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Commutators of some maximal functions with Lipschitz functions on mixed Morrey spaces

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Abstract. Let $0 \leq \alpha < n$, M_α be the fractional maximal function, $M^\sharp$ be the sharp maximal function and b be the locally integrable function. Denote by $[b, M_\alpha]$ and $[b, M^\sharp]$ be the commutators of the fractional maximal function M_α and the sharp maximal function $M^\sharp$. In this paper, we give some necessary and sufficient conditions for the boundedness of the commutators $[b, M_\alpha]$ and $[b, M^\sharp]$ on mixed Morrey spaces when the function b is the Lipschitz function, by which some new characterizations of the non-negative Lipschitz function are obtained.

1. Introduction and main results

Let T be the classical singular integral operator and b be the locally integrable function, the commutator $[b, T]$ is defined by

$$[b, T]f(x) = bTf(x) - T(bf)(x).$$

In 1976, Coifman, Rochberg and Weiss[2] obtained that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ was introduced by John and Nirenberg [9], which is defined as the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $f_Q := \frac{1}{|Q|} \int_Q f(x) \mathrm{d}x$. In 1978, Janson[7] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the commutator $[b, T]$ and showed that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\,(0 < \beta < 1)$, where $1 < p < \frac{n}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$ (see also Paluszyński[12]). Then, the theory of commutators have been studied intensively by many authors (see, for instance, [4, 13–15, 17, 19]), which plays an important role in harmonic analysis and partial differential equations.

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As usual, $Q = Q(x,r) \subset \mathbb{R}^n$ is a cube with a centre x and a radius r, whose sides are parallel to the coordinate axes. Let $|Q|$ be the Lebesgue measure of Q and χ_Q be the characteristic function of Q. The letters $\vec{p}, \vec{q}, \vec{r}, \dots$ denote n-tuples of the numbers in $[0,\infty](n \geq 1), \vec{p} = (p_1, \dots, p_n), \vec{q} = (q_1, \dots, q_n), \vec{r} = (r_1, \dots, r_n)$. By definition, for example, the inequality $0 < \vec{p} < \infty$ means that $0 < p_i < \infty$ for each i and $\vec{p} \leq \vec{q}$ means that $p_i \leq q_i$ for each i. Furthermore, for $\vec{p} = (p_1, \dots, p_n)$ and $r \in \mathbb{R}$, let

$$\frac{1}{\vec{p}} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right), \quad \frac{\vec{p}}{r} = \left(\frac{p_1}{r}, \dots, \frac{p_n}{r}\right), \quad \vec{p'} = \left(p'_1, \dots, p'_n\right),$$

where $p'_j = \frac{p_j}{p_j - 1}$ is the conjugate exponent of p_j. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \approx g$.

Let $0 \leq \alpha < n$, for a locally integrable function f, the fractional maximal function M_α is given by

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

When $\alpha = 0$, M_0 is the classical Hardy-Littlewood maximal function M, and M_α is the classical fractional maximal function when $0 < \alpha < n$.

The sharp maximal function $M^\sharp$ was introduced by Fefferman and Stein [5], which is defined as

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The maximal commutator of the fractional maximal function M_α with the locally integrable function b is given by

$$M_{\alpha,b}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |b(x) - b(y)||f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The nonlinear commutators of the fractional maximal function M_α and sharp maximal function $M^\sharp$ with the locally integrable function b are defined as

$$[b, M_\alpha](f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x)$$

and

$$[b, M^\sharp](f)(x) = b(x)M^\sharp(f)(x) - M^\sharp(bf)(x).$$

When $\alpha = 0$, we simply write by $[b, M] = [b, M_0]$ and $M_b = M_{0,b}$. We also remark that the commutators $M_{\alpha,b}$ and $[b, M_\alpha]$ essentially differ from each other. For instance, maximal commutator $M_{\alpha,b}$ is positive and sublinear, but nonlinear commutators $[b, M_\alpha]$ and $[b, M^\sharp]$ are neither positive nor sublinear. The mapping properties of commutators of maximal functions have been studied intensively, we refer the readers to see [6, 16, 18, 22] and therein references.

For a fixed cube Q and $0 \leq \alpha < n$, the fractional maximal function with respect to Q of a function f is given by

$$M_{\alpha,Q}(f)(x) = \sup_{Q \supseteq Q_0 \ni x} \frac{1}{|Q_0|^{1-\frac{\alpha}{n}}} \int_{Q_0} |f(y)| dy,$$

where the supremum is taken over all cubes Q_0 with $Q_0 \subseteq Q$ and $Q_0 \ni x$. Moreover, we denote by $M_Q = M_{0,Q}$ when $\alpha = 0$.

We also need to recall the definitions of Lipschitz spaces and mixed Morrey spaces.

Definition 1.1. *Let $0<\beta<1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,*

$$|b(x)-b(y)| \leq C|x-y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of the function b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

Definition 1.2. *[10] Let $\vec{q}=(q_1,\ldots,q_n)\in(0,\infty]^n$ and $p\in(0,\infty]$ satisfy*

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}.$$

The mixed Morrey space $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ is defined as the set of all $f \in L^0(\mathbb{R}^n)$ satisfying the following norm $\|\cdot\|_{\mathcal{M}_{\vec{q}}^p}$ is finite:

$$\|f\|_{\mathcal{M}_{\vec{q}}^p} \equiv \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}\left\|f\chi_Q\right\|_{L^{\vec{q}}} : Q \text{ is a cube in } \mathbb{R}^n\right\},$$

where $L^0(\mathbb{R}^n)$ denotes the set of measureable functions on $\mathbb{R}^n$ and

$$\|f\|_{L^{\vec{q}}} \equiv \left(\int_{\mathbb{R}}\cdots\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(x_1,\ldots,x_n)\right|^{q_1}\,\mathrm{d}x_1\right)^{\frac{q_2}{q_1}}\mathrm{d}x_2\right)^{\frac{q_3}{q_2}}\cdots\mathrm{d}x_n\right)^{\frac{1}{q_n}}.$$

If we take $q_1=q_2=\cdots=q_n=q$, then the mixed Morrey space $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ is the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$. If we take $\frac{1}{p}=\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}$, the mixed Morrey space $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ is the mixed-norm Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$.

The main results we obtained can be stated as follows.

Theorem 1.3. *Let $0<\beta<1$, $0\leq\alpha<n$, $0<\alpha+\beta<n$ and b be a locally integrable function. If $1<\vec{q},\vec{s}<\infty$, $1<p,r<\infty$, $\frac{n}{p}\leq\sum_{j=1}^n\frac{1}{q_j}$, $\frac{n}{r}\leq\sum_{j=1}^n\frac{1}{s_j}$, $\frac{1}{r}=\frac{1}{p}-\frac{\alpha+\beta}{n}$ and $\frac{\vec{q}}{p}=\frac{\vec{s}}{r}$, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\geq 0$.

(2) $[b,M_\alpha]$ is bounded from $\mathcal{M}_{\vec{q}}^p(\mathbb{R}^n)$ to $\mathcal{M}_{\vec{s}}^r(\mathbb{R}^n)$.

(3) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}_{\vec{s}}^r(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}_{\vec{s}}^r(\mathbb{R}^n)}} \leq C. \tag{1.1}$$

(4) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x)-M_Q(b)(x)|dx \leq C. \tag{1.2}$$

If we take $q_1=q_2=\cdots=q_n=q$ and $s_1=s_2=\cdots=s_n=s$, we can get the following corollary.

Corollary 1.4. *Let $0<\beta<1$, $0\leq\alpha<n$, $0<\alpha+\beta<n$ and b be a locally integrable function. If $1<q\leq p<\infty$, $1<s\leq r<\infty$, $\frac{1}{r}=\frac{1}{p}-\frac{\alpha+\beta}{n}$ and $\frac{q}{p}=\frac{s}{r}$, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\geq 0$.

(2) $[b,M_\alpha]$ is bounded from $\mathcal{M}_q^p(\mathbb{R}^n)$ to $\mathcal{M}_s^r(\mathbb{R}^n)$.

(3) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}_s^r(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}_s^r(\mathbb{R}^n)}} \leq C.$$

(4) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |b(x) - M_Q(b)(x)|dx \leq C.$$

If we take $\frac{1}{p} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}$ and $\frac{1}{r} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{s_j}$, then the following corollary holds.

Corollary 1.5. *Let $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable function. If $1 < \vec{q} \leq \vec{s} < \infty$ and $\sum_{j=1}^{n}\frac{1}{q_j} - \sum_{j=1}^{n}\frac{1}{s_j} = \alpha + \beta$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.

(2) $[b, M_\alpha]$ is bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{s}}(\mathbb{R}^n)$.

(3) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - M_Q(b))\chi_Q\|_{L^{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{\vec{s}}(\mathbb{R}^n)}} \leq C.$$

(4) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |b(x) - M_Q(b)(x)|dx \leq C.$$

Theorem 1.6. *Let $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable function. If $1 < \vec{q}, \vec{s} < \infty$, $1 < p, r < \infty$, $\frac{n}{p} \leq \sum_{j=1}^{n}\frac{1}{q_j}$, $\frac{n}{r} \leq \sum_{j=1}^{n}\frac{1}{s_j}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$ and $\frac{\vec{q}}{p} = \frac{\vec{s}}{r}$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) $M_{\alpha,b}$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(3) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - b_Q)\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \leq C. \tag{1.3}$$

(4) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |b(x) - b_Q|dx \leq C. \tag{1.4}$$

If we take $q_1 = q_2 = \cdots = q_n = q$ and $s_1 = s_2 = \cdots = s_n = s$, then the following result holds.

Corollary 1.7. *Let $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable function. If $1 < q \leq p < \infty$, $1 < s \leq r < \infty$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$ and $\frac{q}{p} = \frac{s}{r}$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) $M_{\alpha,b}$ is bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^r_s(\mathbb{R}^n)$.

(3) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - b_Q)\chi_Q\|_{\mathcal{M}^r_s(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_s(\mathbb{R}^n)}} \leq C.$$

(4) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |b(x) - b_Q|dx \leq C.$$

If we take $\frac{1}{p}=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}$ and $\frac{1}{r}=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{s_j}$, we have the following result.

Corollary 1.8. *Let $0<\beta<1$, $0\le\alpha<n$, $0<\alpha+\beta<n$ and b be a locally integrable function. If $1<\vec{q}\le\vec{s}<\infty$ and $\sum_{j=1}^{n}\frac{1}{q_j}-\sum_{j=1}^{n}\frac{1}{s_j}=\alpha+\beta$, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) $M_{\alpha,b}$ is bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{s}}(\mathbb{R}^n)$.

(3) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-b_Q)\chi_Q\|_{L^{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{\vec{s}}(\mathbb{R}^n)}}\le C.$$

(4) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x)-b_Q|dx\le C.$$

Theorem 1.9. *Let $0<\beta<1$ and b be a locally integrable function. If $1<\vec{q},\vec{s}<\infty$, $1<p,r<\infty$, $\frac{n}{p}\le\sum_{j=1}^{n}\frac{1}{q_j}$, $\frac{n}{r}\le\sum_{j=1}^{n}\frac{1}{s_j}$, $\frac{1}{r}=\frac{1}{p}-\frac{\beta}{n}$ and $\frac{\vec{q}}{p}=\frac{\vec{s}}{r}$, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\ge 0$.

(2) $[b,M^\sharp]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(3) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-2M^\sharp(b\chi_Q))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\le C. \tag{1.5}$$

(4) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x)-2M^\sharp(b\chi_Q)(x)|dx\le C. \tag{1.6}$$

If we take $q_1=q_2=\cdots=q_n=q$ and $s_1=s_2=\cdots=s_n=s$, then we have the following result.

Corollary 1.10. *Let $0<\beta<1$ and b be a locally integrable function. If $1<q\le p<\infty$, $1<s\le r<\infty$, $\frac{1}{r}=\frac{1}{p}-\frac{\beta}{n}$ and $\frac{q}{p}=\frac{s}{r}$, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\ge 0$.

(2) $[b,M^\sharp]$ is bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^r_s(\mathbb{R}^n)$.

(3) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-2M^\sharp(b\chi_Q))\chi_Q\|_{\mathcal{M}^r_s(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_s(\mathbb{R}^n)}}\le C.$$

(4) There exists a constant $C>0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x)-2M^\sharp(b\chi_Q)(x)|dx\le C.$$

If we take $\frac{1}{p}=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}$ and $\frac{1}{r}=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{s_j}$, we have the following conclusion.

Corollary 1.11. *Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable function. If $1 < \vec{q} \le \vec{s} < \infty$ and $\sum_{j=1}^{n} \frac{1}{q_j} - \sum_{j=1}^{n} \frac{1}{s_j} = \beta$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) $[b, M^\sharp]$ is bounded from $L^{\vec{q}}(\mathbb{R}^n)$ to $L^{\vec{s}}(\mathbb{R}^n)$.

(3) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - 2M^\sharp(b\chi_Q))\chi_Q\|_{L^{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{\vec{s}}(\mathbb{R}^n)}} \le C.$$

(4) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |b(x) - b_Q| dx \le C.$$

2. Preliminaries

To prove our main results, we present some necessary definitions and lemmas in this section.

We first need to introduce the predual spaces of mixed Morrey spaces following the idea of Zorko [23](see also Nogayama [11]).

Definition 2.1. *Let $1 \le p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. A measurable function A is said to be a $(p, \vec{q})$-block if there exists a cube Q that supports A such that*

$$\|A\|_{\vec{q}} \le |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right) - \frac{1}{p}}.$$

Definition 2.2. *Let $1 \le p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Define the function space $\mathcal{H}_{\vec{q'}}^{p'}(\mathbb{R}^n)$ as the set of all $f \in L^p(\mathbb{R}^n)$ for which f is realized as the sum $f = \sum_{j=0}^{\infty} \lambda_j A_j$ with some $\lambda = \{\lambda_j\}_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ and a sequence $\{A_j\}_{j\in\mathbb{N}_0}$ of $(p', \vec{q'})$-blocks. The norm $\|f\|_{\mathcal{H}_{\vec{q'}}^{p'}}$ for $f \in \mathcal{H}_{\vec{q'}}^{p'}(\mathbb{R}^n)$ is defined as*

$$\|f\|_{\mathcal{H}_{\vec{q'}}^{p'}} \equiv \inf_\lambda \|\lambda\|_{\ell^1}$$

where $\lambda = \{\lambda_j\}_{j\in\mathbb{N}_0}$ runs over all admissible expressions

$$f = \sum_{j=0}^{\infty} \lambda_j A_j, \{\lambda_j\}_{j\in\mathbb{N}_0} \in \ell^1, A_j \text{ is a } (p', \vec{q'})\text{-block for all } j \in \mathbb{N}_0.$$

Thus, we have the following Lemma, which was introduced by Nogayama [11].

Lemma 2.3. *Let $1 < p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Then*

$$\|\chi_Q\|_{\mathcal{M}_{\vec{q}}^{p}} = |Q|^{\frac{1}{p}}, \quad \|\chi_Q\|_{\mathcal{H}_{\vec{q'}}^{p'}} = |Q|^{\frac{1}{p'}}.$$

Lemma 2.4. *[10] Let $0 \le \alpha < n, 1 < \vec{q}, \vec{s} < \infty$ and $1 < p, r < \infty$. If $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$, $\frac{n}{r} \le \sum_{j=1}^{n} \frac{1}{s_j}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\vec{q}}{p} = \frac{\vec{s}}{r}$. Then, for $f \in \mathcal{M}_{\vec{q}}^{p}(\mathbb{R}^n)$,*

$$\left\|M_\alpha f\right\|_{\mathcal{M}_{\vec{s}}^{r}} \lesssim \|f\|_{\mathcal{M}_{\vec{q}}^{p}}.$$

It is well-known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey-Companato spaces (see, for example, [8]) and can be characterized by mean oscillation as the following lemma.

Lemma 2.5. *[3] Let* $0 < \beta < 1$ *and* $1 \le q < \infty$. *The space* $\dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ *is defined as the set of all locally integrable functions* f *such that*

$$\|f\|_{\dot{\Lambda}_{\beta,q}} = \sup_{Q} \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|} \int_Q \left| f(x) - f_Q \right|^q dx \right)^{\frac{1}{q}} < \infty.$$

Then, for all $0 < \beta < 1$ *and* $1 \le q < \infty$, $\dot{\Lambda}_{\beta}(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ *with equivalent norms.*

Lemma 2.6. *[21] Let* $0 \le \alpha < n$, Q *be a cube in* $\mathbb{R}^n$ *and* f *be the locally integrable function. Then, for any* $x \in Q$,

$$M_\alpha(f\chi_Q)(x) = M_{\alpha,Q}(f)(x).$$

Lemma 2.7. *[1] For any fixed cube Q, let* $E = \{x \in Q : b(x) \le b_Q\}$ *and* $F = \{x \in Q : b(x) > b_Q\}$. *Then*

$$\int_E |b(x) - b_Q| dx = \int_F |b(x) - b_Q| dx.$$

3. Proofs of main results

Proof of Theorem 1.3. (1) ⇒ (2): Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$. For all locally integral function f, we have

$$\begin{aligned}
|[b, M_\alpha](f)(x)| &= |b(x)M_\alpha(f)(x) - M_\alpha(bf)(x)| \\
&\le \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |b(x) - b(y)||f(y)| dy \\
&\le C\|b\|_{\dot{\Lambda}_\beta} \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha+\beta}{n}}} \int_Q |f(y)| dy \\
&\le C\|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta}(f)(x).
\end{aligned}$$

By Lemma 2.4, we conclude that $[b, M_\alpha]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(2) ⇒ (3): We divide the proof into two cases based on the scope of α.

Case 1. Assume $0 < \alpha < n$. For any fixed cube Q,

$$\begin{aligned}
&\frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&\le \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&\quad + \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(|Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b) - M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&:= I + II.
\end{aligned}$$

By the definition of $M_{\alpha,Q}$, we get, for any $x \in Q$,

$$M_{\alpha,Q}(\chi_Q)(x) = |Q|^{\frac{\alpha}{n}}. \tag{3.1}$$

Using Lemma 2.6, for any $x \in Q$, we have

$$M_\alpha(\chi_Q)(x) = M_{\alpha,Q}(\chi_Q)(x) = |Q|^{\frac{\alpha}{n}}, \quad M_\alpha(b\chi_Q)(x) = M_{\alpha,Q}(b)(x).$$

Thus,

$$\begin{aligned} b(x) - |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b)(x) &= |Q|^{-\frac{\alpha}{n}} (b(x)|Q|^{\frac{\alpha}{n}} - M_{\alpha,Q}(b)(x)) \\ &= |Q|^{-\frac{\alpha}{n}} (b(x)M_\alpha(\chi_Q)(x) - M_\alpha(b\chi_Q)(x)) \\ &= |Q|^{-\frac{\alpha}{n}} [b, M_\alpha](\chi_Q)(x). \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$, then by Lemma 2.3 and noting that $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$, we obtain

$$\begin{aligned} I &= \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\ &= \frac{1}{|Q|^{\frac{\alpha+\beta}{n}}} \frac{\|[b, M_\alpha](\chi_Q)\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\ &\le C \frac{1}{|Q|^{\frac{\alpha+\beta}{n}}} \frac{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\ &\le C. \end{aligned}$$

Similar to (3.1), by Lemma 2.5 and noting that

$$M_Q(\chi_Q)(x) = \chi_Q(x), \text{ for all } x \in Q,$$

we have, for any $x \in Q$,

$$M(\chi_Q)(x) = \chi_Q(x), \quad M(b\chi_Q)(x) = M_Q(b)(x). \tag{3.2}$$

Then, by (3.1) and (3.2), we can see that

$$\begin{aligned} &\left| |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b)(x) - M_Q(b)(x) \right| \\ &\le |Q|^{-\frac{\alpha}{n}} \left| M_\alpha(b\chi_Q)(x) - |b(x)| M_\alpha(\chi_Q)(x) \right| \\ &\quad + |Q|^{-\frac{\alpha}{n}} \left| |b(x)| M_\alpha(\chi_Q)(x) - M_\alpha(\chi_Q)(x) M(b\chi_Q)(x) \right| \\ &= |Q|^{-\frac{\alpha}{n}} \left| M_\alpha(|b|\chi_Q)(x) - |b(x)| M_\alpha(\chi_Q)(x) \right| \\ &\quad + |Q|^{-\frac{\alpha}{n}} M_\alpha(\chi_Q)(x) \left| |b(x)| M(\chi_Q)(x) - M(b\chi_Q)(x) \right| \\ &= |Q|^{-\frac{\alpha}{n}} \left| [|b|, M_\alpha](\chi_Q)(x) \right| + \left| [|b|, M](\chi_Q)(x) \right|. \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$ and we conclude that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ implies $|b| \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. By the definitions of $[b, M_\alpha]$ and M_α, we obtain, for any $x \in Q$,

$$\begin{aligned} \left| [|b|, M_\alpha](\chi_Q)(x) \right| &\le \sup_{Q' \ni x} \frac{1}{|Q'|^{1-\frac{\alpha}{n}}} \int_{Q'} |b(x) - b(y)||\chi_Q(y)| dy \\ &\le \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{Q' \ni x} \frac{1}{|Q'|^{1-\frac{\alpha+\beta}{n}}} \int_{Q'} |\chi_Q(y)| dy \\ &\le \|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta}(\chi_Q)(x) \\ &= \|b\|_{\dot{\Lambda}_\beta} |Q|^{\frac{\alpha+\beta}{n}} \chi_Q(x). \end{aligned}$$

Similarly, for any $x \in Q$, we have

$$\left| [|b|, M](\chi_Q)(x) \right| \le \|b\|_{\dot{\Lambda}_\beta} |Q|^{\frac{\beta}{n}} \chi_Q(x),$$

and hence,

$$\left||Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}(b)(x)-M_Q(b)(x)\right|\le C\|b\|_{\dot{\Lambda}_\beta}|Q|^{\frac{\beta}{n}}\chi_Q(x).$$

Then, by Lemma 2.6, we get

$$\begin{aligned}II&=\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(|Q|^{-\frac{\alpha}{n}}M_{\alpha,Q}(b)-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\\&\le C\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{|Q|^{\frac{\beta}{n}}\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\\&\le C.\end{aligned}$$

Thus, we have

$$\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\le C,$$

which leads to (1.1) since Q is arbitrary and the constant C is independent of Q.

Case 2. Assume $\alpha=0$. For any fixed cube Q and any $x\in Q$, by (3.2), we have

$$b(x)-M_Q(b)(x)=b(x)M(\chi_Q)(x)-M(b\chi_Q)(x)=[b,M](\chi_Q)(x).$$

Assume that $[b,M]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$ and $\frac{1}{r}=\frac{1}{p}-\frac{\beta}{n}$, then by Lemma 2.3, we deduce that

$$\begin{aligned}\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}&=\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|[b,M](\chi_Q)\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\\&\le C\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\\&\le C,\end{aligned}$$

which implies (1.1) holds.

(3) ⇒ (4): Assume (1.1) holds, then for any fixed cube Q, by Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}&\frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q\left|b(x)-M_Q(b)(x)\right|dx\\&\le C\frac{1}{|Q|^{1+\frac{\beta}{n}}}\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}\|\chi_Q\|_{\mathcal{H}^{r'}_{\vec{s}'}(\mathbb{R}^n)}\\&\le C\frac{1}{|Q|^{1+\frac{\beta}{n}}}|Q|^{\frac{1}{r'}}\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}\\&\le C\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{|Q|^{\frac{1}{r}}}\\&\le C\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b-M_Q(b))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}\\&\le C,\end{aligned}$$

where the constant C is independent of Q. This deduces (1.2).

(4) ⇒ (1): To prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, by Lemma 2.5, it suffices to show that there exists a constant $C > 0$ such that for any fixed cube Q,

$$\frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x) - b_Q|dx \le C.$$

For any fixed cube Q, let $E = \left\{x \in Q : b(x) \le b_Q\right\}$ and $F = \left\{x \in Q : b(x) > b_Q\right\}$. Since for any $x \in E$, we have $b(x) \le b_Q \le M_Q(b)(x)$, then

$$|b(x) - b_Q| \le |b(x) - M_Q(b)(x)|. \tag{3.3}$$

By Lemma 2.7 and (3.3), we have

$$\begin{aligned}\frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x) - b_Q|dx &= \frac{2}{|Q|^{1+\frac{\beta}{n}}}\int_E |b(x) - b_Q|dx\\ &\le \frac{2}{|Q|^{1+\frac{\beta}{n}}}\int_E |b(x) - M_Q(b)(x)|dx\\ &\le \frac{2}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x) - M_Q(b)(x)|dx\\ &\le C.\end{aligned}$$

Thus we deduce that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Next, we will show $b \ge 0$, it suffices to prove $b^- = 0$, where $b^- = -\min\{b, 0\}$. Let $b^+ = |b| - b^-$, then $b = b^+ - b^-$. For any fixed cube Q and $x \in Q$, we obtain

$$0 \le b^+(x) \le |b(x)| \le M_Q(b)(x),$$

then we can get

$$0 \le b^-(x) \le M_Q(b)(x) - b^+(x) + b^-(x) = M_Q(b)(x) - b(x).$$

Combining with the above estimates and (1.2) deduces that

$$\begin{aligned}\frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x &\le \frac{1}{|Q|}\int_Q \left|M_Q(b)(x) - b(x)\right|\\ &\le |Q|^{\frac{\beta}{n}}\left(\frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x) - M_Q(b)(x)|dx\right)\\ &\le C|Q|^{\frac{\beta}{n}}.\end{aligned}$$

Thus, Lebesgue's differentiation theorem implies that $b^- = 0$.

The proof of Theorem 1.3 is completed. □

Proof of Theorem 1.6. (1) ⇒ (2): Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q \subset \mathbb{R}^n$, it is easy to see

$$\begin{aligned}M_{\alpha,b}(f)(x) &= \sup_{Q \ni x}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |b(x) - b(y)||f(y)|dy\\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_{\alpha+\beta}f(x).\end{aligned}$$

By Lemma 2.4, we obtain that $M_{\alpha,b}$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(2) ⇒ (3): For any fixed cube $Q \subset \mathbb{R}^n$ and any $x \in Q$, we get

$$\begin{aligned}|b(x) - b_Q| &\le \frac{1}{|Q|}\int_Q |b(x) - b(y)|dy \\ &= \frac{1}{|Q|^{\frac{\alpha}{n}}}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |b(x) - b(y)|\chi_Q(y)dy \\ &\le |Q|^{-\frac{\alpha}{n}} M_{\alpha,b}(\chi_Q)(x).\end{aligned}$$

Since $M_{\alpha,b}$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$, then by Lemma 2.3 and noting that $\frac{1}{r} = \frac{1}{p} - \frac{\alpha+\beta}{n}$, we have

$$\begin{aligned}\frac{1}{|Q|^{\frac{\beta}{n}}}\frac{\|(b - b_Q)\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} &\le \frac{1}{|Q|^{\frac{\alpha+\beta}{n}}}\frac{\|M_{\alpha,b}(\chi_Q)\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\ &\le \frac{C}{|Q|^{\frac{\alpha+\beta}{n}}}\frac{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\ &\le C,\end{aligned}$$

which implies (1.3) holds since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3) ⇒ (4): Assume (1.3) holds, we will prove (1.4). For any fixed cube Q, by Hölder's inequality and Lemma 2.3, we can see

$$\begin{aligned}\frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |b(x) - b_Q|dx &\le \frac{C}{|Q|^{1+\frac{\beta}{n}}}\|(b - b_Q)\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}\|\chi_Q\|_{\mathcal{H}^{r'}_{\vec{s}'}(\mathbb{R}^n)} \\ &\le \frac{C}{|Q|^{\frac{\beta}{n}}}\frac{\|(b - b_Q)\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\ &\le C.\end{aligned}$$

(4) ⇒ (1): It follows from Lemma 2.5 directly, thus we omit the details.

This finishes the proof of Theorem 1.6. □

Proof of Theorem 1.9. (1) ⇒ (2): Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$. For any locally integral function f, we have the following estimate given in [20],

$$\left|[b, M^\sharp]f(x)\right| \le C\|b\|_{\dot{\Lambda}_\beta} M_\beta(f)(x).$$

Then, by Lemma 2.4, we obtain that $[b, M^\sharp]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(2) ⇒ (3): Assume $[b, M^\sharp]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$, we will prove (1.5). For any fixed cube Q and any $x \in Q$, we have (see [1] for details),

$$M^\sharp(\chi_Q)(x) = \frac{1}{2},$$

which implies that

$$\begin{aligned}b(x) - 2M^\sharp(b\chi_Q)(x) &= 2\left(b(x)M^\sharp(\chi_Q)(x) - M^\sharp(b\chi_Q)(x)\right) \\ &= 2[b, M^\sharp](\chi_Q)(x).\end{aligned}$$

Since $[b, M^\sharp]$ is bounded from $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$, then by Lemma 2.3 and noting that $\frac{1}{r} = \frac{1}{p} - \frac{\beta}{n}$, we obtain

$$\begin{aligned}
&\frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - 2M^\sharp(b\chi_Q))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&= \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|[b, M^\sharp](\chi_Q)\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&\leq C \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&\leq C,
\end{aligned}$$

where the constant C is independent of Q. This deduces (1.5).

(3) ⇒ (4): Assume (1.5) holds, we will prove (1.6). For any fixed cube Q, combining Hölder's inequality with Lemma 2.3 deduces that

$$\begin{aligned}
&\frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q \left| b(x) - 2M^\sharp(b\chi_Q)(x) \right| dx \\
&\leq \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - 2M^\sharp(b\chi_Q))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&\leq \frac{1}{|Q|^{\frac{\beta}{n}}} \frac{\|(b - 2M^\sharp(b\chi_Q))\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}}{\|\chi_Q\|_{\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)}} \\
&\leq C,
\end{aligned}$$

which implies (1.6) holds since the constant C is independent of Q.

(4) ⟹ (1): We first prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube Q, the following estimate was given in [1]:

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q| dx \leq \frac{2}{|Q|} \int_Q \left| b(x) - 2M^\sharp(b\chi_Q)(x) \right| dx.$$

It follows from (1.6) that

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \leq \frac{2}{|Q|^{1+\beta/n}} \int_Q \left| b(x) - 2M^\sharp(b\chi_Q)(x) \right| dx \leq C,$$

which leads to $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ by Lemma 2.5.

Now, let us show $b \geq 0$. It suffices to prove $b^- = 0$, where $b^- = -\min\{b, 0\}$ and let $b^+ = |b| - b^-$. For any fixed cube Q and any $x \in Q$, we have (see [1] for details),

$$|b_Q| \leq 2M^\sharp(b\chi_Q)(x),$$

which implies that

$$2M^\sharp(b\chi_Q)(x) - b(x) \geq |b_Q| - b(x) = |b_Q| - b^+(x) + b^-(x).$$

By (1.6), we have

$$|b_Q| - \frac{1}{|Q|} \int_Q b^+(x) dx + \frac{1}{|Q|} \int_Q b^-(x) dx \leq C|Q|^{\frac{\beta}{n}}, \tag{3.4}$$

where the constant C is independent of Q.

Let the side length of Q tends to 0 (then $|Q| \to 0$) with $x \in Q$. Lebesgue's differentiation theorem implies that the limit of the left-hand side of (3.4) equals to

$$|b(x)| - b^+(x) + b^-(x) = 2b^-(x) = 2|b^-(x)|.$$

Moreover, the right-hand side of (3.4) tends to 0. Thus, we conclude that $b^- = 0$.

The proof of Theorem 1.9 is completed. □

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