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Gabor transform in the generalized Weinstein theory setting

Hassen Ben Mohamed[a]

[a] *University of Gabes, Department of Mathematics, Faculty of Sciences, Tunisia*

Abstract. In this paper, we consider the generalized Weinstein operator $\Delta_W^{d,\alpha,n}$, we introduce and study the continuous generalized Weinstein Gabor Transform denoted by $\mathscr{G}_g$ associated with the operator $\Delta_W^{d,\alpha,n}$. We prove a Plancherel formula and a weak uncertainty principle for it. We obtain analogous of Heisenbeg's inequality for the generalized Weinstein Gabor Transform.

1. Introduction

In this paper, we consider the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0,\ \infty)$, by :

$$\Delta_W^{\alpha,d,n} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n\,(\alpha+n)}{x_{d+1}^2} = \Delta_W^{\alpha,d} - \frac{4n\,(\alpha+n)}{x_{d+1}^2}, \tag{1}$$

where $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$ and $\Delta_W^{\alpha,d}$ the classical Weinstein operator given by :

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}}. \tag{2}$$

(See [2], [3], [8], [9], [10], [11], [13], [14] and [15]).
The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by :

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}\,(x,y) = x_{d+1}^{2n} e^{-i\langle x', y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}), \tag{3}$$

where $x = (x', x_{d+1}),\ x' = (x_1, x_2, ..., x_d)$ and j_α is the normalized Bessel function of index α defined by :

$$\forall \xi \in \mathbb{C},\ j_\alpha(\xi) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (\frac{\xi}{2})^{2n}. \tag{4}$$

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Email address: hassenbenmohamed@yahoo.fr (Hassen Ben Mohamed)

(See [1] and [5]-[7]).
Using the Weinstein kernel $\Lambda_{\alpha,d,n}$, we define the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ by :

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda)d\nu_{\alpha,d}(x),$$

where $f \in L^1(\mathbb{R}_+^{d+1}, \nu_{\alpha,d}(x))$ and $\nu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by :

$$d\nu_{\alpha,d}(x) = x_{d+1}^{2\alpha+1}dx. \tag{5}$$

The contents of the paper are as follows :
In the section 2, we recapitulate some results related to the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ given by the relation (1).

The section 3, is devoted to studied the analogue of the continuous Gabor transform associated with the operator $\Delta_W^{\alpha,d,n}$ and we give some harmonic properties for it.

In the section 4, we prove the analogous of Heisenberg's inequality for the generalized Weinstein continuous Gabor transform.

In the section 5, we will show that the portion of the continuous generalized Weinstein Gabor transform lying outside some sufficiently small set of finite measure cannot be arbitrarily too small. Using the kernel reproducing theory given by Saitoh (see [12]), we study the problem of approximative concentration.

2. Preliminaires

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1}$ by the relation (1).

Notations. In what follows, we need the following notations :

- $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathbb{L}_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\begin{aligned} \|f\|_{\alpha,p} &= \left[\int_{\mathbb{R}_+^{d+1}} |f(x)|^p d\nu_{\alpha,d}(x)\right]^{\frac{1}{p}} < +\infty, \text{ if } 1 \le p < +\infty, \\ \|f\|_{\alpha,\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^{d+1}} \left|f(x)\right| < +\infty, \end{aligned}$$

where $\nu_{\alpha,d}$ is the measure given by the relation (5).

- $\mathscr{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.
- $\mathscr{M}_n$, the map defined by :

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{M}_n\left(f\right)(x) = x_{d+1}^{2n} f\left(x\right). \tag{6}$$

where $x = (x', x_{d+1})$ and $x' = (x_1, x_2, ..., x_d)$

- $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that $\|f\|_{\alpha,n,p} < +\infty$ where

$$\begin{aligned} \|f\|_{\alpha,n,p} &= \left[\int_{\mathbb{R}_+^{d+1}} |\mathscr{M}_n^{-1} f(x)|^p d\nu_{\alpha+2n,d}(x)\right]^{\frac{1}{p}}, \text{ if } 1 \le p < +\infty \\ \|f\|_{\alpha,n,\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^{d+1}} \left|\mathscr{M}_n^{-1} f(x)\right|. \end{aligned}$$

• $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ repespectively stand for the subspace of $\mathscr{E}_*(\mathbb{R}^{d+1})$, $\mathscr{D}_*(\mathbb{R}^{d+1})$ and $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, ..., 2n-1\}, \ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

Let us begin by the following result.

Lemma 2.1. *(See [5]-[7])*
i) The map $\mathscr{M}_n$ is an isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})$ $\left(resp.\ \mathscr{S}_*(\mathbb{R}^{d+1})\right)$ onto $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ $\left(resp.\ \mathscr{S}_{n,*}(\mathbb{R}^{d+1})\right)$.
ii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$\Delta_W^{\alpha,d,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ \Delta_W^{\alpha+2n}(f). \tag{7}$$

iii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$ and $g \in \mathscr{D}_{n,*}(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} \Delta_W^{\alpha,d,n} f(x)\, g(x)\, d\nu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} f(x)\, \Delta_W^{\alpha,d,n} g(x)\, d\nu_{\alpha,d}(x). \tag{8}$$

Definition 2.2. *i) The generalized Weinstein intertwining operator is the operator $\mathscr{R}_W^{\alpha,d,n}$ defined on $\mathscr{E}_*(\mathbb{R}^{d+1})$ by :*

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{R}_W^{\alpha,d,n} f(x) = a_{\alpha+2n} x_{d+1}^{2n} \int_0^1 \left(1-t^2\right)^{\alpha+2n-\frac{1}{2}} f(x', t x_{d+1}) dt, \tag{9}$$

where a_α is the constant given by the relation :

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}. \tag{10}$$

ii) The dual of the Weinstein intertwining operator is defined on $\mathscr{D}_*(\mathbb{R}^{d+1})$ by :

$$\forall y \in \mathbb{R}_+^{d+1},\ {}^t\mathscr{R}_W^{\alpha,d,n}(f)(y) = a_{\alpha+2n} \int_{y_{d+1}}^{+\infty} \left(s^2 - y_{d+1}^2\right)^{\alpha+2n-\frac{1}{2}} f(y', s) s^{1-2n} ds. \tag{11}$$

Proposition 2.3. *(See [5]-[7])*
i) The operator $\mathscr{R}_W^{\alpha,d,n}$ is a topological isomorphism from $\mathscr{E}_*(\mathbb{R}_+^{d+1})$ onto $\mathscr{E}_{n,*}(\mathbb{R}_+^{d+1})$ satisfying the following transmutation relation

$$\triangle_W^{\alpha,d,n}(\mathscr{R}_W^{\alpha,d,n} f) = \mathscr{R}_W^{\alpha,d,n}(\triangle_{d+1} f), f \in \mathscr{E}_*(\mathbb{R}_+^{d+1}), \tag{12}$$

where $\triangle_{d+1} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on $\mathbb{R}^{d+1}$.
ii) ${}^t\mathscr{R}_W^{\alpha,d,n}$ can be extended to a topological isomorphism from $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{S}_*(\mathbb{R}^{d+1})$ and satisfies the following transmutation relation

$${}^t\mathscr{R}_W^{\alpha,d,n}(\triangle_W^{\alpha,d,n} f) = \triangle_{d+1}({}^t\mathscr{R}_W^{\alpha,d,n} f), f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1}). \tag{13}$$

iii) Let $f \in \mathscr{D}_{n,*}(\mathbb{R}_+^{d+1})$ and $g \in \mathscr{E}_*(\mathbb{R}_+^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} {}^t\mathscr{R}_W^{\alpha,d,n}(f)(x) g(x)\, dx = \int_{\mathbb{R}_+^{d+1}} f(x) \mathscr{R}_W^{\alpha,d,n}(g)(x)\, d\nu_{\alpha,d}(x). \tag{14}$$

Definition 2.4. *The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by :*

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}), \tag{15}$$

where $x = (x', x_{d+1}),\ x' = (x_1, x_2, ..., x_d)$ and j_α is the function given by the relation (4).

Proposition 2.5. *(See [5]-[7])*
i) We have

$$\forall x, y \in \mathbb{R}_+^{d+1},\ |\Lambda_{\alpha,d,n}(x,y)| \leq x_{d+1}^{2n}. \tag{16}$$

ii) The function $x \mapsto \Lambda_{\alpha,d,n}(x,y)$ satisifies the differentiel equation

$$\triangle_W^{\alpha,d,n}\left(\Lambda_{\alpha,d,n}(.,y)\right)(x) = -\left\|y\right\|^2 \Lambda_{\alpha,d,n}(x,y). \tag{17}$$

iii) For all $x,\ y \in \mathbb{C}^{d+1}$, we have

$$\Lambda_{\alpha,d,d}(x,y) = a_{\alpha+2n} e^{-i\langle x',y'\rangle} x_{d+1}^{2n} \int_0^1 \left(1-t^2\right)^{\alpha+2n-\frac{1}{2}} \cos(t x_{d+1} y_{d+1}) dt, \tag{18}$$

where a_α is the constant given by the relation (10).

Definition 2.6. *The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ is given for $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ by :*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d,n}(x,\lambda) d\nu_{\alpha,d}(x), \tag{19}$$

where $\nu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by the relation (5).

Lemma 2.7. *The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ can be written in the form :*

$$\mathscr{F}_W^{\alpha,d,n} = \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1}, \tag{20}$$

where $\mathscr{F}_W^{\alpha,d}$ is the classical Weinstein transform given by :

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\beta,d}(x,\lambda) d\nu_{\alpha,d}(x). \tag{21}$$

Proof. Let $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}_+^{d+1})$. For all $\lambda \in \mathbb{R}_+^{d+1}$, using the relations (15) and (19), we obtain

$$\begin{aligned}\mathscr{F}_W^{\alpha,d,n}(f)(\lambda) &= \int_{\mathbb{R}_+^{d+1}} f(x) e^{-i\langle x',y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}) x_{d+1}^{2n} d\nu_{\alpha,d}(x)\\ &= \int_{\mathbb{R}_+^{d+1}} \mathscr{M}_n^{-1} f(x) \Lambda_{\alpha+2n,d}(x,\lambda) d\nu_{\alpha+2n,d}(x)\\ &= \mathscr{F}_W^{\alpha+2n,d}\left(\mathscr{M}_n^{-1} f\right)(\lambda),\end{aligned}$$

where $\mathscr{F}_W^{\alpha,d}$ is the classical Weinstein transform given by (21). □

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d,n}$ are summarized in the following results.

Proposition 2.8. *(See [5]-[7])*
i) For all $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$*, we have*

$$\|\mathscr{F}^{\alpha,d,n}_W(f)\|_{\alpha,n,\infty} \leq \|f\|_{\alpha,n,1}. \tag{22}$$

ii) For $m \in \mathbb{N}$ *and* $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$*, we have*

$$\forall x \in \mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d,n}_W\left[\left(\triangle^{\alpha,d,n}_W\right)^m f\right](x) = (-1)^m \|x\|^{2m} \mathscr{F}^{\alpha,d,n}_W(f)(x). \tag{23}$$

iii) For all f *in* $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *and* $m \in \mathbb{N}$*, we have*

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \left(\triangle^{\alpha,d,n}_W\right)^m \left[\mathscr{M}_n \mathscr{F}^{\alpha,d,n}_W(f)\right](\lambda) = \mathscr{M}_n \mathscr{F}^{\alpha,d,n}_W(P_m f)(\lambda), \tag{24}$$

where $P_m(\lambda) = (-1)^m \|\lambda\|^{2m}$.
iv) For all $f \in \mathscr{S}_{n,*}\left(\mathbb{R}^{d+1}\right)$*, we have*

$$\forall y \in \mathbb{R}^{d+1}_+,\ \mathscr{F}^{\alpha,d,n}_W(f)(y) = \mathscr{F}_o \circ\ {}^t\mathscr{R}^{\alpha,d,n}_W(f)(y), \tag{25}$$

where $\mathscr{F}_o$ *is the classical Fourier transform defined for* $f \in L^1(\mathbb{R}^{d+1}_+, dx)$ *by*

$$\forall\ y \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_o(f)(y) = \int_{\mathbb{R}^{d+1}_+} f(x) e^{-i\langle y',x'\rangle} \cos(x_{d+1} y_{d+1}) dx.$$

v) Let $f \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$*. If* $\mathscr{F}^{\alpha,d,n}_W(f) \in \mathbb{L}^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$*, then we have*

$$f(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(y)\, \Lambda_{\alpha,d,n}(-x,y) d\nu_{\alpha+2n,d}(y),\ a.e\ x \in \mathbb{R}^{d+1}_+, \tag{26}$$

where $C_{\alpha,d}$ *is the constant given by :*

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}}\, 2^\alpha \Gamma(\alpha+1)}. \tag{27}$$

vi) The Weinstein transform $\mathscr{F}^{\alpha,d,n}_W$ *is a topological isomorphism from* $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *onto itself and from* $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ *onto* $\mathscr{H}_*(\mathbb{C}^{d+1})$.

Theorem 2.9. *(See [5]-[7])*
i) For all $f, g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$*, we have the following Parseval formula*

$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)} d\nu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda) \overline{\mathscr{F}^{\alpha,d,n}_W(g)(\lambda)} d\nu_{\alpha+2n,d}(\lambda), \tag{28}$$

where $C_{\alpha,d}$ *is the constant given by the relation (27).*
ii) (Plancherel formula).
For all $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$*, we have :*

$$\int_{\mathbb{R}^{d+1}_+} \left|f(x)\right|^2 d\nu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\right|^2 d\nu_{\alpha+2n,d}(\lambda). \tag{29}$$

iii) (Plancherel Theorem) :
The transform $\mathscr{F}^{\alpha,d,n}_W$ *extends uniquely to an isometric isomorphism from* $L^2(\mathbb{R}^{d+1}_+,\ d\nu_{\alpha,d}(x))$ *onto* $L^2(\mathbb{R}^{d+1}_+,\ C^2_{\alpha+2n,d} d\nu_{\alpha+2n,d}(x))$.

Definition 2.10. *The translation operator $T_x^{\alpha,d,n}$, $x \in \mathbb{R}_+^{d+1}$, associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ is defined on $\mathscr{E}_{n,*}(\mathbb{R}_+^{d+1})$ by :*

$$\forall y \in \mathbb{R}_+^{d+1},\ T_x^{\alpha,d,n} f(y) = x_{d+1}^{2n} \mathscr{M}_n T_x^{\alpha+2n,d} \mathscr{M}_n^{-1} f(y), \tag{30}$$

where

$$T_x^{\alpha,d} f(y) = \frac{a_\alpha}{2} \int_0^\pi f\left(x' + y',\ \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha}\, d\theta, \tag{31}$$

$x' + y' = (x_1 + y_1, ..., x_d + y_d)$ and a_α is the constant given by (10).

The following proposition summarizes some properties of the generalized Weinstein translation operator.

Proposition 2.11. *(see [5]-[7])*
i) We have

$$\forall x \in \mathbb{R}_+^{d+1},\ \Delta_W^{\alpha,d,n} \circ T_x^{\alpha,d,n} = T_x^{\alpha,d,n} \circ \Delta_W^{\alpha,d,n}. \tag{32}$$

ii) Let $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$ and $x \in \mathbb{R}_+^{d+1}$. Then $T_x^{\alpha,d,d} f$ belongs to $\mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$ and we have

$$\|T_x^{\alpha,d,n} f\|_{\alpha,p,n} \le x_{d+1}^{2n} \|f\|_{\alpha,p,n}. \tag{33}$$

iii) The function $\Lambda_{\alpha,d,n}(.,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}_+^{d+1}$ the following product formula:

$$\forall y \in \mathbb{R}_+^{d+1},\ \Lambda_{\alpha,d,n}(x,\lambda)\, \Lambda_{\alpha,d,n}(y,\lambda) = T_x^{\alpha,d,n}\left[\Lambda_{\alpha,d,n}(.,\lambda)\right](y). \tag{34}$$

iv) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ and $x \in \mathbb{R}_+^{d+1}$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}\left(T_x^{\alpha,d,n} f\right)(y) = \Lambda_{\alpha,d,n}(-x,y)\, \mathscr{F}_W^{\alpha,d,n}(f)(y). \tag{35}$$

Definition 2.12. *The generalized Weinstein convolution product of $f, g \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$ is given by :*

$$\forall x \in \mathbb{R}_+^{d+1},\ f *_{\alpha,n} g(x) = \int_{\mathbb{R}_+^{d+1}} T_x^{\alpha,d,n} f(-y)\, g(y)\, d\nu_{\alpha,d}(y). \tag{36}$$

Proposition 2.13. *(See [5]-[7])*
i) Let $p, q, r \in [1,\ +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $f \in \mathbb{L}_{\alpha,n}^p(\mathbb{R}_+^{d+1})$ and $g \in \mathbb{L}_{\alpha,n}^q(\mathbb{R}_+^{d+1})$, the function $f *_{\alpha,n} g \in \mathbb{L}_{\alpha,n}^r(\mathbb{R}_+^{d+1})$ and we have

$$\|f *_{\alpha,n} g\|_{\alpha,r,n} \le \|f\|_{\alpha,p,n} \|g\|_{\alpha,q,n}. \tag{37}$$

ii) For all $f, g \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$, $f *_{\alpha,n} g \in \mathbb{L}_{\alpha,n}^1(\mathbb{R}_+^{d+1})$ and we have

$$\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g). \tag{38}$$

iii) Let $f, g \in \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$. Then $f *_{\alpha,n} g \in \mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$ if and only if $\mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g)$ belongs to $\mathbb{L}_{\alpha,n}^2(\mathbb{R}_+^{d+1})$ and in this case, we have

$$\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g).$$

3. The generalized continuous Weinstein Gabor transform

In this section, we introduce the analogue of the continuous Weinstein Gabor transform associated with the generalized Weinstein operator $\triangle_W^{\alpha,d,n}$ and we give some harmonic analysis properties for it.
In what follows, we need the following notations :
Notations. We denote by :
• $X^p_{\alpha,d,n}$, $1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}_+^{d+1} \times \mathbb{R}_+^{d+1}$ with respect to the measure $dw_{\alpha,d,n}(x,y) = d\nu_{\alpha+2n,d}(x)d\nu_{\alpha+2n,d}(y)$ such that $\|f\|_{w_{\alpha,d,n},p} < +\infty$, where

$$\begin{aligned} \|f\|_{w_{\alpha,d,n},p} &= \left[\int_{\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}} |\mathscr{M}_{n,x}^{-1}\mathscr{M}_{n,y}^{-1} f(x,y)|^p dw_{\alpha,d,n}(x,y)\right]^{\frac{1}{p}}, \text{ if } 1 \leq p < +\infty, \\ \|f\|_{w_{\alpha,d,n},\infty} &= \operatorname*{ess\,sup}_{x,y\in\mathbb{R}_+^{d+1}} \left|\mathscr{M}_{n,x}^{-1}\mathscr{M}_{n,y}^{-1} f(x,y)\right|. \end{aligned}$$

• $X^p_{\alpha,d} = X^p_{\alpha,d,0}$.
• $\|f\|_{w_{\alpha,d},p} = \|f\|_{w_{\alpha,d,0},p}$ and $dw_{\alpha,d}(x,y) = d\nu_{\alpha,d}(x)d\nu_{\alpha,d}(y)$.

Definition 3.1. *Let $t \in \mathbb{R}_+^{d+1}$ and $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, the modulation of the function g by t is given by the relation :*

$$M_t g := g_t := C^2_{\alpha+2n,d}\mathscr{M}_n \mathscr{F}_W^{\alpha,d,n}\left(\sqrt{\mathscr{M}_n t_{d+1}^{2n} T_t^{\alpha,d,n}\left(\mathscr{M}_n^{-1}\left|g\right|^2\right)}\right), \tag{39}$$

where $T_t^{\alpha,d,n}$ are generalized Weinstein translation operators given by the relation (30).

Remark 3.2. *Let $t \in \mathbb{R}_+^{d+1}$ and $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, we have*

$$\forall y \in \mathbb{R}_+^{d+1},\ \left(\mathscr{F}_W^{\alpha,d,n}(g_t)(y)\right)^2 = t_{d+1}^{2n}\mathscr{M}_n^{-1}T_t^{\alpha,d,n}\left(\mathscr{M}_n^{-1}\left|g\right|^2\right)(-y). \tag{40}$$

Proposition 3.3. *Let $t \in \mathbb{R}_+^{d+1}$ and $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$.*
i) The function g_t belongs to $\mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\left\|g_t\right\|_{\alpha,2,n} = C_{\alpha+2n,d}t_{d+1}^{2n}\left\|g\right\|_{\alpha,2,n}. \tag{41}$$

ii) We have

$$\int_{\mathbb{R}_+^{d+1}}\left|\mathscr{F}_W^{\alpha,d,n}(g_x)(y)\right|^2 d\nu_{\alpha,d}(x) = \left\|g\right\|^2_{\alpha,2,n}. \tag{42}$$

iii) We have

$$\int_{\mathbb{R}_+^{d+1}}\left|\mathscr{F}_W^{\alpha,d,n}(g_t)(y)\right|^2 d\nu_{\alpha+2n,d}(y) = t_{d+1}^{4n}\left\|g\right\|^2_{\alpha,2,n}. \tag{43}$$

Proof. i) Let $t \in \mathbb{R}_+^{d+1}$ and $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, using the relation (29) and (40), we obtain

$$\begin{aligned} \left\|g_t\right\|^2_{\alpha,2,n} &= C^2_{\alpha+2n,d}t_{d+1}^{2n}\int_{\mathbb{R}_+^{d+1}}\mathscr{M}_n^{-1}T_t^{\alpha,d,n}\left(\mathscr{M}_n^{-1}\left|g\right|^2\right)(y)\,d\nu_{\alpha+2n,d}(y) \\ &= C^2_{\alpha+2n,d}t_{d+1}^{2n}\left\|T_t^{\alpha,d,n}\left(\mathscr{M}_n^{-1}\left|g\right|^2\right)\right\|_{\alpha,1,n} \\ &= C^2_{\alpha+2n,d}t_{d+1}^{4n}\left\|\mathscr{M}_n^{-1}\left|g\right|^2\right\|_{\alpha,1,n} \\ &= C^2_{\alpha+2n,d}t_{d+1}^{4n}\left\|g\right\|^2_{\alpha,2,n}. \end{aligned}$$

ii) An easily combination of the relations (40), (29) and (41), we obtain the result.
iii) The result follows from the relations (29) and (41). □

Definition 3.4. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. We consider the family $g_{x,y}$, $x, y \in \mathbb{R}^{d+1}_+$ defined by :*

$$\forall \xi \in \mathbb{R}^{d+1}_+,\ g_{x,y}(\xi) = T^{\alpha,d,n}_y g_x(-\xi). \tag{44}$$

For any function $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we define its continuous Weinstein Gabor transform by :

$$\forall x, y \in \mathbb{R}^{d+1}_+,\ \mathscr{G}_g f(x,y) := \int_{\mathbb{R}^{d+1}_+} f(\xi)\, g_{x,y}(\xi)\, d\nu_{\alpha,d}(\xi). \tag{45}$$

Remark 3.5. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. The transform can $\mathscr{G}_g$ can be written in the form*

$$\mathscr{G}_g f(x,y) = f *_{\alpha,n} g_x(y). \tag{46}$$

The following Lemmas will plays an important role in the sequel.

Lemma 3.6. *Let $x \in \mathbb{R}^{d+1}_+$ and $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}\left(\mathbb{R}^{d+1}_+\right)$. We have*

$$\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(g_x)\right|^4(\xi)\, d\nu_{\alpha+2n,d}(\xi) \le x^{8n}_{d+1} \left\|g\right\|^2_{\alpha,\infty,n} \left\|g\right\|^2_{\alpha,2,n}. \tag{47}$$

Proof. Let $x \in \mathbb{R}^{d+1}_+$ and $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Using the relations (40) and (33), we obtain :

$$\begin{aligned}
\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(g_x)\right|^4(\xi)\, d\nu_{\alpha+2n,d}(\xi) &= x^{4n}_{d+1} \int_{\mathbb{R}^{d+1}_+} \left|T^{\alpha,d,n}_x\left(\mathscr{M}^{-1}_n \left|g\right|^2\right)\right|^2(-\xi)\, d\nu_{\alpha,d}(\xi)\\
&= x^{4n}_{d+1} \left\|T^{\alpha,d,n}_x\left(\mathscr{M}^{-1}_n \left|g\right|^2\right)\right\|^2_{\alpha,2,n}\\
&\le x^{8n}_{d+1} \left\|\mathscr{M}^{-1}_n \left|g\right|^2\right\|^2_{\alpha,2,n}\\
&\le x^{8n}_{d+1} \left\|g\right\|^2_{\alpha,\infty,n} \left\|g\right\|^2_{\alpha,2,n}.
\end{aligned}$$

□

Lemma 3.7. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+) \setminus \{0\}$. For any positive integer m, we consider the two functions given by :*

$$h_m(\xi) = C^2_{\alpha+2n,d} \int_{B^{d+1}_+(0,\ m)} \int_{\mathbb{R}^{d+1}_+} \Lambda_{\alpha,d,n}(-\xi, x) \left|\mathscr{F}^{\alpha,d,n}_W(g_t)(x)\right|^2 d\nu_{\alpha+2n,d}(x)\, d\nu_{\alpha,d}(t), \tag{48}$$

and

$$k_m(\xi) = \frac{1}{\left\|g\right\|^2_{\alpha,2,n}} \int_{B^{d+1}_+(0,\ m)} \left|\mathscr{F}^{\alpha,d,n}_W(g_t)(\xi)\right|^2 d\nu_{\alpha,d}(t), \tag{49}$$

where $B^{d+1}_+(0,\ m) = \left\{x \in \mathbb{R}^{d+1}_+,\ \|x\| \le m\right\}$.
Then for all $m \in \mathbb{N}$, $h_m \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$, $k_m \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have $\mathscr{F}^{\alpha,d,n}_W(h_m) = k_m$.

Proof. Using Cauchy-Schwartz inequality, we obtain

$$|h_m(\xi)|^2 \le C_1 \int_{B^{d+1}_+(0,\ m)} \left|\int_{\mathbb{R}^{d+1}_+} \Lambda_{\alpha,d,n}(-\xi, x) \left|\mathscr{F}^{\alpha,d,n}_W(g_t)(x)\right|^2 d\nu_{\alpha+2n,d}(x)\right|^2 d\nu_{\alpha,d}(t)$$

where

$$C_1 = C_1(\alpha, d, n, m) = C^4_{\alpha+2n,d} \int_{B^{d+1}_+(0,\, m)} d\nu_{\alpha,d}(t).$$

According to the Fubini theorem, the relations (26), (29) and (38), we get

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}_+} |h_m(\xi)|^2 \, d\nu_{\alpha,d}(\xi)\\ &\leq C_1 \int_{B^{d+1}_+(0,\, m)} \int_{\mathbb{R}^{d+1}_+} \left| \left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1} \left(\left| \mathscr{F}^{\alpha,d,n}_W(g_t)\right|^2 (\xi)\right)\right|^2 d\nu_{\alpha,d}(\xi)\, d\nu_{\alpha,d}(t).\end{aligned}$$

Now, from the relation (29) and (47), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}_+} \left| \left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1} \left(\left| \mathscr{F}^{\alpha,d,n}_W(g_t)\right|^2 (\xi)\right)\right|^2 d\nu_{\alpha,d}(\xi)\\ &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left| \mathscr{F}^{\alpha,d,n}_W(g_t)\right|^4 (\xi)\, d\nu_{\alpha+2n,d}(\xi)\\ &\leq C^2_{\alpha+2n,d} t^{8n}_{d+1} \left\|g\right\|^2_{\alpha,\infty,n} \left\|g\right\|^2_{\alpha,2,n}.\end{aligned}$$

Then

$$\int_{\mathbb{R}^{d+1}_+} |h_m(\xi)|^2 \, d\nu_{\alpha,d}(\xi) \leq C_1 C^2_{\alpha+2n,d} \left\|g\right\|^2_{\alpha,\infty,n} \left\|g\right\|^2_{\alpha,2,n} \int_{B^{d+1}_+(0,\, m)} t^{8n}_{d+1} d\nu_{\alpha,d}(t) < +\infty.$$

On the other hand, it is clear that $k_m \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\begin{aligned}\left(\mathscr{F}^{\alpha,d,n}_W\right)^{-1}(k_m)(\xi) &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} k_m(x)\, \Lambda_{\alpha,d,n}(-\xi, x) d\nu_{\alpha+2n,d}(x)\\ &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \Lambda_{\alpha,d,n}(-\xi, x) \int_{B^{d+1}_+(0,\, m)} \left| \mathscr{F}^{\alpha,d,n}_W(g_t)(x)\right|^2 d\nu_{\alpha,d}(t)\, d\nu_{\alpha+2n,d}(x)\\ &= C^2_{\alpha+2n,d} \int_{B^{d+1}_+(0,\, m)} \int_{\mathbb{R}^{d+1}_+} \Lambda_{\alpha,d,n}(-\xi, x) \left| \mathscr{F}^{\alpha,d,n}_W(g_t)(x)\right|^2 d\nu_{\alpha,d}(t)\, d\nu_{\alpha+2n,d}(x)\\ &= h_m(\xi).\end{aligned}$$

□

Lemma 3.8. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$. The function $h_m,\ m \in \mathbb{N}$ given by the relation (48) can also be written in the form*

$$\forall \xi \in \mathbb{R}^{d+1}_+,\ h_m(\xi) = \int_{B^{d+1}_+(0,\, m)} \widetilde{g_t} *_{\alpha,n} g_t(\xi)\, d\nu_{\alpha,d}(t), \tag{50}$$

where $\widetilde{g_t}(\xi) = g_t(-\xi)$.

Proof. From the relations (26) and (38), we obtain the result. □

Lemma 3.9. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. We consider the sequence $(f_m)_{m\in\mathbb{N}}$, given by :*

$$\forall \xi \in \mathbb{R}^{d+1}_+,\ f_m(\xi) = \int_{B^{d+1}_+(0,\, m)} \int_{\mathbb{R}^{d+1}_+} \mathscr{G}_g f(x, y)\, g_{x,-y}(-\xi)\, dw_{\alpha,d}(x, y). \tag{51}$$

Then for all $m \in \mathbb{N}$, we have

$$f_m = h_m *_{\alpha,n} f. \tag{52}$$

Proof. From the relations (36) and (46), we have

$$\begin{aligned} f_m(\xi) &= \int_{B_+^{d+1}(0,\, m)} \int_{\mathbb{R}_+^{d+1}} \mathcal{G}_g(f)(x,y)\, T_\xi^{\alpha,d,n} \widetilde{g_x}(-y)\, d\nu_{\alpha,d}(y)\, d\nu_{\alpha,d}(x) \\ &= \int_{B_+^{d+1}(0,\, m)} \mathcal{G}_g(f)(x,.) *_{\alpha,n} \widetilde{g_x}(\xi)\, d\nu_{\alpha,d}(x) \\ &= \int_{B_+^{d+1}(0,\, m)} f *_{\alpha,n} g_x *_{\alpha,n} \widetilde{g_x}(\xi)\, d\nu_{\alpha,d}(x) \\ &= \int_{B_+^{d+1}(0,\, m)} \int_{\mathbb{R}_+^{d+1}} T_\xi^{\alpha,d,n} f(-y)\, \widetilde{g_x} *_{\alpha,n} g_x(y)\, d\nu_{\alpha,d}(y)\, d\nu_{\alpha,d}(x). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_{B_+^{d+1}(0,\, m)} \int_{\mathbb{R}_+^{d+1}} \left| T_\xi^{\alpha,d,n} f(y)\, \widetilde{g_x} *_{\alpha,n} g_x(y) \right| d\nu_{\alpha,d}(x)\, d\nu_{\alpha,d}(y) \\ &\le \int_{B_+^{d+1}(0,\, m)} \left| |f| *_{\alpha,n} \widetilde{g_x} *_{\alpha,n} g_x(\xi) \right| d\nu_{\alpha,d}(x). \end{aligned}$$

Using the fact that $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1}) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}_+^{d+1})$, we deduce that $\widetilde{g_x} *_{\alpha,n} g_x \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$.
From Young's inequality and Parseval theorem and the relation (47) , we obtain

$$\begin{aligned} \left\| |f| *_{\alpha,n} \widetilde{g_x} *_{\alpha,n} g_x \right\|_{\alpha,n,\infty} &\le \|f\|_{\alpha,n,2} \left\| \widetilde{g_x} *_{\alpha,n} g_x \right\|_{\alpha,n,2} \\ &\le C_{\alpha+2n,d}\, x_{d+1}^{4n} \|f\|_{\alpha,n,2} \|g\|_{\alpha,n,2} \|g\|_{\alpha,n,\infty}. \end{aligned}$$

Thus

$$\int_{B_+^{d+1}(0,\, m)} \left| |f| *_{\alpha,n} \widetilde{g_x} *_{\alpha,n} g_x(\xi) \right| d\nu_{\alpha,d}(x) \le C_2 \|f\|_{\alpha,n,2} \|g\|_{\alpha,n,2} \|g\|_{\alpha,n,\infty},$$

where

$$C_2 = C_2(\alpha, d, n, m) = C_{\alpha+2n,d} \int_{B_+^{d+1}(0,\, m)} d\nu_{\alpha+2n,d}(x).$$

Now applying the Fubini's theorem, we get

$$\begin{aligned} f_m(\xi) &= \int_{\mathbb{R}_+^{d+1}} T_\xi^{\alpha,d,n} f(-y) \left(\int_{B_+^{d+1}(0,\, m)} \widetilde{g_x} *_{\alpha,n} g_x(y)\, d\nu_{\alpha,d}(x) \right) d\nu_{\alpha,d}(y) \\ &= \int_{\mathbb{R}_+^{d+1}} T_\xi^{\alpha,d,n} f(-y)\, h_m(y)\, d\nu_{\alpha,d}(y) \\ &= h_m *_{\alpha,n} f(\xi). \end{aligned}$$

□

Theorem 3.10. (*Inversion formula*)
Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1}) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}_+^{d+1})$ such that $\|g\|_{\alpha,n,2} = 1$. For any function $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, we consider the sequence $(f_m)_{m\in\mathbb{N}}$, given by :

$$f_m(\xi) = \int_{B_+^{d+1}(0,\, m)} \int_{\mathbb{R}_+^{d+1}} \mathcal{G}_g f(x,y)\, g_{x,-y}(-\xi)\, dw_{\alpha,d}(x,y).$$

Then for all $m \in \mathbb{N}$, $f_m \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$ and we have

$$\lim_{m\to+\infty} \|f_m - f\|_{\alpha,n,2} = 0. \tag{53}$$

Proof. Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$ such that $\|g\|_{\alpha,n,2} = 1$.
From Lemma 3.7, we deduce that $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\mathscr{F}^{\alpha,d,n}_W(f_m) = k_m \mathscr{F}^{\alpha,d,n}_W(f). \tag{54}$$

From the relation (54) and the Plancherel formula, we obtain

$$\begin{aligned}\|f_m - f\|^2_{\alpha,n,2} &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(x) - k_m(x)\mathscr{F}^{\alpha,d,n}_W(f)(x)\right|^2 d\nu_{\alpha+2n,d}(x)\\ &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(x)(1 - k_m(x))\right|^2 d\nu_{\alpha+2n,d}(x).\end{aligned}$$

Then using the fact that $k_m \to 1$ pointwise as $m \to +\infty$ and the dominated convergence theorem, we deduce that

$$\lim_{m\to+\infty} \|f_m - f\|_{\alpha,n,2} = 0.$$

Which finishes the proof. □

Now, we give some properties of the generalized continuous Weinstein Gabor transform $\mathscr{G}_g$.

Proposition 3.11. *Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then $\mathscr{G}_g f \in \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+)$ and we have*

$$\|\mathscr{G}_g f\|_{w_{\alpha,d,n},\infty} \le C_{\alpha+2n,d}\|g\|_{\alpha,n,2}\|f\|_{\alpha,n,2}. \tag{55}$$

Proof. The result is a direct consequence of the relations (46), (37) and (41) □

Proposition 3.12. *(Plancherel formula)*
Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then $\mathscr{G}_g f \in L^2_{\alpha,n}(\mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+)$ and we have

$$\|\mathscr{G}_g f\|_{w_{\alpha,d,n},2} = \|g\|_{\alpha,n,2}\|f\|_{\alpha,n,2}. \tag{56}$$

Proof. From the relations (46), (29), (42) and Fubini's theorem, we obtain

$$\begin{aligned}\|\mathscr{G}_g f\|^2_{w_{\alpha,d,n},2} &= \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left|f *_{\alpha,n} \widetilde{g_x}(y)\right|^2(y)\, d\nu_{\alpha,d}(x)d\nu_{\alpha,d}(y)\\ &= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\right|^2 \left|\mathscr{F}^{\alpha,d,n}_W(g_x)(\lambda)\right|^2 d\nu_{\alpha,d}(x)d\nu_{\alpha+2n,d}(\lambda)\\ &= C^2_{\alpha+2n,d}\left\|g\right\|^2_{\alpha,2,n}\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\right|^2 d\nu_{\alpha+2n,d}(\lambda)\\ &= \left\|g\right\|^2_{\alpha,2,n}\int_{\mathbb{R}^{d+1}_+}\left|f(x)\right|^2 d\nu_{\alpha,d}(x)\\ &= \|g\|^2_{\alpha,n,2}\|f\|^2_{\alpha,n,2}.\end{aligned}$$

Which achieves the proof. □

In the following resultat, we can see that the generalized continuous Weinstein Gabor transform preserves the orthogonality relation.

Corollary 3.13. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then for all $f, h \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we have*

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \mathscr{G}_g f(x,y)\overline{\mathscr{G}_g h(x,y)} d\nu_{\alpha,d}(x)d\nu_{\alpha,d}(y) = \|g\|^2_{\alpha,n,2}\int_{\mathbb{R}^{d+1}_+} f(x)\overline{h(x)} d\nu_{\alpha,d}(x). \tag{57}$$

Proof. Let $g, f, h \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Using the relations (46), (28) and (43), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \mathscr{G}_g f(x,y)\overline{\mathscr{G}_g h(x,y)} d\nu_{\alpha,d}(x) d\nu_{\alpha,d}(y)\\
&= \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} f *_{\alpha,n} \widetilde{g_x}(y) \overline{h *_{\alpha,n} \widetilde{g_x}}(y)\, d\nu_{\alpha,d}(x) d\nu_{\alpha,d}(y)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W (f *_{\alpha,n} \widetilde{g_x})(\lambda) \overline{\mathscr{F}^{\alpha,d,n}_W (h *_{\alpha,n} \widetilde{g_x})}(\lambda)\, d\nu_{\alpha+2n,d}(\lambda)\, d\nu_{\alpha,d}(x)\\
&= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(h)(\lambda)} \left|\mathscr{F}^{\alpha,d,n}_W(g_x)(\lambda)\right|^2 d\nu_{\alpha+2n,d}(\lambda)\, d\nu_{\alpha,d}(x)\\
&= C^2_{\alpha+2n,d}\|g\|^2_{\alpha,n,2}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(h)(\lambda)} d\nu_{\alpha+2n,d}(\lambda)\\
&= \|g\|^2_{\alpha,n,2}\int_{\mathbb{R}^{d+1}_+} f(x)\overline{h(x)} d\nu_{\alpha,d}(x).
\end{aligned}$$

□

Remark 3.14. *Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. We have*

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{G}_g f(x,y)\right|^2 dw_{\alpha,d}(x,y) = \|f\|^2_{\alpha,n,2}\|g\|^2_{\alpha,n,2}. \tag{58}$$

In the following, we show the weak uncertainty principle for the generalized continuous Weinstein Gabor transform.

Proposition 3.15. *Let $\varepsilon > 0$ and $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ such that $\|g\|_{\alpha,n,2} = \|f\|_{\alpha,n,2} = 1$. Then for all $U \subset \mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+$ satisfying :*

$$\int_U dw_{\alpha,d,n}(x,y) < \infty \text{ and } \int_U \left|\mathscr{G}_g f(x,y)\right|^2 dw_{\alpha,d}(x,y) \geq 1-\varepsilon,$$

we have

$$C^2_{\alpha+2n,d}\int_U dw_{\alpha,d,n}(x,y) \geq 1-\varepsilon.$$

Proof. Using the relation (55), we obtain $\|\mathscr{G}_g f\|_{w_{\alpha,d,n},\infty} \leq C_{\alpha+2n,d}$. Then

$$\begin{aligned}
1-\varepsilon &\leq \int_U \left|\mathscr{G}_g f(x,y)\right|^2 dw_{\alpha,d}(x,y) \leq \|\mathscr{G}_g f\|^2_{w_{\alpha,d,n},\infty}\int_U dw_{\alpha,d,n}(x,y)\\
&\leq C^2_{\alpha+2n,d}\int_U dw_{\alpha,d,n}(x,y).
\end{aligned}$$

Which finishes the proof. □

Proposition 3.16. *Let $2 \leq p < \infty$ and $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ such that $\|g\|_{\alpha,n,2} = 1$. Then*

$$\int_{\mathbb{R}^{d+1}_+\times\mathbb{R}^{d+1}_+} \left|\mathscr{M}^{-1}_{n,x}\mathscr{M}^{-1}_{n,y}\mathscr{G}_g f(x,y)\right|^p dw_{\alpha,d,n}(x,y) \leq C^{p-2}_{\alpha+2n,d}\|f\|^p_{\alpha,n,2}. \tag{59}$$

Proof. The result follows immediatly from the relations (55) and (56) . □

4. Uncertainty Principale of Heisenberg Type

In this section, we will to prove the Heisenberg inequality for the generalized continuous Weinstein Gabor transform.
In the following proposition, we give an Uncertainty Principale of Heisenberg type for the generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$.

Lemma 4.1. *Let $f \in \mathbb{L}^2_\alpha(\mathbb{R}^{d+1}_+)$. Then we have*

$$\left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d}(f)(y)\right|^2 d\nu_{\alpha,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|f(y)\right|^2 d\nu_{\alpha,d}(y)\right)^{\frac{1}{2}} \geq C^o_{\alpha,d}\|f\|^2_{\alpha,2}, \tag{60}$$

where $C^o_{\alpha,d} = C^{-1}_{\alpha,d}\left(\alpha + \frac{d}{2} + 1\right)$ and $C_{\alpha,d}$ is the constant given by the relation (27).

Proof. We obtain the result by combing the Heisenberg inequality for the classical Fourier and Fourier-Bessel Transform. (See [11]). □

Proposition 4.2. *Let $f \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then we have*

$$\left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}(f)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|f(y)\right|^2 d\nu_{\alpha,d}(y)\right)^{\frac{1}{2}} \tag{61}$$
$$\geq C_{\alpha+2n,d}\left(\alpha + 2n + \frac{d}{2} + 1\right)\|f\|^2_{\alpha,n,2}.$$

Proof. Using the relations (20) and (60), we obtain :

$$\left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}(f)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|f(y)\right|^2 d\nu_{\alpha,d}(y)\right)^{\frac{1}{2}}$$
$$= \left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha+2n,d}\left(\mathscr{M}_n^{-1}f\right)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{M}_n^{-1}f(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}}$$
$$\geq C^{-1}_{\alpha+2n,d}\left(\alpha + 2n + \frac{d}{2} + 1\right)\|\mathscr{M}_n^{-1}f\|^2_{\alpha+2n,2} = C^{-1}_{\alpha+2n,d}\left(\alpha + 2n + 1 + \frac{d}{2}\right)\|f\|^2_{\alpha,n,2}.$$

□

Lemma 4.3. *Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. We have*

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{G}_g f\right)(x,.)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\, d\nu_{\alpha,d}(x) \tag{62}$$
$$= \|g\|^2_{\alpha,n,2}\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}(f)(y)\right|^2 d\nu_{\alpha+2n,d}(y).$$

Proof. Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. From the relations (46), (42) and Fubini's theorem, we obtain :

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}\left(\mathscr{G}_g f\right)(x,.)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\, d\nu_{\alpha,d}(x)$$
$$= \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}(f)(y)\right|^2 \left|\mathscr{F}_W^{\alpha,d,n}(\widetilde{g_x})(y)\right|^2 d\nu_{\alpha+2n,d}(y)\, d\nu_{\alpha,d}(x)$$
$$= \left\|g\right\|^2_{\alpha,2,n}\int_{\mathbb{R}^{d+1}_+} \left\|y\right\|^2 \left|\mathscr{F}_W^{\alpha,d,n}(f)(y)\right|^2 d\nu_{\alpha+2n,d}(y).$$

□

Theorem 4.4. *(Uncertainty Principale of Heisenberg Type for $\mathscr{G}_g$)*

Let $f, g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$, the following inequality holds

$$\left(\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{F}^{\alpha,d,n}_W(f)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{G}_g f(x,y)\right|^2 dw_{\alpha,d}(x,y)\right)^{\frac{1}{2}} \tag{63}$$
$$\geq C^{-1}_{\alpha+2n,d}\left(\alpha+2n+\frac{d}{2}+1\right)\|g\|_{\alpha,n,2}\|f\|^2_{\alpha,n,2}.$$

Proof. Let us assume the non-trivial case that both integrals on the left hand side of the relation (63) are finite. Using the relation (61) for all $y \in \mathbb{R}^{d+1}_+$, we have

$$\left(\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{F}^{\alpha,d,n}_W\left(\mathscr{G}_g f\right)(x,.)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{G}_g f(x,y)\right|^2 d\nu_{\alpha,d}(y)\right)^{\frac{1}{2}}$$
$$\geq C^{-1}_{\alpha+2n,d}\left(\alpha+2n+\frac{d}{2}+1\right)\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{G}_g f(x,y)\right|^2 d\nu_{\alpha,d}(y).$$

Integrating over x and using Cauchy- Schwartz inequality, we obtain

$$\left(\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{F}^{\alpha,d,n}_W\left(\mathscr{G}_g f\right)(x,.)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\, d\nu_{\alpha,d}(x)\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{G}_g f(x,y)\right|^2 d\nu_{\alpha,d}(x)\, d\nu_{\alpha,d}(y)\right)^{\frac{1}{2}}$$
$$\geq C^{-1}_{\alpha+2n,d}\left(\alpha+2n+\frac{d}{2}+1\right)\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \left|\mathscr{G}_g f(x,y)\right|^2 d\nu_{\alpha,d}(x)\, d\nu_{\alpha,d}(y).$$

Then, using the relations (62) and (58), we obtain

$$\|g\|_{\alpha,n,2}\left(\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{F}^{\alpha,d,n}_W(f)(y)\right|^2 d\nu_{\alpha+2n,d}(y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \|y\|^2 \left|\mathscr{G}_g f(x,y)\right|^2 dw_{\alpha,d}(x,y)\right)^{\frac{1}{2}}$$
$$\geq C^{-1}_{\alpha+2n,d}\left(\alpha+2n+\frac{d}{2}+1\right)\|f\|^2_{\alpha,n,2}\|g\|^2_{\alpha,n,2}.$$

Thus the proof is completed. □

5. Reproducing kernel

Proposition 5.1. *Let $g \in \mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+) \cap \mathbb{L}^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$ such that $g \neq 0$. Then $\mathscr{G}_g\left(\mathbb{L}^2_{\alpha,n}(\mathbb{R}^{d+1}_+)\right)$ is a reproducing kernel Hilbert space in $X^2_{\alpha,n}$ with kernel function :*

$$\Theta^{\alpha,d,n}_g(x,y;z,t) := \frac{\mathscr{M}^{-1}_{n,x}\mathscr{M}^{-1}_{n,y}\mathscr{M}^{-1}_{n,z}\mathscr{M}^{-1}_{n,t}}{C_{\alpha+2n,d}\|g\|^2_{\alpha,n,2}}\int_{\mathbb{R}^{d+1}_+} T^{\alpha,d,n}_\xi(g_x)(-y)\, T^{\alpha,d,n}_\xi\overline{g_z}(-t)\, d\nu_{\alpha,d}(\xi). \tag{64}$$

The kernel is pointwise bounded and we have

$$\forall x,y,z,t \in \mathbb{R}^{d+1}_+,\ \left|\Theta^{\alpha,d,n}_g(x,y;\, z,t)\right| \leq 1. \tag{65}$$

Proof. Using the relation (45) and (57), we obtain

$$\mathscr{G}_g f\left(x,y\right)=\frac{1}{\|g\|^2_{\alpha,n,2}}\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}}\mathscr{G}_g f\left(z,t\right)\overline{\mathscr{G}_g g_{x,y}\left(z,t\right)}dw_{\alpha,d,n}\left(z,t\right).$$

Then for all $x,y,z,t\in\mathbb{R}_+^{d+1}$, we obtain

$$\Theta_g^{\alpha,d,n}(x,y;z,t)=\frac{\mathscr{M}_{n,x}^{-1}\mathscr{M}_{n,y}^{-1}\mathscr{M}_{n,z}^{-1}\mathscr{M}_{n,t}^{-1}}{C_{\alpha+2n,d}\|g\|^2_{\alpha,n,2}}\mathscr{G}_g g_{x,y}\left(z,t\right). \tag{66}$$

Now, according the relations (33), (41), (46) and (55), we obtain the result. □

We need the following notations :
Notations. We denote by :
$\cdot\, P_g\ : X^2_{\alpha,n}\to X^2_{\alpha,n}$ the orthogonal projection from $X^2_{\alpha,n}$ onto $\mathscr{G}_g\left(\mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})\right)$.
$\cdot\, P_U\ : X^2_{\alpha,n}\to X^2_{\alpha,n}$ the orthogonal projection from $X^2_{\alpha,n}$ onto the subspace of function supported in the subset $U\subset\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}$ with $w_{\alpha,d,n}\left(U\right)<\infty$.
We put

$$\left\|P_UP_g\right\|:=\sup\left\{\left\|P_UP_gf\right\|_{w_{\alpha,d,n},2},\ f\in X^2_{\alpha,n}\text{ and }\left\|f\right\|_{w_{\alpha,d,n},2}=1\right\}. \tag{67}$$

The main result in this section is the following :

Theorem 5.2. *(Concentration of $\mathscr{G}_g f$ in small sets)*
Let $g\in\mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})\cap\mathbb{L}^\infty_{\alpha,n}(\mathbb{R}_+^{d+1})$ such that $g\neq 0$ and $U\subset\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}$ with $w_{\alpha,d,n}\left(U\right)<1$. Then for all $f\in\mathbb{L}^2_{\alpha,n}(\mathbb{R}_+^{d+1})$, we have

$$\left\|\mathscr{G}_g f-\chi_U\mathscr{G}_g f\right\|_{w_{\alpha,d,n},2}\geq\left(1-\sqrt{w_{\alpha,d,n}\left(U\right)}\right)\|f\|_{\alpha,n,2}\|g\|_{\alpha,n,2}, \tag{68}$$

where χ_U is the characteristic function of U.

Proof. From the definition of P_g and P_U, we have

$$\left\|\mathscr{G}_g f-\chi_U\mathscr{G}_g f\right\|_{w_{\alpha,d,n},2}=\left\|\left(1-P_gP_U\right)\mathscr{G}_g f\right\|_{w_{\alpha,d,n},2}.$$

Then from the relation (56), we obtain

$$\begin{aligned}\left\|\mathscr{G}_g f-\chi_U\mathscr{G}_g f\right\|_{w_{\alpha,d,n},2}&\geq\left(1-\left\|P_UP_g\right\|\right)\left\|\mathscr{G}_g f\right\|_{w_{\alpha,d,n},2}\\&\geq\left(1-\left\|P_UP_g\right\|\right)\|f\|_{\alpha,n,2}\|g\|_{\alpha,n,2}.\end{aligned}$$

On the other hand P_g is the projection onto a reproducing kernel Hilbert space, then from Saitoh see [12], P_g can be represented by :

$$P_gF(x,y)=\int_{\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}}F\left(z,t\right)\Theta_g^{\alpha,d,n}(x,y;z,t)dw_{\alpha,d,n}\left(z,t\right),$$

with $\Theta_g^{\alpha,d,n}$ is given by the relation (64)
Hence for all $F\in X^2_{\alpha,n}$, we have

$$P_UP_gF(x,y)=\int_{\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}}\chi_U(x,y)F\left(z,t\right)\Theta_g^{\alpha,d,n}(x,y;z,t)dw_{\alpha,d,n}\left(z,t\right),$$

and its Hilbert-Schmidt norm

$$\left\|P_U P_g\right\|_{HS} := \left(\int_{\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}} \left|\chi_U(x,y)\right|^2 \left|\Theta_g^{\alpha,d,n}(x,y;\ z,t)\right|^2 dw_{\alpha,d,n}(x,y)\, dw_{\alpha,d,n}(z,t)\right)^{\frac{1}{2}}.$$

Using the Cauchy-Schwartz inequality, we see that

$$\left\|P_U P_g\right\|_{HS} \geq \left\|P_U P_g\right\|. \tag{69}$$

On the other hand, from the relation (66) and Fubini's theorem, it easy to see that

$$\left\|P_U P_g\right\|_{HS} \leq \sqrt{w_{\alpha,d,n}(U)}. \tag{70}$$

Thus from the relations (69) and (70), we obtain the result. □

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6. Declarations

Ethical Approval
Not applicable.
Competing interest
The author declares that there is no competing interest.
Autor's Contributions
Conceptualization, methodology, validation, formal analysis, investigation, resources, data curation, writing-original, draft preparation, writing-review and editing, visualization, supervision and project administration, H.Ben Mohamed have read and agreed to the published version of manuscript.
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