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The regularity criteria for the 3D magnetohydrodynamics equations

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Abstract. In this paper, we establish several regularity criteria for weak solutions of the 3D magnetohydrodynamics equations. Compared to previous results, we extend these criteria to Lorentz spaces, Lebesgue sum spaces, and anisotropic Lebesgue spaces.

1. Introduction

The 3D magnetohydrodynamics (MHD) equations take the form

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - \Delta b = (b\cdot\nabla)u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x),\ b(x,0) = b_0(x), \end{cases} \tag{1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, $b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t))$ and p represent the fluid velocity, the magnetic field and the scalar pressure, respectively. The MHD equations are fundamental tools for investigating the behavior of conducting fluids under the influence of magnetic fields. These equations seamlessly integrate principles from fluid dynamics and electromagnetic theory, thereby offering a comprehensive mathematical framework to understand physical phenomena in plasmas, liquid metals, and other conductive fluids. The MHD equations are pivotal in diverse fields such as astrophysics, nuclear fusion reactor design, geophysics, and materials processing. Their applications extend to elucidating the interactions between the solar wind and the Earth's magnetosphere, addressing control issues in magnetically confined nuclear fusion, and exploring magnetohydrodynamic effects in high-temperature superconductors. Through these applications, the MHD equations not only advance theoretical understanding but also drive technological innovations and practical solutions in various scientific and engineering domains.

For any given initial values $(u_0, b_0)\in H^s(\mathbb{R}^3)$, $s\geq 3$, Sermange and Temam [1] established the local well-posedness of the system (1). However, the global existence of strong solutions, or equivalently, the smoothness of global weak solutions, remains an open problem. It is well known that if (u,b) is a solution of the system (1), then for any $\lambda>0$, the scaled solution $(u_\lambda(x,t), b_\lambda(x,t)) = (\lambda u(\lambda x, \lambda^2 t), \lambda b(\lambda x, \lambda^2 t))$ is also

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a solution of the system (1). This type of Serrin condition under the perspective of scale invariance is very important, meaning that for any $\lambda > 0$, we have $\|u_\lambda\|_{L^\beta(0,T;L^\alpha(\mathbb{R}^3))} = \|u\|_{L^\beta(0,T;L^\alpha(\mathbb{R}^3))}$ if and only if $\frac{2}{\beta} + \frac{3}{\alpha} = 1$. He and Xin [2], as well as Zhou [3], obtained regularity conditions for the velocity field u and its gradient field ∇u. They proved that the solution (u, b) is regular if

$$u \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} \leq 1, \quad 3 < \alpha \leq \infty,$$

or

$$\nabla u \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} \leq 2, \quad \frac{3}{2} < \alpha \leq \infty.$$

This indirectly indicates that in the study of the regularity theory of the 3D MHD equations, the fluid velocity plays a more crucial role than the magnetic field. It has been found that the reduction in the velocity component is more interesting than the reduction in the magnetic field component.

An interesting question is whether regularity requires all velocity components. Since the third component can be computed from the divergence-free condition, two components of the velocity field should suffice. Previous studies have shown that even one component of the velocity field is sufficient. Jia and Zhou [4] proved this. More precisely, they obtained that if

$$u_3,\ b \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}, \quad \frac{10}{3} < \alpha \leq \infty,$$

then the weak solution is regular. They further refined this result by imposing certain scale-invariant conditions on b_h and b_3, see [5]. Additionally, Han and Xiong [6] extended the results of [4] to Lorentz spaces by utilizing the symmetric structure of the MHD equations.

When $b = 0$, the system (1) reduces to the classical 3D incompressible Navier-Stokes equations. The classical Prodi-Serrin condition [7] states that if

$$u \in L^\beta\left(0,T;L^\alpha\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} = 1, \quad 3 \leq \alpha \leq \infty, \tag{2}$$

then the weak solution u is regular on $(0,T]$. The Prodi-Serrin condition (2) was later extended to the gradient field ∇u by Da Veiga [8], specifically

$$\nabla u \in L^\beta\left(0,T;L^\alpha\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} = 2 \quad \text{and} \quad \frac{3}{2} < \alpha \leq \infty. \tag{3}$$

This corresponds to the regularity criteria for the MHD system. Over the past few decades, there have been many refinements to conditions (2) and (3). Penel and Pokorny [9] were the first to establish a regularity criterion for the 3D Navier-Stokes equations based solely on one directional derivative of the velocity field. They showed that if

$$\partial_3 u \in L^\beta\left(0,T;L^\alpha\left(\mathbb{R}^3\right)\right) \quad \text{with} \quad \frac{2}{\beta} + \frac{3}{\alpha} = \frac{3}{2} \quad \text{and} \quad 2 \leq \alpha \leq \infty, \tag{4}$$

then the solution u is smooth. Clearly, compared to (3), there is a discrepancy in (4). Kukavica and Ziane [10] optimized this condition. Recently, Miller [11] extended the regularity criterion (2) to Lebesgue sum spaces. More specifically, if $u = \phi + \varphi$, such that

$$\phi \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \varphi \in L^2(0,T;L^\infty(\mathbb{R}^3)),$$

with $\frac{2}{\beta} + \frac{3}{\alpha} = 1$ and $\beta > 3$, then the weak solution u is regular on $[0,T]$. Additionally, Ragusa and Wu [12] extended the regularity criterion (4) to anisotropic Lebesgue spaces. For more regularity criteria, see [13]-[21] and the references therein.

Inspired by the aforementioned studies, the goal of this paper is to establish regularity criteria for the 3D MHD equations (1) in Lorentz spaces, Lebesgue sum spaces and anisotropic Lebesgue spaces. The study of the 3D MHD equations not only contributes to a deeper understanding of theoretical physics but also actively promotes innovative developments in practical engineering technologies.

Theorem 1.1. *Let (u,b) be a local strong solution to the 3D MHD equations* (1) *with initial data $(u_0,b_0)\in H^s(\mathbb{R}^3)$, $s\geq 3$, and $\nabla\cdot u_0=\nabla\cdot b_0=0$. If*

$$u_3,\ b_3\in L^{\beta_1}(0,T;L^{\alpha_1,\infty}(\mathbb{R}^3)),\quad b_h\in L^{\beta_2}(0,T;L^{\alpha_2,\infty}(\mathbb{R}^3)), \tag{5}$$

where $\frac{3}{\alpha_1}+\frac{2}{\beta_1}\leq\frac{3}{4}+\frac{3}{2\alpha_1}$, $\alpha_1>\frac{10}{3}$, and $\frac{3}{\alpha_2}+\frac{2}{\beta_2}\leq 1$, $\alpha_2>3$, then (u,b) remains smooth on $[0,T]$.

Theorem 1.2. *Let (u,b) be a local strong solution to the 3D MHD equations* (1) *with initial data $(u_0,b_0)\in H^s(\mathbb{R}^3)$, $s\geq 3$, and $\nabla\cdot u_0=\nabla\cdot b_0=0$. If $u_3=\phi_1+\varphi_1$ and $b=\phi_2+\varphi_2$, such that*

$$\phi_1,\ \phi_2\in L^{\beta}(0,T;L^{\alpha}(\mathbb{R}^3)),\ \ \varphi_1,\ \varphi_2\in L^{\frac{8}{3}}(0,T;L^{\infty}(\mathbb{R}^3)), \tag{6}$$

where $\frac{3}{\alpha}+\frac{2}{\beta}\leq\frac{3}{4}+\frac{1}{2s}$ and $s>\frac{10}{3}$, then (u,b) remains smooth on $[0,T]$.

Theorem 1.3. *Let (u,b) be a local strong solution to the 3D MHD equations* (1) *with initial data $(u_0,b_0)\in H^s(\mathbb{R}^3)$, $s\geq 3$, and $\nabla\cdot u_0=\nabla\cdot b_0=0$. If*

$$\int_0^T\left\|\left\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\right\|_{L^{q_1}_{x_2}}\right\|_{L^{r_1}_{x_3}}^{\alpha}\mathrm{d}t+\int_0^T\left\|\left\|\|\partial_3 b\|_{L^{p_2}_{x_1}}\right\|_{L^{q_2}_{x_2}}\right\|_{L^{r_2}_{x_3}}^{\beta}\mathrm{d}t<\infty, \tag{7}$$

where $\frac{2}{\alpha}+\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}\leq 1$, $\frac{2}{\beta}+\frac{1}{p_2}+\frac{1}{q_2}+\frac{1}{r_2}\leq 1$, and $2<p_1,q_1,r_1,p_2,q_2,r_2\leq\infty$, then (u,b) remains smooth on $[0,T]$.

Remark 1.4. *Notice that, on the one hand, for $1\leq p<q<+\infty$, we have*

$$L^p(\mathbb{R}^3)=L^{p,p}(\mathbb{R}^3)\hookrightarrow L^{p,q}(\mathbb{R}^3)\hookrightarrow L^{p,\infty}(\mathbb{R}^3).$$

On the other hand, Theorem 1.1 by relaxing the conditions on b_h and b_3 to be scaling invariant. Therefore, Theorem 1.1 can be viewed as a further improvement of [5, 6, 22].

Remark 1.5. *Notice that for $1\leq p<q<+\infty$, we have the embedding*

$$L^{q,\infty}(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)+L^{\infty}(\mathbb{R}^3).$$

This relationship indicates that functions in the Lorentz space $L^{q,\infty}(\mathbb{R}^3)$ can be represented as sums of functions from $L^p(\mathbb{R}^3)$ and $L^{\infty}(\mathbb{R}^3)$. Therefore, Theorem 1.2 extends the results obtained by Jia and Zhou [4] to the context of Lebesgue sum spaces, providing a broader framework for understanding the regularity of solutions.

Remark 1.6. *On one hand, when $b=0$, the system* (1) *reduces to the Navier-Stokes equations. In this case, Theorem 1.3 aligns with the findings of Ragusa and Wu [12]. On the other hand, Theorem 1.3 broadens the scope of the results presented by Ni and Zhou [23], extending them to anisotropic Lebesgue spaces. This extension provides a more comprehensive framework for analyzing the regularity of solutions in various function spaces.*

The rest of this paper is organized as follows. Section 2 reviews some preliminaries. Section 3 is devoted to the proof of Theorem 1.1. Section 4 demonstrates the proof of Theorem 1.2. Finally, the proof of Theorem 1.3 is provided in Section 5.

2. Preliminaries

In this section, we introduce several important function spaces and lemmas that will be utilized throughout our analysis. First, we review the definition of Lorentz spaces [24].

Definition 2.1. *For $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz spaces $L^{p,q}(\mathbb{R}^3)$ consist of all measurable functions f such that*

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} := \begin{cases} \left[\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & \text{if } q = \infty, \end{cases}$$

is finite. Here, $f^*(t)$ denotes the non-increasing rearrangement of f.

Next, we recall the definition of the Lebesgue sum spaces [11], which are critical in the operations between multiple function spaces.

Definition 2.2. *Let X and Y be Banach function spaces, and let V be a vector space such that $X, Y \subset V$. Then, we have*

$$X + Y = \{g + h : g \in X, h \in Y\}.$$

Furthermore, $X + Y$ is a Banach function space with the norm

$$\|f\|_{X+Y} = \inf_{g+h=f} (\|g\|_X + \|h\|_Y).$$

Additionally, we review the definition of anisotropic Lebesgue spaces as introduced in [25].

Definition 2.3. *Let f be a measurable function defined on $\mathbb{R}^3 = \mathbb{R}_{x_1} \times \mathbb{R}_{x_2} \times \mathbb{R}_{x_3}$. If*

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{p_1} \, dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}} dx_3\right)^{\frac{1}{p_3}} < \infty,$$

then we say that $f \in L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_{x_3}$, and its norm is defined as

$$\left\|\left\|\|f\|_{L^{p_1}_{x_1}}\right\|_{L^{p_2}_{x_2}}\right\|_{L^{p_3}_{x_3}} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{p_1} \, dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}} dx_3\right)^{\frac{1}{p_3}}.$$

After introducing the definitions of Lorentz spaces, Lebesgue sum spaces and anisotropic Lebesgue spaces, we proceed to review several critical lemmas that will play essential roles in our subsequent analysis and proofs. These lemmas provide fundamental tools for our theoretical results and facilitate the establishment of important inequalities in different function spaces.

We begin by recalling Hölder inequality and Gagliardo-Nirenberg inequality in Lorentz spaces, as introduced in [26].

Lemma 2.4. *Let $1 \leq p, q, p_1, p_2, q_1, q_2 \leq \infty$. For any functions $f \in L^{p_1,q_1}(\mathbb{R}^3)$ and $g \in L^{p_2,q_2}(\mathbb{R}^3)$, the following inequality holds*

$$\|fg\|_{L^{p,q}(\mathbb{R}^3)} \leq C\|f\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^3)},$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Lemma 2.5. *Let $f \in L^{p,q}(\mathbb{R}^3)$ with $1 \leq p, q, p_1, q_1, p_2, q_2 \leq \infty$. Then, the Gagliardo-Nirenberg inequality for Lorentz spaces is given by*

$$\|f\|_{L^{p,q}} \leq C\|f\|^{\theta}_{L^{p_1,q_1}}\|f\|^{1-\theta}_{L^{p_2,q_2}},$$

where C is a positive constant, and the exponents satisfy

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad \theta \in (0, 1).$$

Following this, we give a crucial proposition concerning norm estimation in Lorentz spaces, which will be instrumental in our analysis.

Proposition 2.6. *Let $3 \le p \le \infty$. There exists a constant $C > 0$ such that for every $f \in C_0^\infty(\mathbb{R}^3)$, we have*

$$\|f\|_{L^{\frac{2p}{p-2},2}} \le C\|\partial_1 f\|_{L^2}^{\frac{1}{p}}\|\partial_2 f\|_{L^2}^{\frac{1}{p}}\|\partial_3 f\|_{L^2}^{\frac{1}{p}}\|f\|_{L^2}^{1-\frac{3}{p}}.$$

Proof: According to Lemma 2.5 and multiplicative Sobolev inequality, it follows that

$$\begin{aligned}
\|f\|_{L^{\frac{2p}{p-2},2}} &\le C\|f\|_{L^{2,2}}^{1-\frac{3}{p}}\|f\|_{L^{6,6}}^{\frac{3}{p}}\\
&= C\|f\|_{L^2}^{1-\frac{3}{p}}\|f\|_{L^6}^{\frac{3}{p}}\\
&\le C\|\partial_1 f\|_{L^2}^{\frac{1}{p}}\|\partial_2 f\|_{L^2}^{\frac{1}{p}}\|\partial_3 f\|_{L^2}^{\frac{1}{p}}\|f\|_{L^2}^{1-\frac{3}{p}}.
\end{aligned}$$

Finally, we present an important lemma in anisotropic Lebesgue spaces, which plays a critical role in the proof of Theorem 1.3, as detailed in [27, 28].

Lemma 2.7. *Let $p_1, p_2, p_3 \in [2, \infty)$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2} \ge 0$. Then, there exists a constant $C > 0$ such that for any $f \in L^2 \cap C^\infty$, the following inequality holds*

$$\left\|\left\|\|f\|_{L_{x_1}^{\frac{2p_1}{p_1-2}}}\right\|_{L_{x_2}^{\frac{2p_2}{p_2-2}}}\right\|_{L_{x_3}^{\frac{2p_3}{p_3-2}}} \le C\left\|\partial_1 f\right\|_{L^2}^{\frac{1}{p_1}}\left\|\partial_2 f\right\|_{L^2}^{\frac{1}{p_2}}\left\|\partial_3 f\right\|_{L^2}^{\frac{1}{p_3}}\left\|f\right\|_{L^2}^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}.$$

3. The proof of Theorem 1.1

The proof of Theorem 1.1: Multiplying $(1)_{1-2}$ by $-\Delta u, -\Delta b$ respectively, integrating over $\mathbb{R}^3$, and summing up, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\\
&= -\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_i\partial_k u_j\partial_k u_j\,\mathrm{d}x\\
&\quad - \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_j\partial_k u_j\,\mathrm{d}x + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_3 u_3\partial_k u_j\partial_k u_j\,\mathrm{d}x\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3}\partial_3 u_3\partial_3 u_j\partial_3 u_j\,\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\,\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\,\mathrm{d}x\\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k\partial_k u_i\partial_i b_j b_j\,\mathrm{d}x + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_k\partial_i b_j b_j\,\mathrm{d}x\\
&\quad - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k\partial_i u_i\partial_k b_j b_j\,\mathrm{d}x - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_i\partial_k\partial_k b_j b_j\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\nabla_h u||\nabla u|^2\,\mathrm{d}x + C\int_{\mathbb{R}^3}|b||\nabla b||\Delta u|\,\mathrm{d}x + C\int_{\mathbb{R}^3}|b||\nabla u||\Delta b|\,\mathrm{d}x\\
&=: \sum_{i=1}^{3} I_i.
\end{aligned} \tag{8}$$

Using Hölder inequality and multiplicative Sobolev inequality, it follows that

$$\begin{aligned} I_1 &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^6}^{\frac{3}{2}} \\ &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}. \end{aligned} \tag{9}$$

For I_2, using Lemma 2.4, Proposition 2.6 and Young inequality, one gets

$$\begin{aligned} I_2 &\le C\|b_3\|_{L^{\alpha_1,\infty}}\|\nabla b\|_{L^{\frac{2\alpha_1}{\alpha_1-2},2}}\|\Delta u\|_{L^2} \\ &\quad + C\|b_h\|_{L^{\alpha_2,\infty}}\|\nabla b\|_{L^{\frac{2\alpha_2}{\alpha_2-2},2}}\|\Delta u\|_{L^2} \\ &\le C\|b_3\|_{L^{\alpha_1,\infty}}\|\nabla b\|_{L^2}^{1-\frac{3}{\alpha_1}}\|\Delta b\|_{L^2}^{\frac{3}{\alpha_1}}\|\Delta u\|_{L^2} \\ &\quad + C\|b_h\|_{L^{\alpha_2,\infty}}\|\nabla b\|_{L^2}^{1-\frac{3}{\alpha_2}}\|\Delta b\|_{L^2}^{\frac{3}{\alpha_2}}\|\Delta u\|_{L^2} \\ &\le \frac{1}{8}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\right) + C(\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-3}} + \|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})\|\nabla b\|_{L^2}^2. \end{aligned} \tag{10}$$

Similarly, we see that

$$I_3 \le \frac{1}{8}\left(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\right) + C(\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-3}} + \|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})\|\nabla u\|_{L^2}^2. \tag{11}$$

Substituting (9)-(11) into (8), yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \\ &\le C(\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-3}} + \|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ &\quad + C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}. \end{aligned} \tag{12}$$

Using Gronwall inequality and Hölder inequality, we get

$$\begin{aligned} &\frac{1}{2}\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4}\int_0^T (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)\mathrm{d}t \\ &\le \|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + C\int_0^T (\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-3}} + \|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)\mathrm{d}t \\ &\quad + C\int_0^T \|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}\mathrm{d}t \\ &\le \|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + C\int_0^T (\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{8\alpha_2}{3\alpha_2-10}} + \|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)\mathrm{d}t \\ &\quad + C\sup_{t\in[0,T]}\|\nabla_h u\|_{L^2}\Big(\int_0^T \|\nabla u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac{1}{4}}\Big(\int_0^t \|\nabla_h \nabla u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^t \|\Delta u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac{1}{4}} \\ &\le C\int_0^T (\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{8\alpha_2}{3\alpha_2-10}} + \|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)\mathrm{d}t \\ &\quad + C\Big(\sup_{0\le t\le T}\|\nabla_h u\|_{L^2}^2 + \int_0^T \|\nabla\nabla_h u\|_{L^2}^2\,\mathrm{d}t\Big)\Big(\int_0^T \|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{1}{4}}. \end{aligned} \tag{13}$$

Multiplying $(1)_{1-2}$ by $-\Delta_h u, -\Delta_h b$ respectively, integrating over $\mathbb{R}^3$ and summing up, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + \|\nabla_h \nabla u\|_{L^2}^2 + \|\nabla_h \nabla b\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} (u\cdot\nabla)u \cdot \Delta_h u \,\mathrm{d}x - \int_{\mathbb{R}^3} (b\cdot\nabla)b \cdot \Delta_h u \,\mathrm{d}x + \int_{\mathbb{R}^3} (u\cdot\nabla)b \cdot \Delta_h b \,\mathrm{d}x - \int_{\mathbb{R}^3} (b\cdot\nabla)u \cdot \Delta_h b \,\mathrm{d}x \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i u_j \partial_k\partial_k u_j \mathrm{d}x + \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i u_3 \partial_k\partial_k u_3 \mathrm{d}x + \sum_{j=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_3\partial_3 u_j \partial_k\partial_k u_j \mathrm{d}x \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i b_j \partial_k\partial_k u_j \mathrm{d}x - \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i b_3 \partial_k\partial_k u_3 \mathrm{d}x + \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i b_j \partial_k\partial_k b_j \mathrm{d}x \\
&\quad + \sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} u_i\partial_i b_3 \partial_k\partial_k b_3 \mathrm{d}x - \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i u_j \partial_k\partial_k b_j \mathrm{d}x \\
&\quad - \sum_{i=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_i\partial_i u_3 \partial_k\partial_k b_3 \mathrm{d}x - \sum_{j=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3} b_3\partial_3 u_j \partial_k\partial_k b_j \mathrm{d}x \\
&\le C\int_{\mathbb{R}^3} |u_3||\nabla u||\nabla\nabla_h u| \,\mathrm{d}x + C\int_{\mathbb{R}^3} |u_3||\nabla_h b||\nabla\nabla_h b| \,\mathrm{d}x + C\int_{\mathbb{R}^3} |b_h||\nabla_h b||\nabla\nabla_h u| \,\mathrm{d}x \\
&\quad + C\int_{\mathbb{R}^3} |b_h||\nabla_h u||\nabla\nabla_h b| \,\mathrm{d}x + C\int_{\mathbb{R}^3} |b_3||\nabla_h b||\nabla\nabla_h u| \,\mathrm{d}x + C\int_{\mathbb{R}^3} |b_3||\nabla u||\nabla\nabla_h b| \,\mathrm{d}x \\
&=: \sum_{i=1}^{6} J_i.
\end{aligned}\tag{14}$$

By Lemma 2.4, Proposition 2.6 and Young inequality, we deduce that

$$\begin{aligned}
J_1 &\le C\|u_3\|_{L^{\alpha_1,\infty}}\|\nabla u\|_{L^{\frac{2\alpha_1}{\alpha_1-2},2}}\|\nabla\nabla_h u\|_{L^2} \\
&\le C\|u_3\|_{L^{\alpha_1,\infty}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha_1}}\|\Delta u\|_{L^2}^{\frac{1}{\alpha_1}}\|\nabla\nabla_h u\|_{L^2}^{\frac{2}{\alpha_1}+1} \\
&\le \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2 + C\|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha_1}{\alpha_1-2}}\|\nabla u\|_{L^2}^{\frac{2(\alpha_1-3)}{\alpha_1-2}}\|\Delta u\|_{L^2}^{\frac{2}{\alpha_1-2}}.
\end{aligned}\tag{15}$$

For J_2, we have

$$\begin{aligned}
J_2 &\le C\|u_3\|_{L^{\alpha_1,\infty}}\|\nabla_h b\|_{L^{\frac{2\alpha_1}{\alpha_1-2},2}}\|\nabla\nabla_h b\|_{L^2} \\
&\le \frac{1}{4}\|\nabla\nabla_h b\|_{L^2}^2 + C\|u_3\|_{L^{\alpha,\infty}}^{\frac{2\alpha_1}{\alpha_1-2}}\|\nabla b\|_{L^2}^{\frac{2(\alpha_1-3)}{\alpha_1-2}}\|\Delta b\|_{L^2}^{\frac{2}{\alpha_1-2}}.
\end{aligned}\tag{16}$$

Similarly, it can be deduced that

$$\begin{aligned}
&J_3 + J_4 \\
&\le \frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h b\|_{L^2}^2) + C\|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-2}}(\|\nabla u\|_{L^2}^{\frac{2(\alpha_2-3)}{\alpha_2-2}} + \|\nabla b\|_{L^2}^{\frac{2(\alpha_2-3)}{\alpha_2-2}}) \\
&\quad \times(\|\Delta u\|_{L^2}^{\frac{2}{\alpha_2-2}} + \|\Delta b\|_{L^2}^{\frac{2}{\alpha_2-2}}),
\end{aligned}\tag{17}$$

and

$$
\begin{aligned}
&J_5+J_6\\
&\le C\|b_3\|_{L^{\alpha_1,\infty}}\|\nabla_h b\|_{L^{\frac{2\alpha_1}{\alpha_1-2},2}}\|\nabla\nabla_h u\|_{L^2}+C\|b_3\|_{L^{\alpha_1,\infty}}\|\nabla u\|_{L^{\frac{2\alpha_1}{\alpha_1-2},2}}\|\nabla\nabla_h b\|_{L^2}\\
&\le C\|b_3\|_{L^{\alpha_1,\infty}}\|\nabla b\|_{L^2}^{1-\frac{3}{\alpha_1}}\|\Delta b\|_{L^2}^{\frac{1}{\alpha_1}}\|\nabla\nabla_h b\|_{L^2}^{\frac{2}{\alpha_1}}\|\nabla\nabla_h u\|_{L^2}\\
&\quad+C\|b_3\|_{L^{\alpha_1,\infty}}\|\nabla u\|_{L^2}^{1-\frac{3}{\alpha_1}}\|\Delta u\|_{L^2}^{\frac{1}{\alpha_1}}\|\nabla\nabla_h u\|_{L^2}^{\frac{2}{\alpha_1}}\|\nabla\nabla_h b\|_{L^2}\\
&\le\frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+C\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-2}}(\|\nabla u\|_{L^2}^{\frac{2(\alpha_1-3)}{\alpha_1-2}}+\|\nabla b\|_{L^2}^{\frac{2(\alpha_1-3)}{\alpha_1-2}})\\
&\quad\times(\|\Delta u\|_{L^2}^{\frac{2}{\alpha_1-2}}+\|\Delta b\|_{L^2}^{\frac{2}{\alpha_1-2}}). \qquad (18)
\end{aligned}
$$

Combining (14)-(18) and using Gronwall inequality, it yields

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2\right)+\int_0^T(\|\nabla_h\nabla u\|_{L^2}^2+\|\nabla_h\nabla b\|_{L^2}^2)\mathrm{d}t\\
&\le C\int_0^T(\|u_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-2}}+\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-2}})(\|\nabla u\|_{L^2}^{\frac{2(\alpha_1-3)}{\alpha_1-2}}+\|\nabla b\|_{L^2}^{\frac{2(\alpha_1-3)}{\alpha_1-2}})\\
&\quad\times(\|\Delta u\|_{L^2}^{\frac{2}{\alpha_1-2}}+\|\Delta b\|_{L^2}^{\frac{2}{\alpha_1-2}})\,\mathrm{d}t+C\int_0^T\|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-2}}(\|\nabla u\|_{L^2}^{\frac{2(\alpha_2-3)}{\alpha_2-2}}+\|\nabla b\|_{L^2}^{\frac{2(\alpha_2-3)}{\alpha_2-2}})(\|\Delta u\|_{L^2}^{\frac{2}{\alpha_2-2}}+\|\Delta b\|_{L^2}^{\frac{2}{\alpha_2-2}})\,\mathrm{d}t. \qquad (19)
\end{aligned}
$$

Substituting (19) into (13), we obtain

$$
\begin{aligned}
&\frac{1}{2}\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{3}{4}\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\mathrm{d}t\\
&\le C\int_0^T(\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{8\alpha_1}{3\alpha_1-10}}+\|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\mathrm{d}t\\
&\quad+C\Big[\int_0^T(\|u_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-3}}+\|b_3\|_{L^{\alpha_1,\infty}}^{\frac{2\alpha_1}{\alpha_1-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\mathrm{d}t\Big]^{\frac{\alpha_1-3}{\alpha_1-2}}\\
&\quad\times\Big[\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\mathrm{d}t\Big]^{\frac{1}{4}+\frac{1}{\alpha_1-2}}\\
&\quad+C\Big[\int_0^T\|b_h\|_{L^{\alpha_2,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\mathrm{d}t\Big]^{\frac{\alpha_2-3}{\alpha_2-2}}\\
&\quad\times\Big[\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\mathrm{d}t\Big]^{\frac{1}{4}+\frac{1}{\alpha_2-2}}\\
&\le\frac{1}{4}\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\mathrm{d}t\\
&\quad+C\int_0^T(\|u_3\|_{L^{\alpha,\infty}}^{\frac{8\alpha_1}{3\alpha_1-10}}+\|b_3\|_{L^{\alpha,\infty}}^{\frac{8\alpha_1}{3\alpha_1-10}}+\|b_h\|_{L^{\alpha,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t. \qquad (20)
\end{aligned}
$$

Applying Gronwall inequality and condition (5), it follows that

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\exp\Big[C\int_0^T(\|u_3\|_{L^{\alpha,\infty}}^{\frac{8\alpha_1}{3\alpha_1-10}}+\|b_3\|_{L^{\alpha,\infty}}^{\frac{8\alpha_1}{3\alpha_1-10}}+\|b_h\|_{L^{\alpha,\infty}}^{\frac{2\alpha_2}{\alpha_2-3}})\,\mathrm{d}t\Big]<\infty.
\end{aligned}
$$

This implies that

$$u, b \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)),$$

i.e., the solution (u,b) is smooth on $[0,T]$.

4. The proof of Theorem 1.2

The proof of Theorem 1.2: Multiplying $(1)_{1-2}$ by $-\Delta u, -\Delta b$ respectively, integrating over $\mathbb{R}^3$, and summing up, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta b\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\,\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\,\mathrm{d}x \\
&\le C\int_{\mathbb{R}^3}|\nabla_h u||\nabla u|^2\,\mathrm{d}x + C\int_{\mathbb{R}^3}|b||\nabla b||\Delta u|\,\mathrm{d}x + C\int_{\mathbb{R}^3}|b||\nabla u||\Delta b|\,\mathrm{d}x \\
&=: \sum_{i=1}^3 K_i.
\end{aligned} \tag{21}$$

By using Hölder inequality, Gagliardo-Nirenberg inequality and multiplicative Sobolev inequality, we have

$$\begin{aligned}
K_1 &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^3}\|\nabla u\|_{L^6} \\
&\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^6}^{\frac{3}{2}} \\
&\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}.
\end{aligned} \tag{22}$$

From Hölder inequality and Young inequality, it follows that

$$\begin{aligned}
K_2 &\le \int_{\mathbb{R}^3}|\phi_2||\nabla b||\Delta u|\,\mathrm{d}x + \int_{\mathbb{R}^3}|\varphi_2||\nabla b||\Delta u|\,\mathrm{d}x \\
&\le C\|\phi_2\|_{L^\alpha}\|\nabla b\|_{L^{\frac{2\alpha}{\alpha-2}}}\|\Delta u\|_{L^2} + C\|\varphi_2\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta u\|_{L^2} \\
&\le C\|\phi_2\|_{L^\alpha}\|\nabla b\|_{L^2}^{\frac{\alpha-3}{\alpha}}\|\Delta b\|_{L^2}^{\frac{3}{\alpha}}\|\Delta u\|_{L^2} + C\|\varphi_2\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta u\|_{L^2} \\
&\le \frac{1}{16}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + C\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla b\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} + C\|\varphi_2\|_{L^\infty}^2\|\nabla b\|_{L^2}^2.
\end{aligned} \tag{23}$$

Similar to the analysis of K_2, we obtain

$$K_3 \le \frac{1}{16}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + C\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}\|\nabla u\|_{L^2}^2 + C\|\varphi_2\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \tag{24}$$

Substituting (22)-(24) into (21), gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \frac{7}{8}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \\
&\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}} + C\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\
&\quad + C\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).
\end{aligned} \tag{25}$$

By Gronwall inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\sup_{0\le t\le T}\Big(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\Big)+\frac{7}{8}\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\sup_{0\le t\le T}\|\nabla_h u\|_{L^2}\Big(\int_0^T\|\nabla\nabla_h u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^T\|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{1}{4}}\\
&\quad+C\int_0^T\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t+C\int_0^T\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\Big(\sup_{0\le t\le T}\|\nabla_h u\|_{L^2}^2+\int_0^T\|\nabla\nabla_h u\|_{L^2}^2\,\mathrm{d}t\Big)\Big(\int_0^T\|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{1}{4}}\\
&\quad+C\int_0^T\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t+C\int_0^T\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t. \qquad (26)
\end{aligned}$$

Multiplying $(1)_{1-2}$ by $-\Delta_h u, -\Delta_h b$ respectively, integrating over $\mathbb{R}^3$ and summing up, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+\|\nabla_h\nabla u\|_{L^2}^2+\|\nabla_h\nabla b\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta_h u\,\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta_h u\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta_h b\,\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta_h b]\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|u_3||\nabla u||\nabla\nabla_h u|\,\mathrm{d}x+C\int_{\mathbb{R}^3}|b||\nabla_h b||\nabla\nabla_h u|\,\mathrm{d}x\\
&\quad+C\int_{\mathbb{R}^3}|b||\nabla_h u||\nabla\nabla_h b|\,\mathrm{d}x+C\int_{\mathbb{R}^3}|b||\nabla u||\nabla\nabla_h b|\,\mathrm{d}x\\
&=:\sum_{i=1}^4 L_i. \qquad (27)
\end{aligned}$$

By the Hölder inequality, Gagliardo-Nirenberg inequality and Young inequality, we can deduce that

$$\begin{aligned}
L_1&\le C\int_{\mathbb{R}^3}|\phi_1||\nabla u||\nabla\nabla_h u|\,\mathrm{d}x+C\int_{\mathbb{R}^3}|\varphi_1||\nabla u||\nabla\nabla_h u|\,\mathrm{d}x\\
&\le C(\|\phi_1\|_{L^\alpha}\|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2}}}\|\nabla\nabla_h u\|_{L^2}+\|\varphi_1\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\nabla_h u\|_{L^2})\\
&\le C(\|\phi_1\|_{L^\alpha}\|\nabla u\|_{L^2}^{\frac{\alpha-3}{\alpha}}\|\nabla\nabla_h u\|_{L^2}^{\frac{\alpha+2}{\alpha}}\|\Delta u\|_{L^2}^{\frac{1}{\alpha}}+\|\varphi_1\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\nabla_h u\|_{L^2})\\
&\le\frac{1}{8}\|\nabla\nabla_h u\|_{L^2}^2+C\|\phi_1\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}}\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}+C\|\varphi_1\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \qquad (28)
\end{aligned}$$

Similar to the estimate of L_1, we obtain

$$\begin{aligned}
&L_2+L_3+L_4\\
&\le\frac{1}{4}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h b\|_{L^2}^2)+C\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}(\|\nabla_h u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}}+\|\nabla_h b\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}})(\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}+\|\Delta b\|_{L^2}^{\frac{2}{\alpha-2}})\\
&\quad+C\|\varphi_2\|_{L^\infty}^2(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2+\|\nabla u\|_{L^2}^2). \qquad (29)
\end{aligned}$$

Combining (27)-(29), gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+\|\nabla_h\nabla u\|_{L^2}^2+\|\nabla_h\nabla b\|_{L^2}^2\\
&\le C(\|\phi_1\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}+\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}})(\|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}}+\|\nabla b\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}})(\|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}+\|\Delta b\|_{L^2}^{\frac{2}{\alpha-2}})\\
&\quad+C(\|\varphi_1\|_{L^\infty}^2+\|\varphi_2\|_{L^\infty}^2)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\\
&\le C(\|\phi_1\|_{L^\alpha}+\|\phi_2\|_{L^\alpha})^{\frac{2\alpha}{\alpha-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{2(\alpha-3)}{\alpha-2}}(\|\Delta u\|_{L^2}+\|\Delta b\|_{L^2})^{\frac{2}{\alpha-2}}\\
&\quad+C(\|\varphi_1\|_{L^\infty}^2+\|\varphi_2\|_{L^\infty}^2)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \qquad (30)
\end{aligned}
$$

Substituting (30) into (26), it yields that

$$
\begin{aligned}
&\frac{1}{2}\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\frac{7}{8}\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\int_0^T(\|\phi_1\|_{L^\alpha}+\|\phi_2\|_{L^\alpha})^{\frac{2\alpha}{\alpha-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{2(\alpha-3)}{\alpha-2}}(\|\Delta u\|_{L^2}+\|\Delta b\|_{L^2})^{\frac{2}{\alpha-2}}\,\mathrm{d}t\Big(\int_0^T\|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac14}\\
&\quad+C\int_0^T(\|\varphi_1\|_{L^\infty}^2+\|\varphi_2\|_{L^\infty}^2)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big(\int_0^T\|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac14}\\
&\quad+C\int_0^T\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t+C\int_0^T\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\Big[\int_0^T(\|\phi_1\|_{L^\alpha}+\|\phi_2\|_{L^\alpha})^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\,\mathrm{d}t\Big]^{\frac{\alpha-3}{\alpha-2}}\Big(\int_0^T\|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{\alpha+2}{4(\alpha-2)}}\\
&\quad+C\int_0^T(\|\varphi_1\|_{L^\infty}^2+\|\varphi_2\|_{L^\infty}^2)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big(\int_0^T\|\Delta u\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac14}\\
&\quad+\frac18\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t+C\int_0^T\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t+C\int_0^T\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\Big[\int_0^T(\|\phi_1\|_{L^\alpha}+\|\phi_2\|_{L^\alpha})^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\,\mathrm{d}t\Big]^{\frac{3\alpha-10}{4(\alpha-3)}}+\frac38\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
&\quad+C\Big[\int_0^T(\|\varphi_1\|_{L^\infty}^2+\|\varphi_2\|_{L^\infty}^2)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big]^{\frac43}+C\int_0^T\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\\
&\quad+C\int_0^T\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\int_0^T(\|\phi_1\|_{L^\alpha}+\|\phi_2\|_{L^\alpha})^{\frac{8\alpha}{3\alpha-10}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big[\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big]^{\frac{\alpha-2}{3\alpha-10}}\\
&\quad+\frac38\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
&\quad+C\int_0^T(\|\varphi_1\|_{L^\infty}^{\frac83}+\|\varphi_2\|_{L^\infty}^{\frac83})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big[\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\Big]^{\frac14}\\
&\quad+C\int_0^T\|\phi_2\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t+C\int_0^T\|\varphi_2\|_{L^\infty}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t\\
&\le\frac38\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
&\quad+C\int_0^T(\|\phi_1\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\phi_2\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\varphi_1\|_{L^\infty}^{\frac83}+\|\varphi_2\|_{L^\infty}^{\frac83})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t.
\end{aligned}
$$

Furthermore, we can obtain

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t$$
$$\le C\int_0^T(\|\phi_1\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\phi_2\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\varphi_1\|_{L^\infty}^{\frac{8}{3}}+\|\varphi_2\|_{L^\infty}^{\frac{8}{3}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}t.$$

Utilizing Gronwall inequality and condition (6), it follows that

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t$$
$$\le C\exp\left[C\int_0^T(\|\phi_1\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\phi_2\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}}+\|\varphi_1\|_{L^\infty}^{\frac{8}{3}}+\|\varphi_2\|_{L^\infty}^{\frac{8}{3}})\,\mathrm{d}t\right]<\infty.$$

This implies that

$$u,b\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)),$$

i.e., the solution (u,b) is smooth on $[0,T]$.

5. The proof of Theorem 1.3

The proof of Theorem 1.3: Take the L^2 inner product of $(1)_{1-2}$ with u,b respectively, integrating over $\mathbb{R}^3$, and summing them up, one obtains

$$\|u\|_{L^2}^2+\|b\|_{L^2}^2+2\int_0^T(\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\mathrm{d}t\le\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2. \tag{31}$$

Multiplying $(1)_{1-2}$ by $-\Delta u,-\Delta b$ respectively, integrating over $\mathbb{R}^3$ and summing up, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\mathrm{d}x\\
&=-\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\partial_k u_i\partial_i u_j\partial_k u_j\mathrm{d}x+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\partial_k b_i\partial_i b_j\partial_k u_j\mathrm{d}x\\
&\quad-\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\partial_k u_i\partial_i b_j\partial_k b_j\mathrm{d}x+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\nabla u|^3\mathrm{d}x+C\int_{\mathbb{R}^3}|\nabla u||\nabla b|^2\mathrm{d}x\\
&=:M_1+M_2.
\end{aligned}\tag{32}$$

By Hölder inequality, multiplicative Sobolev inequality and Young inequality, it can be deduced that

$$\begin{aligned}
M_1&\le C\|\nabla u\|_{L^3}^3\\
&\le C(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}^{\frac{1}{3}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{6}})^3\\
&=C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla_h\nabla u\|_{L^2}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
&\le\frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^3\|\nabla\partial_3 u\|_{L^2}\\
&\le\frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2(\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla u\|_{L^2}^2).
\end{aligned}\tag{33}$$

For M_2. Similar to M_1, we have

$$\begin{aligned}
M_2 \leq & C\|\nabla u\|_{L^3}\|\nabla b\|_{L^3}^2 \\
\leq & C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}^{\frac{1}{3}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{6}}\|\nabla b\|_{L^2}\|\nabla_h\nabla b\|_{L^2}^{\frac{2}{3}}\|\nabla\partial_3 b\|_{L^2}^{\frac{1}{3}} \\
\leq & \frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{2}\|\nabla_h\nabla b\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla b\|_{L^2}^2\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{3}}\|\nabla\partial_3 b\|_{L^2}^{\frac{2}{3}} \\
\leq & \frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{2}\|\nabla_h\nabla b\|_{L^2}^2+C\|\nabla b\|_{L^2}^2(\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2+\|\nabla u\|_{L^2}^2). \qquad (34)
\end{aligned}$$

Combining (33) with (34), gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2 \\
\leq & C(\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2+\|\nabla u\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \qquad (35)
\end{aligned}$$

Multiplying $(1)_{1-2}$ by $-\partial_3\partial_3 u, -\partial_3\partial_3 b$ respectively, integrating over $\mathbb{R}^3$ and summing up, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)+\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2 \\
=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\partial_3\partial_3 u\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\partial_3\partial_3 u\mathrm{d}x+\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\partial_3\partial_3 b\mathrm{d}x-\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\partial_3\partial_3 b\mathrm{d}x \\
=&-\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_3 u_i\partial_i u_j\partial_3 u_j\mathrm{d}x+\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_i u_i\partial_3 u_j\partial_3 u_j\mathrm{d}x+\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_3 b_i\partial_i b_j\partial_3 u_j\mathrm{d}x \\
&-\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_i b_i\partial_3 b_j\partial_3 u_j\mathrm{d}x-\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_3 u_i\partial_i b_j\partial_3 b_j\mathrm{d}x+\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_i u_i\partial_3 b_j\partial_3 b_j\mathrm{d}x \\
&+\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_3 b_i\partial_i u_j\partial_3 b_j\mathrm{d}x-\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_{\mathbb{R}^3}\partial_i b_i\partial_3 u_j\partial_3 b_j\mathrm{d}x \\
\leq & C\int_{\mathbb{R}^3}|\partial_3 u||\nabla u||\partial_3 u|\mathrm{d}x+C\int_{\mathbb{R}^3}|\partial_3 b||\nabla b||\partial_3 u|\mathrm{d}x+\int_{\mathbb{R}^3}|\partial_3 b||\nabla u||\partial_3 b|\mathrm{d}x \\
=:&\sum_{i=1}^3 N_i(t). \qquad (36)
\end{aligned}$$

By Hölder inequality, Lemma 2.7 and Young inequality, we deduce that

$$\begin{aligned}
N_1(t)\leq & C\|\nabla u\|_{L^2}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}\Big\|\Big\|\|\partial_3 u\|_{L^{\frac{2p_1}{p_1-2}}_{x_1}}\Big\|_{L^{\frac{2q_1}{q_1-2}}_{x_2}}\Big\|_{L^{\frac{2r_1}{r_1-2}}_{x_3}} \\
\leq & C\|\nabla u\|_{L^2}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{p_1}}\|\partial_2\partial_3 u\|_{L^2}^{\frac{1}{q_1}}\|\partial_3\partial_3 u\|_{L^2}^{\frac{1}{r_1}}\|\partial_3 u\|_{L^2}^{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})} \\
\leq & C\|\nabla u\|_{L^2}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}}\|\partial_3 u\|_{L^2}^{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})} \\
\leq & \frac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2+C\|\nabla u\|_{L^2}^{\frac{2}{2-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{2-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}\|\partial_3 u\|_{L^2}^{\frac{2-2(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}{2-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}} \\
\leq & \frac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2+C(e+\|\partial_3 u\|_{L^2}^2)\left(\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}+\|\nabla u\|_{L^2}^2\right), \qquad (37)
\end{aligned}$$

Applying Hölder inequality, Lemma 2.7 and Young inequality, N_2 can be estimated as

$$
\begin{aligned}
N_2(t) &\le C\|\nabla b\|_{L^2}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}\Big\|\Big\|\|\partial_3 b\|_{L^{\frac{2p_1}{p_1-2}}_{x_1}}\Big\|_{L^{\frac{2q_1}{q_1-2}}_{x_2}}\Big\|_{L^{\frac{2r_1}{r_1-2}}_{x_3}}\\
&\le C\|\nabla b\|_{L^2}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}\|\partial_1\partial_3 b\|_{L^2}^{\frac{1}{p_1}}\|\partial_2\partial_3 b\|_{L^2}^{\frac{1}{q_1}}\|\partial_3\partial_3 b\|_{L^2}^{\frac{1}{r_1}}\|\partial_3 b\|_{L^2}^{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}\\
&\le C\|\nabla b\|_{L^2}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}\|\nabla\partial_3 b\|_{L^2}^{\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}}\|\partial_3 b\|_{L^2}^{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}\\
&\le \frac14\|\nabla\partial_3 b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^{\frac{2}{2-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{2-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}\|\partial_3 b\|_{L^2}^{\frac{2-2(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}{2-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}\\
&\le \frac14\|\nabla\partial_3 b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2 + C\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}\|\partial_3 b\|_{L^2}^2\\
&\le \frac14\|\nabla\partial_3 b\|_{L^2}^2 + C(e+\|\partial_3 b\|_{L^2}^2)\bigg(\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}} + \|\nabla b\|_{L^2}^2\bigg).
\end{aligned}
\tag{38}
$$

For N_3. Similar to N_2, it is easy to get

$$
N_3(t) \le \frac14\|\nabla\partial_3 b\|_{L^2}^2 + C(e+\|\partial_3 b\|_{L^2}^2)\bigg(\Big\|\Big\|\|\partial_3 b\|_{L^{p_2}_{x_1}}\Big\|_{L^{q_2}_{x_2}}\Big\|_{L^{r_2}_{x_3}}^{\frac{2}{1-(\frac{1}{p_2}+\frac{1}{q_2}+\frac{1}{r_2})}} + \|\nabla u\|_{L^2}^2\bigg),
\tag{39}
$$

Combining (36)-(39), it yields that

$$
\begin{aligned}
&\frac{d}{dt}(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)+\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2\\
&\le C\bigg(\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}} + \Big\|\Big\|\|\partial_3 b\|_{L^{p_2}_{x_1}}\Big\|_{L^{q_2}_{x_2}}\Big\|_{L^{r_2}_{x_3}}^{\frac{2}{1-(\frac{1}{p_2}+\frac{1}{q_2}+\frac{1}{r_2})}} + \|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\bigg)\\
&\quad\times(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)\\
&\le C\left(\frac{\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}} + \Big\|\Big\|\|\partial_3 b\|_{L^{p_2}_{x_1}}\Big\|_{L^{q_2}_{x_2}}\Big\|_{L^{r_2}_{x_3}}^{\frac{2}{1-(\frac{1}{p_2}+\frac{1}{q_2}+\frac{1}{r_2})}}}{1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)} + \|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)\\
&\quad\times(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)(1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\frac{d}{dt}[1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)]+\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2\\
&\le C\left(\frac{\Big\|\Big\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\Big\|_{L^{q_1}_{x_2}}\Big\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}} + \Big\|\Big\|\|\partial_3 b\|_{L^{p_2}_{x_1}}\Big\|_{L^{q_2}_{x_2}}\Big\|_{L^{r_2}_{x_3}}^{\frac{2}{1-(\frac{1}{p_2}+\frac{1}{q_2}+\frac{1}{r_2})}}}{1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)} + \|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right)\\
&\quad\times[1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)].
\end{aligned}
\tag{40}
$$

From Gronwall inequality, it can be deduced that

$$\begin{aligned}
&\sup_{0\le t\le T}[1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)]+\int_0^T(\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2)\,\mathrm{d}t\\
\le& C\exp\left(\frac{\left\|\left\|\|\partial_3 u\|_{L^{p_1}_{x_1}}\right\|_{L^{q_1}_{x_2}}\right\|_{L^{r_1}_{x_3}}^{\frac{2}{1-(\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1})}}+\left\|\left\|\|\partial_3 b\|_{L^{p_2}_{x_1}}\right\|_{L^{q_2}_{x_2}}\right\|_{L^{r_2}_{x_3}}^{\frac{2}{1-(\frac{1}{p_2}+\frac{1}{q_2}+\frac{1}{r_2})}}}{1+\ln(e+\|\partial_3 u\|_{L^2}^2+\|\partial_3 b\|_{L^2}^2)}+\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\right).
\end{aligned}\tag{41}$$

Substituting (41) into (35), and combining with condition (7), we conclude that

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\,\mathrm{d}t\\
\le& C\exp\left[C\int_0^T(\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3 b\|_{L^2}^2+\|\nabla u\|_{L^2}^2)\mathrm{d}t\right]<\infty.
\end{aligned}$$

This indicates that

$$u,b\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)),$$

which means the solution (u,b) is regular on $[0,T]$.

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