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Existence result for a new class of Ψ-Caputo fractional differential equation involving the p-Laplacian operator

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Abstract. This paper investigates the existence result for a new class of Ψ-Caputo-type fractional differential equation involving the p-Laplacian operator. By making use of some basic proprieties of fractional calculus and the p-Laplacian operator and by applying Schaefer's fixed point theorem we established the existence result. As application, we give an example to demonstrate our theoretical result.

1. Introduction

Newly, fractional differential equations have attracted the curiosity of numerous mathematicians, due to the fact that it can accurately model a wide range of scientific phenomena, and has been proven to be effective in physics, mechanics, biology, chemistry, and control theory, and other domains for exemple, see [1, 4, 5, 8, 12, 14, 15, 17–19, 27–31].

There are several ways to define fractional integrals and derivatives, however the most well-known ones are the Riemann-Liouville and the Caputo fractional integrals and derivatives, in [14], Almeida introduce the generalization of these derivatives under the name of Ψ-Caputo fractional derivative, for more details for Ψ-Caputo and Caputo fractional derivative, we direct readers to the papers [13, 21–24]. In the previous few decades, differential equations with p-Laplacian operator commonly used in a wide range of scientific domains, such as dynamical systems and mathematical models of mechanics. To investigate these kinds of situations, in [11] Leibenson, introduced the p-Laplacian equation as follows

$$\Big(\phi_p\big(u'(\tau)\big)\Big)' = f(\tau, u(\tau), u'(\tau)).$$

Where $\phi_p(s) = |s|^{s-2}s$, $p > 1$, ϕ_p is invertible and its inverse operator is ϕ_q, with $q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For the fractional differential equations with the p-Laplacian operator, we refers to [2, 3, 7, 10, 16]

Liu and Jia [6], discussed the existence and uniqueness of solutions to some fractional differential equation involving the p-Laplacian operator, by means of the Banach contraction mapping principle.

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Su et al. [9], studied the existence criteria of non-negative solutions of nonlinear p-Laplacian fractional differential equations with first order derivative, the existence result are obtained by making use of the nonlinear alternative of Leray-Schauder type and Banach fixed point theorems.

Motivated by the mentioned works, in this paper, we combine their ideas to investigate the existence result for the problem of the form

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi}\Big(\phi_p\Big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau)\Big]\Big) = h(\tau, u(\tau)),\ \tau \in \Lambda := [a, b], \\ u(a) = \mu u(\xi)\ ,\ {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(a) = 0\ ,\ {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(b) = \kappa\,{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\eta). \end{cases} \tag{1}$$

Where ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi}$ and ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}$ are the Ψ-Caputo fractional derivative of order α, $0 < \alpha \leq 2$, and β, $0 < \beta \leq 1$, respectively, $a \geq 0$, $\mu, \kappa \in \mathbb{R}$, $\eta, \xi \in \Lambda$, $h \in C\Big(\Lambda \times \mathbb{R}, \mathbb{R}\Big)$ and $\phi_p(\tau)$ is the p-Laplacian operator (i.e $\phi_p(\tau) = |\tau|^{p-2}\tau$, $p > 1$).

The originality of this work is studing a new and a challenging case of fractional derivative named the Ψ-Caputo fractional derivative [14], this kind of fractional derivative generalize the well-known fractional derivatives, for different values of function Ψ such as

- ⋆ If $\Psi(\tau) = \tau$, then the problem (1) reduces to Caputo-type fractional derivative.
- ⋆ If $\Psi(\tau) = log(\tau)$, then the problem (1) reduces to Caputo-Hadamard-type fractional derivative.
- ⋆ If $\Psi(\tau) = \tau^\rho$, then the problem (1) reduces to Caputo-Katugampola-type fractional derivative.

The rest of this paper is organized as follows : In section 2, we recall some notations, definitions, and lemmas from fractional calculus and important results of p-Laplacian operator that will be used in our study. In section 3, we discuss the existence result for the problem (1), by making use of Schaefer's fixed point theorem. In section 4, an example is provided to illustrate the main result.

2. Preliminaries

In this section, we introduce some definitions, useful notations of fractional calculus and some basic properties of the p-Laplacian operator which will be used throughout this paper.

$C(\Lambda, \mathbb{R})$ denote the Banach space of all continuous functions from Λ into $\mathbb{R}$ with the norm defined by $\|h\| = sup_{\tau\in\Lambda}\{|h(\tau)|\}$. We denote by $C^n(\Lambda, \mathbb{R})$ the n-times absolutely continuous functions given by $C^n(\Lambda, \mathbb{R}) = \Big\{h : \Lambda \longrightarrow \mathbb{R}\ ;\ h^{(n-1)} \in C(\Lambda, \mathbb{R})\Big\}$. $\mathcal{B}_\rho$ denote the closed ball centered at 0 with radius ρ. We denote by $\mathbb{L}^1(\Lambda, \mathbb{R})$ the space of Lebesgue integrable real-valued functions on Λ equipped with the norm $\|h\|_{\mathbb{L}^1} = \int_\Lambda |h(\tau)|d\tau$.

Definition 2.1. *[14] For $\alpha > 0$, $h \in \mathbb{L}^1(\Lambda, \mathbb{R})$ and $\Psi \in C^n(\Lambda, \mathbb{R})$, with $\Psi'(\tau) > 0$, for all $\tau \in \Lambda$, the Ψ-Riemann-Liouville fractional integral of order α of a function h is defined by*

$$\mathfrak{I}_{a^+}^{\alpha;\Psi}h(\tau) = \frac{1}{\Gamma(p)}\int_a^t \Psi'(\tau)(\Psi(\tau) - \Psi(s))^{\alpha-1}h(s)ds, \tag{2}$$

where $\Gamma(.)$ represents the gamma function, $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, $\Re e(z) > 0$.

Definition 2.2. *[14] For $\alpha > 0$, $h \in C^{n-1}(\Lambda, \mathbb{R})$ and $\Psi \in C^n(\Lambda, \mathbb{R})$, with $\Psi'(\tau) > 0$, for all $\tau \in \Lambda$, the Ψ-Caputo fractional derivative of order α of a function h is defined by*

$$\begin{aligned} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi} h(\tau) &= \mathfrak{I}_{a^+}^{n-\alpha;\Psi} h_{\Psi}^{[n]}(\tau) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^\tau \Psi'(\tau)(\Psi(\tau)-\Psi(s))^{n-\alpha-1} h_{\Psi}^{[n]}(s)ds, \end{aligned} \tag{3}$$

where $h_{\Psi}^{[n]}(\tau) = \left(\frac{1}{\Psi'(\tau)}\frac{d}{d\tau}\right)^n$, $n-1 < \alpha < n$, $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *[14] Let $\alpha, \beta > 0$. Then we have the following semigroup property given by*

$$\mathfrak{I}_{a^+}^{\alpha;\Psi}\mathfrak{I}_{a^+}^{\beta;\Psi} h(\tau) = \mathfrak{I}_{a^+}^{\alpha+\beta;\Psi} h(\tau), \tau > a. \tag{4}$$

Proposition 2.4. *[14] Let $\alpha > 0$, $\upsilon > 0$ and $\tau \in \Lambda$. Then*

(i) $\mathfrak{I}_{a^+}^{\alpha;\Psi}(\Psi\tau) - \Psi(a))^{\upsilon-1} = \dfrac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)}(\Psi(\tau)-\Psi(a))^{\upsilon+\alpha-1}$.

(ii) ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi}(\Psi(\tau) - \Psi(a))^{\upsilon-1} = \dfrac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)}(\Psi(\tau)-\Psi(a))^{\upsilon-\alpha-1}$.

(iii) ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi}(\Psi(\tau) - \Psi(a))^{k} = 0,\ \forall k < n \in \mathbb{N}$.

Lemma 2.5. *[14] If $h \in C^n(\Lambda, \mathbb{R})$, $n-1 < \alpha < n$, then*

$$\mathfrak{I}_{a^+}^{\alpha;\Psi}({}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi} h)(\tau) = h(\tau) - \sum_{k=0}^{n-1} \frac{h_{\Psi}^{[k]}(a)}{k!}(\Psi(\tau)-\Psi(a))^k, \tag{5}$$

for all $\tau \in \Lambda$, where $h_{\Psi}^{[k]}(\tau) := \left(\frac{1}{\Psi'(\tau)}\frac{d}{d\tau}\right)^k h(\tau)$.

Lemma 2.6. *[20]*
Let $\phi_p : \mathbb{R} \longrightarrow \mathbb{R}$ be a p-Laplacian operator defined by $\phi_p(u) = |u|^{p-2}u$, then we have

- ⋆ *If $1 < p < 2$ and $u \neq 0$ then $(\phi_p(u))' = (p-1)|u|^{p-2}$.*
- ⋆⋆ *If $1 < p < 2$, $uv > 0$ and $|u|, |v| \geq l > 0$, then*

$$|\phi_p(u) - \phi_p(v)| \leq (p-1)l^{p-2}|u-v|.$$

- ⋆ ⋆ ⋆ *If $p > 2$, and $|u|, |v| \leq L$, then*

$$|\phi_p(u) - \phi_p(v)| \leq (p-1)L^{p-2}|u-v|.$$

- ⋆ ⋆ ⋆⋆ *ϕ_p is revertible such that $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 2.7. *[25, 26]*
Let $\mathbb{X}$ be a Banach space and $\mathcal{W} : \mathbb{X} \longrightarrow \mathbb{X}$, be a completely continuous operator. If the set $\Upsilon_\varepsilon = \{u \in \mathbb{X} \mid u = \varepsilon\mathcal{W}u\ ;\ 0 \leq \varepsilon \leq 1\}$ is bounded, then $\mathcal{W}$ has at least a fixed point in $\mathbb{X}$.

3. Main results

Definition 3.1. *A function $u \in C(\Lambda, \mathbb{R})$ is said to be a solution of problem (1), if u satisfies the equation ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi}\Big(\phi_p\big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau)\big]\Big) = h(\tau, u(\tau))$, a.e on Λ and the conditions $u(a) = \mu u(\xi)$, ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(a) = 0$, ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(b) = \kappa{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\eta)$.*

Lemma 3.2. *Let $a \geq 0$, $0 < \beta \leq 1$, $y \in C(\Lambda, \mathbb{R})$ and $\mu \in \mathbb{R}$, such that $\mu \neq 1$. Then the function u is a solution of the following boundary value problem:*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau) = y(\tau), \quad \tau \in \Lambda := [a,b] \\ u(a) = \mu u(\xi),\ a < \xi < b, \end{cases} \tag{6}$$

if and only if

$$u(\tau) = \frac{1}{\Gamma(\beta)}\int_a^\tau \Psi'(s)(\Psi(\tau)-\Psi(s))^{\beta-1}y(s)ds + \frac{\mu}{(1-\mu)\Gamma(\beta)}\int_a^\xi \Psi'(s)(\Psi(\xi)-\Psi(s))^{\beta-1}y(s)ds. \tag{7}$$

Proof. Applying the Ψ-Riemann-Liouville fractional integral of order β to both sides of (6) we obtain by using Lemma 2.5

$$u(\tau) = \mathfrak{I}_{a^+}^{\beta;\psi}y(\tau) + d_0, \tag{8}$$

where d_0 is a constant. Next, by using the boundary condition $u(a) = \mu u(\xi)$ in (8) we obtain

$$u(a) = d_0 = \mu\mathfrak{I}_{a^+}^{\beta;\Psi}h(\xi) + \mu d_0, \tag{9}$$

then

$$d_0 = \frac{\mu}{(1-\mu)\Gamma(\beta)}\int_{a^+}^\xi \Psi'(s)(\Psi(\xi)-\Psi(s))^{\beta-1}y(s)ds. \tag{10}$$

Substituting the value of d_0 in (8) we obtain the integral equation in (7), defined by

$$u(\tau) = \frac{1}{\Gamma(\beta)}\int_a^\tau \Psi'(s)(\Psi(\tau)-\Psi(s))^{\beta-1}y(s)ds + \frac{\mu}{(1-\mu)\Gamma(\beta)}\int_a^\xi \Psi'(s)(\Psi(\xi)-\Psi(s))^{\beta-1}y(s)ds.$$

□

Lemma 3.3. *Let $a \geq 0$, $0 < \beta \leq 1$, $1 < \alpha \leq 2$, $w \in C(\Lambda, \mathbb{R})$ and $\mu, \kappa \in \mathbb{R}$, such that $\mu \neq 1$. Then the function u is a solution of the following boundary value problem:*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha;\Psi}\Big(\phi_p\big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau)\big]\Big) = w(\tau),\ \tau \in \Lambda := [a,b], \\ u(a) = \mu u(\xi)\ ,\ {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(a) = 0\ ,\ {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(b) = \kappa{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\eta);\ \eta, \xi \in \Lambda, \end{cases} \tag{11}$$

if and only if

$$u(\tau) = \mathfrak{I}_{a^+}^{\beta;\Psi}\phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi}w(\tau) + G_1w(\tau)\Big) + G_2w(\tau)$$
$$= \frac{1}{\Gamma(\beta)}\int_a^\tau \Psi'(s)(\Psi(\tau)-\Psi(s))^{\beta-1}\phi_q\Bigg(\frac{1}{\Gamma(\alpha)}\int_a^s \Psi'(\varsigma)(\Psi(s)-\Psi(\varsigma))^{\alpha-1}w(\varsigma)d\varsigma$$
$$+ G_1w(s)\Bigg)ds + G_2w(\tau), \tag{12}$$

where

$$G_1w(\tau) = \frac{\Psi(s)-\Psi(a)}{\Theta\Gamma(\alpha)}\Bigg[\kappa^{p-1}\int_a^\eta \Psi'(s)(\Psi(\eta)-\Psi(s))^{\alpha-1}w(s)ds$$
$$- \int_a^b \Psi'(s)(\Psi(b)-\Psi(s))^{\alpha-1}w(s)ds\Bigg], \tag{13}$$

$$G_2w(\tau) = \frac{\mu}{(1-\mu)\Gamma(\beta)}$$
$$\times \int_a^\xi \Psi'(s)(\Psi(\xi)-\Psi(s))^{\beta-1}\phi_q\Bigg(\frac{1}{\Gamma(\alpha)}\int_a^s \Psi'(\varsigma)(\Psi(s)-\Psi(\varsigma))^{\alpha-1}w(\varsigma)d\varsigma + G_1w(s)\Bigg)ds, \tag{14}$$

$$\Theta = (\Psi(b)-\Psi(a)) - \kappa^{p-1}(\Psi(\eta)-\Psi(a)) \neq 0. \tag{15}$$

Proof. Applying the Ψ-Riemann-Liouville fractional integral of order α to both sides of (11) we obtain by using Lemma 2.5

$$\phi_p\Big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau)\Big] = \mathfrak{I}_{a^+}^{\alpha;\Psi}w(\tau) + d_1 + d_2(\Psi(\tau)-\Psi(a)), \tag{16}$$

where d_1, d_2 are constants. Next, by using the boundary condition ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(a) = 0$ in (16) we obtain

$$\phi_p\Big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(a)\Big] = \phi_p(0) = 0 = d_1,$$

it follows that

$$\phi_p\Big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau)\Big] = \mathfrak{I}_{a^+}^{\alpha;\Psi}w(\tau) + d_2(\Psi(\tau)-\Psi(a)), \tag{17}$$

by using the boundary condition ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(b) = \kappa\,{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\eta)$ and the proprieties of ϕ_p in (17) we get

$$\mathfrak{I}_{a^+}^{\alpha;\Psi}w(b) + d_2(\Psi(b)-\Psi(a)) = \kappa^{p-1}\mathfrak{I}_{a^+}^{\alpha;\Psi}w(\eta) + d_2\kappa^{p-1}(\Psi(\eta)-\Psi(a)), \tag{18}$$

then

$$d_2 = \frac{\kappa^{p-1}\mathfrak{I}_{a^+}^{\alpha;\Psi}w(\eta) - \mathfrak{I}_{a^+}^{\alpha;\Psi}w(b)}{(\Psi(b)-\Psi(a)) - \kappa^{p-1}(\Psi(\eta)-\Psi(a))} = \frac{\kappa^{p-1}\mathfrak{I}_{a^+}^{\alpha;\Psi}w(\eta) - \mathfrak{I}_{a^+}^{\alpha;\Psi}w(b)}{\Theta}, \tag{19}$$

it follows that

$$\phi_p\Big[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\beta;\Psi}u(\tau)\Big] = \mathfrak{I}_{a^+}^{\alpha;\Psi}w(\tau) + (\Psi(\tau)-\Psi(a))\frac{\kappa^{p-1}\mathfrak{I}_{a^+}^{\alpha;\Psi}w(\eta) - \mathfrak{I}_{a^+}^{\alpha;\Psi}w(b)}{\Theta}, \tag{20}$$

then

$$\begin{aligned} {}^{\mathfrak{C}}\mathfrak{D}^{\beta;\Psi}_{a^+}u(\tau) &= \phi_q\left[\mathfrak{I}^{\alpha;\Psi}_{a^+}w(\tau) + (\Psi(\tau)-\Psi(a))\frac{\kappa^{p-1}\mathfrak{I}^{\alpha;\Psi}_{a^+}w(\eta) - \mathfrak{I}^{\alpha;\Psi}_{a^+}w(b)}{\Theta}\right] \\ &= \phi_q\left[\mathfrak{I}^{\alpha;\Psi}_{a^+}w(\tau) + G_1 w(\tau)\right] \end{aligned} \tag{21}$$

Using Lemma 3.2 and setting $y(\tau) = \phi_q\left[\mathfrak{I}^{\alpha;\Psi}_{a^+}w(\tau) + G_1 w(\tau)\right]$ we get

$$\begin{aligned} u(\tau) &= \mathfrak{I}^{\beta;\Psi}_{a^+}\phi_q\left(\mathfrak{I}^{\alpha;\Psi}_{a^+}w(\tau) + G_1 w(\tau)\right) + G_2 w(\tau) \\ &= \frac{1}{\Gamma(\beta)}\int_a^\tau \Psi'(s)(\Psi(\tau)-\Psi(s))^{\beta-1}\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_a^s \Psi'(\varsigma)(\Psi(s)-\Psi(\varsigma))^{\alpha-1}w(\varsigma)d\varsigma \right. \\ &\left. + G_1 w(s)\right)ds + G_2 w(\tau), \end{aligned}$$

□

Now, we deal with the existence result for the problem (1), for that to simplify the computations, we use the following notations

$$\mathfrak{A} = \frac{(\Psi(b)-\Psi(a))^\beta}{\Gamma(\beta+1)} + \frac{|\mu|(\Psi(\xi)-\Psi(a))^\beta}{|1-\mu|\Gamma(\beta+1)}. \tag{22}$$

$$\mathfrak{B} = \frac{(\Psi(b)-\Psi(a))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))}{|\Theta|\Gamma(\alpha+1)}\left[\kappa^{p-1}(\psi(\eta)-\psi(a))^\alpha + (\Psi(b)-\Psi(a))^\alpha\right]. \tag{23}$$

We assume the following hypotheses throughout the rest of our paper

(H_1): there exist nonnegative functions $\gamma, \delta \in C(\Lambda, \mathbb{R})$ such that

$$|h(t,u)| \le \gamma(\tau) + \delta(\tau)|u|^{p-1}, \textit{for each } t \in \Lambda \textit{ and } u \in C(\Lambda, \mathbb{R}). \tag{24}$$

(H_2): $\mathfrak{A}^{p-1}\mathfrak{B}\|\delta\| < 1$ where $\mathfrak{A}, \mathfrak{B}$ are given by (22), (23).

From Lemma 3.3 we define the operator $\mathcal{W} : C(\Lambda, \mathbb{R}) \longrightarrow C(\Lambda, \mathbb{R})$ by

$$\begin{aligned} \mathcal{W}u(\tau) &= \mathfrak{I}^{\beta;\Psi}_{a^+}\phi_q\left(\mathfrak{I}^{\alpha;\Psi}_{a^+}h(\tau,u(\tau)) + G_1 h(\tau,u(\tau))\right) + G_2 h(\tau,u(\tau)) \\ &= \frac{1}{\Gamma(\beta)}\int_a^\tau \Psi'(s)(\Psi(\tau)-\Psi(s))^{\beta-1}\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_a^s \Psi'(\varsigma)(\Psi(s)-\Psi(\varsigma))^{\alpha-1}h(\varsigma,u(\varsigma))d\varsigma \right. \\ &+ \frac{\Psi(s)-\Psi(a)}{\Theta\Gamma(\alpha)}\left[\kappa^{p-1}\int_a^\eta \Psi'(s)(\Psi(\eta)-\Psi(s))^{\alpha-1}h(s,u(s))ds \right. \\ &\left.\left. - \int_a^b \Psi'(s)(\Psi(b)-\Psi(s))^{\alpha-1}h(s,u(s))ds\right]\right)ds + \frac{\mu}{(1-\mu)\Gamma(\beta)} \\ &\times \int_a^\xi \Psi'(s)(\Psi(\xi)-\Psi(s))^{\beta-1}\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_a^s \Psi'(\varsigma)(\Psi(s)-\Psi(\varsigma))^{\alpha-1}h(\varsigma,u(\varsigma))d\varsigma \right. \\ &+ \frac{\Psi(s)-\Psi(a)}{\Theta\Gamma(\alpha)}\left[\kappa^{p-1}\int_a^\eta \Psi'(s)(\Psi(\eta)-\Psi(s))^{\alpha-1}h(s,u(s))ds \right. \\ &\left.\left. - \int_a^b \Psi'(s)(\Psi(b)-\Psi(s))^{\alpha-1}h(s,u(s))ds\right]\right)ds, \end{aligned} \tag{25}$$

Theorem 3.4. *Suppose that* (H_1)–(H_2) *are satisfied, then the problem (1) has at least one solution* $u \in C(\Lambda, \mathbb{R})$.

As a means of demonstrating the Theorem 3.4, we will prove that the operator $\mathcal{W}$ satisfies the conditions of Theorem 2.7 (Schaefer's fixed point theorem).

Proof. Consider the operator $\mathcal{W}$ defined in (25), we will show that $\mathcal{W}$ is completely continuous operator.
step 1: $\mathcal{W}$ is continuous.
Let $(u_n) \in C(\Lambda, \mathbb{R})$ be a sequence such that $u_n \longrightarrow u$ in $C(\Lambda, \mathbb{R})$, by using the continuity of the function h and ϕ_q we get $\lim\limits_{n\to+\infty} G_1 h(\tau, u_n(\tau)) = G_1 h(\tau, u(\tau))$ and $\lim\limits_{n\to+\infty} G_1 h(\tau, u_n(\tau)) = G_2 h(\tau, u(\tau))$, furthermore

$$\begin{aligned}
\lim_{n\to+\infty} \mathcal{W} u_n(\tau) &= \lim_{n\to+\infty} \left(\mathfrak{I}_{a^+}^{\beta;\Psi} \phi_q \Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau, u_n(\tau)) + G_1 h(\tau, u_n(\tau)) \Big) + G_2 h(\tau, u_n(\tau)) \right) \\
&= \mathfrak{I}_{a^+}^{\beta;\Psi} \phi_q \Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau, u(\tau)) + G_1 h(\tau, u(\tau)) \Big) + G_2 h(\tau, u(\tau)) \Big) \\
&= \mathcal{W} u(\tau).
\end{aligned}$$

This shows that $\mathcal{W}$ is continuous.
step 2: $\mathcal{W}$ is bounded.
Let $\mathcal{N}$ a bounded set, such that $\mathcal{N} \subset \mathcal{B}_\rho$, we will show that $\mathcal{W}(\mathcal{N})$ is bounded, for that $\forall u \in \mathcal{N}$, we have $\|u\| \leq \rho$, by making use of the continuity of the function h and (H_1), we get $|h(\tau, u)| \leq |\gamma| + |\delta|\rho^{p-1} := N_1$, then for all $\tau \in \Lambda$, $u \in \mathcal{N}$ we have

$$\begin{aligned}
|\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau, u(\tau)) + G_1 h(\tau, u(\tau))| &\leq \frac{1}{\Gamma(\alpha)} \int_a^\tau \Psi'(s)(\Psi(\tau) - \Psi(s))^{\alpha-1} |h(s, u(s))| ds \\
&+ \frac{\Psi(\tau) - \Psi(a)}{|\Theta|\Gamma(\alpha)} \Bigg[\kappa^{p-1} \int_a^\eta \Psi'(s)(\Psi(\eta) - \Psi(s))^{\alpha-1} |h(s, u(s))| ds \\
&- \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{\alpha-1} |h(s, u(s))| ds \Bigg], \\
&\leq \mathfrak{B} N_1.
\end{aligned}$$

And

$$\begin{aligned}
|G_2 h(\tau, u(\tau))| &\leq \frac{|\mu|}{|1-\mu|\Gamma(\beta)} \int_a^\xi \Psi'(s)(\Psi(\xi) - \Psi(s))^{\beta-1} \\
&\times \left| \phi_q \Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau, u(\tau)) + G_1 h(\tau, u(\tau)) \Big) \right| \\
&\leq \frac{|\mu|(\Psi(\xi) - \Psi(a))^\beta}{|1-\mu|\Gamma(\beta+1)} \Big[\mathfrak{B} N_1 \Big]^{q-1},
\end{aligned}$$

it follows that

$$\begin{aligned}
|\mathcal{W} u(\tau)| &\leq \frac{(\Psi(b) - \Psi(a))^\beta}{\Gamma(\beta+1)} \Big[\mathfrak{B} N_1 \Big]^{q-1} + \frac{|\mu|(\Psi(\xi) - \Psi(a))^\beta}{|1-\mu|\Gamma(\beta+1)} \Big[\mathfrak{B} N_1 \Big]^{q-1}, \\
&\leq \mathfrak{A} \Big[\mathfrak{B} N_1 \Big]^{q-1}.
\end{aligned}$$

Taking the supremum over τ, we obtain

$$\|\mathcal{W} u\| \leq \mathfrak{A} \Big[\mathfrak{B} N_1 \Big]^{q-1}.$$

Then $\mathcal{W}(\mathcal{N})$ is bounded.

step 3:$\mathcal{W}$ is equicontinuous.
Let $\tau_1, \tau_2 \in \Lambda$ with $\tau_1 < \tau_2$ and for $u \in \mathcal{N}$; (see that $G_2 u$ given by (14) is independent of τ); then we have

$$
\begin{aligned}
|\mathcal{W}u(\tau_2) - \mathcal{W}u(\tau_1)| &= \Big| \mathfrak{I}_{a^+}^{\beta;\Psi} \phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau_2, u(\tau_2)) + G_1 h(\tau_2, u(\tau_2)) \Big) \\
&- \mathfrak{I}_{a^+}^{\beta;\Psi} \phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau_1, u(\tau_1)) + G_1 h(\tau_1, u(\tau_1)) \Big) \Big|, \\
&\le \Big| \frac{1}{\Gamma(\beta)} \int_a^{\tau_1} \Psi'(s) \Big[(\Psi(\tau_2) - \Psi(s))^{\beta-1} - (\Psi(\tau_1) - \Psi(s))^{\beta-1} \Big] \\
&\times \phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(s, u(s)) + G_1 h(s, u(s)) \Big) ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{\tau_1}^{\tau_2} \Psi'(s)(\Psi(\tau_2) - \Psi(s))^{\beta-1} \\
&\times \phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(s, u(s)) + G_1 h(s, u(s)) \Big) ds \Big|, \\
&\le \frac{\Big[\mathfrak{B} N_1\Big]^{q-1}}{\Gamma(\beta)} \Big\{ \int_a^{\tau_1} \Psi'(s) \Big| (\Psi(\tau_2) - \Psi(s))^{\beta-1} - (\Psi(\tau_1) - \Psi(s))^{\beta-1} \Big| ds \\
&+ \int_{\tau_1}^{\tau_2} \Psi'(s)(\Psi(\tau_2) - \Psi(s))^{\beta-1} ds \Big\} \\
&\le \frac{\Big[\mathfrak{B} N_1\Big]^{q-1}}{\Gamma(\beta+1)} \Big\{ 2(\Psi(\tau_2) - \Psi(\tau_1))^{\beta} \\
&+ \Big| (\Psi(\tau_2) - \Psi(s))^{\beta} - (\Psi(\tau_1) - \Psi(s))^{\beta} \Big| \Big\}.
\end{aligned}
$$

By using the continuity of the function Ψ, the right hand side of the above inequality tends to 0 as τ_2 tends to τ_1 this implies that $\mathcal{W}(\mathcal{N})$ is equicontinuous. From step 2 and step 3 It follows by using the Arzelà-Ascoli theorem that the operator $\mathcal{W}$ is relatively compact, as consequence the operator $\mathcal{W}$ is completely continuous.

Step 4: The set $\Upsilon_\varepsilon = \{u \in C(\Lambda, \mathbb{R}) \mid u = \varepsilon \mathcal{W} u \,;\, 0 < \varepsilon \le 1\}$ is bounded.
We are going to show that the set Υ_ε is bounded. By using (H_1) we have

$$
\begin{aligned}
|\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau, u(\tau)) + G_1 h(\tau, u(\tau))| &\le \frac{1}{\Gamma(\alpha)} \int_a^{\tau} \Psi'(s)(\Psi(\tau) - \Psi(s))^{\alpha-1} |h(s, u(s))| ds \\
&+ \frac{\Psi(\tau) - \Psi(a)}{|\Theta|\Gamma(\alpha)} \Big[\kappa^{p-1} \int_a^{\eta} \Psi'(s)(\Psi(\eta) - \Psi(s))^{\alpha-1} |h(s, u(s))| ds \\
&- \int_a^{b} \Psi'(s)(\Psi(b) - \Psi(s))^{\alpha-1} |h(s, u(s))| ds \Big], \\
&\le \mathfrak{B}(\|\gamma\| + \|\delta\| \|u\|^{p-1}).
\end{aligned}
$$

And

$$
\begin{aligned}
|G_2 h(\tau, u(\tau))| &\le \frac{|\mu|}{|1-\mu|\Gamma(\beta)} \int_a^{\xi} \Psi'(s)(\Psi(\xi) - \Psi(s))^{\beta-1} \\
&\times \Big| \phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau, u(\tau)) + G_1 h(\tau, u(\tau)) \Big) \Big| \\
&\le \frac{|\mu|(\Psi(\xi) - \Psi(a))^{\beta}}{|1-\mu|\Gamma(\beta+1)} \Big[\mathfrak{B}(\|\gamma\| + \|\delta\| \|u\|^{p-1}) \Big]^{q-1},
\end{aligned}
$$

for all $u \in \Upsilon_\varepsilon$ we have $u(\tau) = \varepsilon \mathcal{W} u(\tau)$ then it follows that

$$
\begin{aligned}
|u(\tau)| &\leq \left|\mathfrak{I}_{a^+}^{\beta;\Psi} \phi_q\Big(\mathfrak{I}_{a^+}^{\alpha;\Psi} h(\tau,u(\tau)) + G_1 h(\tau,u(\tau))\Big)\right| + \left|G_2 h(\tau,u(\tau))\right| \\
&\leq \frac{(\Psi(b)-\Psi(a))^\beta}{\Gamma(\beta+1)} \Big[\mathfrak{B}(\|\gamma\| + \|\delta\|\|u\|^{p-1})\Big]^{q-1} \\
&+ \frac{|\mu|(\Psi(\xi)-\Psi(a))^\beta}{|1-\mu|\Gamma(\beta+1)} \Big[\mathfrak{B}(\|\gamma\| + \|\delta\|\|u\|^{p-1})\Big]^{q-1} \\
&\leq \left(\frac{(\Psi(b)-\Psi(a))^\beta}{\Gamma(\beta+1)} + \frac{|\mu|(\Psi(\xi)-\Psi(a))^\beta}{|1-\mu|\Gamma(\beta+1)}\right) \Big[\mathfrak{B}(\|\gamma\| + \|\delta\|\|u\|^{p-1})\Big]^{q-1} \\
&\leq \mathfrak{A}\mathfrak{B}^{q-1}\Big(\|\gamma\| + \|\delta\|\|u\|^{p-1}\Big)^{q-1}.
\end{aligned}
$$

Where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (22) and (23). Thus, we have

$$\|u\| \leq \mathfrak{A}\mathfrak{B}^{q-1}\Big(\|\gamma\| + \|\delta\|\|u\|^{p-1}\Big)^{q-1},$$

then

$$\|u\|^{p-1} \leq \mathfrak{A}^{p-1}\mathfrak{B}\Big(\|\gamma\| + \|\delta\|\|u\|^{p-1}\Big),$$

by using (H_2), we get

$$\|u\|^{p-1} \leq \frac{\mathfrak{A}^{p-1}\mathfrak{B}\|\gamma\|}{1-\mathfrak{A}^{p-1}\mathfrak{B}\|\delta\|} := \mathcal{M},$$

finally

$$\|u\| \leq \mathcal{M}^{q-1}.$$

This proves that the set Υ_ε is bounded in $C(\Lambda,\mathbb{R})$, by using Theorem 2.7, $\mathcal{W}$ has at least one fixed point which is the solution of the problem (1). □

4. Example

Consider the following problem

$$
\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{3}{2};\frac{e^\tau}{3}}\Big(\phi_3\Big[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{2}{5};\frac{e^\tau}{3}} u(\tau)\Big]\Big) = \dfrac{5\tau^2}{cos\tau} + \dfrac{1}{7+e^\tau}u^2(\tau),\ \tau \in \Lambda := [0,1], \\
u(0) = \frac{5}{8}u(\frac{2}{3}),\ {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{2}{5};\frac{e^\tau}{3}} u(a) = 0,\ {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{2}{5};\frac{e^t}{3}} u(1) = \frac{3}{8}\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{2}{5};\frac{e^\tau}{3}} u(\frac{1}{2}).
\end{cases}
\tag{26}
$$

Where $\alpha = \dfrac{3}{2}, \beta = \dfrac{2}{5}, p = 3, q = \dfrac{3}{2}, a = 0, b = 1, \Lambda = [0,1], \xi = \dfrac{2}{3}, \eta = \dfrac{1}{2}, \kappa = \dfrac{3}{8}, \mu = \dfrac{5}{8}$ and $\Psi(\tau) = \dfrac{e^\tau}{3}$.

We define $h(t,u) = \dfrac{5\tau^2}{cos\tau} + \dfrac{1}{7+e^\tau}u^2, \tau \in [0,1]$. h is a continuous function, furthermore for every $\tau \in [0,1]$ we put : $\gamma(\tau) = \dfrac{5\tau^2}{cos\tau}$ and $\delta(\tau) = \dfrac{1}{7+e^\tau}$ such that the condition (H_1) holds. By using the data given above, we get : $|\Theta| = 0.6701$, $\mathfrak{A} = 1.378024$, $\mathfrak{B} = 1.138093$ and $\|\delta\| = \dfrac{1}{8} = 0.125$.

Then $\mathfrak{A}^2\mathfrak{B}\|\delta\| = 1.378024^2 \times 1.138093 \times 0.125 = 0.270147 < 1$.

The problem (26) satisfies all the hypothesis of Theorem 3.4. Thus, The problem (26) has at least one solution on $[0,1]$.

5. Conclusion

In this paper, we have studied and investigated the existence result for a new class of Ψ-Caputo-type fractional differential equation involving the p-Laplacian operator. The novelty of the considered problem is that it has been investigated under the Ψ-Caputo fractional derivatives, which is more general than the works based on the well-known fractional derivatives such as (Caputo fractional derivative, Caputo-Hadamard fractional derivative and Caputo-Katugampola fractional derivative) for different values of the function Ψ. In this article we established the existence results for the problem (1), by using a standard fixed point theorem (Schaefer's fixed point theorem). Finally a numerical example is presented to clarify the obtained result.

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Data Availability

The data used to support the finding of this study are available from the corresponding author upon request.

Conflicts Of Interest

The authors declare that they have no conflicts of interest.

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