\phi_1(\theta)|)d\theta]ds \\
&+[t^\alpha(\alpha+1)e_{1,11}+e_{1,12}]\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\phi_1(\theta)+\phi_2(\theta)|)d\theta]ds \\
\leq\ & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|\varepsilon_i\psi_i(s)|ds \\
&+[t^\alpha(\alpha+1)e_{1,1}+e_{1,2}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\varepsilon_1\psi_1(s)+\varepsilon_2\psi_2(s)+\varepsilon_3\psi_3(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,3}+e_{1,4}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\varepsilon_1\psi_1(s)+\varepsilon_2\psi_2(s)+\varepsilon_3\psi_3(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,5}+e_{1,6}]\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}(|\varepsilon_1\psi_1(s)+\varepsilon_2\psi_2(s)+\varepsilon_3\psi_3(s)|)ds \\
&+[t^\alpha(\alpha+1)e_{1,7}+e_{1,8}]\int_{\eta_1}^{\xi_1}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varepsilon_1\psi_1(\theta)+\varepsilon_2\psi_2(\theta)+\varepsilon_3\psi_3(\theta)|)d\theta]ds \\
&+[t^\alpha(\alpha+1)e_{1,9}+e_{1,10}]\int_{\eta_2}^{\xi_2}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varepsilon_1\psi_1(\theta)+\varepsilon_2\psi_2(\theta)+\varepsilon_3\psi_3(\theta)|)d\theta]ds \\
&+[t^\alpha(\alpha+1)e_{1,11}+e_{1,12}]\int_{\eta_3}^{\xi_3}[\frac{1}{\Gamma(\alpha+\beta)}\int_0^s (s-\theta)^{\alpha+\beta-1}(|\varepsilon_1\psi_1(\theta)+\varepsilon_2\psi_2(\theta)+\varepsilon_3\psi_3(\theta)|)d\theta]ds \\
\leq\ & \varepsilon_1\rho_{\psi_1}\psi_1(t)+m_{11}(t)[\varepsilon_1\rho_{\psi_1}\psi_1(t)+\varepsilon_2\rho_{\psi_2}\psi_2(t)+\varepsilon_3\rho_{\psi_3}\psi_3(t)] \\
&+m_{12}(t)[\varepsilon_1\rho_{\psi_1}\psi_1(t)+\varepsilon_2\rho_{\psi_2}\psi_2(t)+\varepsilon_3\rho_{\psi_3}\psi_3(t)] \\
&+m_{13}(t)[\varepsilon_1\rho_{\psi_1}\psi_1(t)+\varepsilon_2\rho_{\psi_2}\psi_2(t)+\varepsilon_3\rho_{\psi_3}\psi_3(t)] \\
&+m_{14}(t)(\xi_1-\eta_1)[\varepsilon_1\rho_{\psi_1}\psi_1(t)+\varepsilon_2\rho_{\psi_2}\psi_2(t)+\varepsilon_3\rho_{\psi_3}\psi_3(t)] \\
&+m_{15}(t)(\xi_2-\eta_2)[\varepsilon_1\rho_{\psi_1}\psi_1(t)+\varepsilon_2\rho_{\psi_2}\psi_2(t)+\varepsilon_3\rho_{\psi_3}\psi_3(t)] \\
&+m_{16}(t)(\xi_3-\eta_3)[\varepsilon_1\rho_{\psi_1}\psi_1(t)+\varepsilon_2\rho_{\psi_2}\psi_2(t)+\varepsilon_3\rho_{\psi_3}\psi_3(t)] \\
\leq\ & (\|\varsigma_1\|_\infty+1)\varepsilon_1\rho_{\psi_1}\psi_1(t)+\|\varsigma_1\|_\infty\varepsilon_2\rho_{\psi_2}\psi_2(t)+\|\varsigma_1\|_\infty\varepsilon_3\rho_{\psi_3}\psi_3(t),\quad t\in[0,1].
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
|(T_2v)(t)-v_2(t)| &\leq \|\varsigma_2\|_\infty\varepsilon_1\rho_{\psi_1}\psi_1(t)+(\|\varsigma_2\|_\infty+1)\varepsilon_2\rho_{\psi_2}\psi_2(t)+\|\varsigma_2\|_\infty\varepsilon_3\rho_{\psi_3}\psi_3(t), \\
|(T_3v)(t)-v_3(t)| &\leq \|\varsigma_3\|_\infty\varepsilon_1\rho_{\psi_1}\psi_1(t)+\|\varsigma_3\|_\infty\varepsilon_2\rho_{\psi_2}\psi_2(t)+(\|\varsigma_3\|_\infty+1)\varepsilon_3\rho_{\psi_3}\psi_3(t).
\end{aligned}
$$

In the above content,

$$\begin{aligned} \varsigma_i(t) &= m_{i,1}(t) + m_{i,2}(t) + m_{i,3}(t) + m_{i,4}(t)(\xi_1 - \eta_1) + m_{i,5}(t)(\xi_2 - \eta_2) + m_{i,6}(t)(\xi_3 - \eta_3), \\ \varsigma &= \max\{\|\varsigma_1\|_\infty, \ \|\varsigma_2\|_\infty, \ \|\varsigma_3\|_\infty\}, \ i = 1, 2, 3. \end{aligned}$$

In view of the above estimates, we find that

$$\begin{aligned} \| Tv - v\|_X &= \|T_1 v - v_1\|_\infty + \|T_2 v - v_2\|_\infty + \|T_3 v - v_3\|_\infty \\ &\le (3\varsigma + 1)[\varepsilon_1 \rho_{\psi_1} \psi_1(t) + \varepsilon_2 \rho_{\psi_2} \psi_2(t) + \varepsilon_3 \rho_{\psi_3} \psi_3(t)], \ \ t \in [0, 1]. \end{aligned}$$

Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, $\psi(t) = \max\{\psi_1(t), \psi_2(t), \psi_3(t)\}$, which is combined with (4.5), we get

$$\| u - v\|_X \le \frac{\Gamma(\alpha+1)(3\varsigma+1)(\rho_{\psi_1} + \rho_{\psi_2} + \rho_{\psi_3})}{\Gamma(\alpha+1)[1 - (\Delta_1 + \Delta_2 + \Delta_3)] - \lambda - 3\lambda N} \varepsilon \psi(t), \ \ t \in [0, 1].$$

Let

$$c_{f_1, f_2, f_3, \psi} = \frac{\Gamma(\alpha+1)(3\varsigma+1)(\rho_{\psi_1} + \rho_{\psi_2} + \rho_{\psi_3})}{\Gamma(\alpha+1)[1 - (\Delta_1 + \Delta_2 + \Delta_3)] - \lambda - 3\lambda N}.$$

We deduce that the system (1.1) is Ulam-Hyers-Rassias stable. So, we finish the proof of Theorem 4.6. □

Conclusion

In this paper, the cyclic anti-periodic and nonlocal integral boundary value problem to the self-adjoint nonlinear tripled fractional Langevin differential system was investigated. By some fixed point theorems, some sufficient conditions for the existence and uniqueness of solutions to the problem have been presented. And, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the problem have been established. Our main results enrich some previous results. Furthermore, some examples is given to validate our main results. From our view, there few papers discussing the cyclic nonlocal boundary value problems to the self-adjoint nonlinear tripled fractional Langevin differential system. Moreover, the cyclic boundary value problems of the self-adjoint nonlinear tripled system are more difficult and challenging than boundary value problems of the normal coupled systems.

5. Example

Example 5.1 Let $\alpha = \frac{3}{5}$, $\beta = \frac{1}{2}$, $\lambda = 4 \times 10^{-8}$, $\xi_1 = \frac{1}{2}$, $\eta_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{4}$, $\eta_1 = \frac{1}{5}$, $\xi_3 = \frac{1}{6}$, $\eta_3 = \frac{1}{7}$, $A_1 = \frac{3}{7}$, $A_2 = \frac{4}{7}$, $A_3 = \frac{5}{7}$. We consider the following tripled system.

$$\begin{cases} {}^cD_{0+}^{\frac{1}{2}}({}^cD_{0+}^{\frac{3}{5}} + 4 \times 10^{-8})x_i(t) = f_i(t, x_1(t), x_2(t), x_3(t)), t \in (0, 1), \\ x_1(0) + x_2(0) = -(x_2(1) + x_3(1)), \int_{\frac{1}{3}}^{\frac{1}{2}} (x_2(s) + x_3(s))ds = \frac{3}{7}, \\ x_2(0) + x_3(0) = -(x_3(1) + x_1(1)), \int_{\frac{1}{5}}^{\frac{1}{4}} (x_3(s) + x_1(s))ds = \frac{4}{7}, \\ x_3(0) + x_1(0) = -(x_1(1) + x_2(1)), \int_{\frac{1}{7}}^{\frac{1}{6}} (x_1(s) + x_2(s))ds = \frac{5}{7}. \end{cases} \tag{5.1}$$

Where

$$\begin{aligned} f_1(t, x_1(t), x_2(t), x_3(t)) &= t^{\frac{7}{8}} + 8 \times 10^{-6} \varrho + 5 \times 10^{-6} \sigma + 1.7 \times 10^{-7} \sin \varpi, \\ f_2(t, x_1(t), x_2(t), x_3(t)) &= t^{\frac{4}{5}} + 1 \times 10^{-7} \sin \varrho + 1 \times 10^{-7} \sigma + 1 \times 10^{-7} \varpi, \\ f_3(t, x_1(t), x_2(t), x_3(t)) &= t^{\frac{1}{2}} + 2.5 \times 10^{-7} \sin \varrho + 2 \times 10^{-6} \sin \sigma + 3 \times 10^{-6} \varpi. \end{aligned}$$

We choose

$$k_1(t) = t^{\frac{7}{8}},\ k_2(t) = t^{\frac{4}{5}},\ k_3(t) = t^{\frac{1}{2}},$$
$$p_1 = 8\times 10^{-6},\ p_2 = 1\times 10^{-7},\ p_3 = 2.5\times 10^{-7},$$
$$q_1 = 5\times 10^{-6},\ q_2 = 1\times 10^{-7},\ q_3 = 2\times 10^{-6},$$
$$r_1 = 1.7\times 10^{-7},\ r_2 = 1\times 10^{-7},\ r_3 = 3\times 10^{-6},$$

then assumption $(H_1), (H_2)$ are satisfied. In addition, we can calculate

$$\tau + \omega + \gamma \leq 56738010,\ N \leq 19792106,\ \ell_i = p_i + q_i + r_i = 3\times 10^{-7},\ i = 1, 2, 3.$$

Therefore, we find

$$(\tau + \omega + \gamma)\sum_{i=1}^{3} \ell_i + \frac{\lambda(1+3N)}{\Gamma(\alpha+1)} \leq 56738010\times 9\times 10^{-7} + \frac{4\times 10^{-8}\times 59376319}{\Gamma(1.6)} \approx 0.8 < 1,$$

that is, condition (3.11) holds. Therefore, by Theorem 3.2, we conclude that the system (5.1) has at least one solution on $[0, 1]$.

Example 5.2 Let $\alpha = \frac{3}{5},\ \beta = \frac{1}{2},\ \lambda = 8\times 10^{-8},\ \xi_1 = \frac{1}{2},\ \eta_1 = \frac{1}{3},\ \xi_2 = \frac{1}{4},\ \eta_1 = \frac{1}{5},\ \xi_3 = \frac{1}{6},\ \eta_3 = \frac{1}{7},$ $A_1 = \frac{3}{7},\ A_2 = \frac{4}{7},\ A_3 = \frac{5}{7}$. We consider the following tripled system.

$$\begin{cases} {}^cD_{0+}^{\frac{1}{2}}({}^cD_{0+}^{\frac{3}{5}} + 8\times 10^{-8})x_i(t) = f_i(t, x_1(t), x_2(t), x_3(t)), t\in(0,1), \\ x_1(0) + x_2(0) = -(x_2(1) + x_3(1)), \int_{\frac{1}{3}}^{\frac{1}{2}}(x_2(s) + x_3(s))ds = \frac{3}{7}, \\ x_2(0) + x_3(0) = -(x_3(1) + x_1(1)), \int_{\frac{1}{5}}^{\frac{1}{4}}(x_3(s) + x_1(s))ds = \frac{4}{7}, \\ x_3(0) + x_1(0) = -(x_1(1) + x_2(1)), \int_{\frac{1}{7}}^{\frac{1}{6}}(x_1(s) + x_2(s))ds = \frac{5}{7}. \end{cases} \tag{5.2}$$

Where

$$\begin{aligned} f_1(t, x_1(t), x_2(t), x_3(t)) &= 2\times 10^{-7}[\frac{|x_1(t)|}{3+|x_1(t)|} + \frac{|x_2(t)|}{3+|x_2(t)|} + \frac{|x_3(t)|}{3+|x_3(t)|}], \\ f_2(t, x_1(t), x_2(t), x_3(t)) &= 4\times 10^{-7}[\frac{|x_1(t)|}{4+|x_1(t)|} + \sin|x_2(t)| + \sin|x_3(t)|], \\ f_3(t, x_1(t), x_2(t), x_3(t)) &= 3\times 10^{-7}[\sin|x_1(t)| + \sin|x_2(t)| + \sin|x_3(t)|]. \end{aligned}$$

We take

$$L_1 = 2\times 10^{-7},\ L_2 = 4\times 10^{-7},\ L_3 = 3\times 10^{-7},$$

then assumption $(H_1), (H_3)$ are satisfied. In addition, we can find

$$\begin{aligned} \frac{\lambda(3+3N)}{\Gamma(\alpha+1)} + \Delta_1 + \Delta_2 + \Delta_3 &\leq \frac{\lambda(3+3N)}{\Gamma(\alpha+1)} + (\tau+\omega+\gamma)(L_1 + L_2 + L_3) \\ &\leq \frac{8\times 10^{-8}\times 59376321}{\Gamma(1.6)} + 56738010\times 9\times 10^{-7} \approx 0.723 < 1, \end{aligned}$$

that is, condition (3.12) is also satisfied. Then we conclude from Theorem 3.3 that the system (5.2) has a unique solution.

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