Filomat 38:31 (2024), 11097–11109
https://doi.org/10.2298/FIL2431097M

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Digital numerical semigroups

M.A. Moreno-Frías[a,*], J.C. Rosales[b]

[a] *Department of Mathematics, Faculty of Sciences, University of Cádiz, E-11510, Puerto Real, Cádiz, Spain*
[b] *Department of Algebra, Faculty of Sciences, University of Granada, E-18071, Granada, Spain*

Abstract. The number of digits in base ten system of a positive integer n, is denoted by $\mathrm{l}(n)$. A digital semigroup is a subsemigroup D of $(\mathbb{N}\backslash\{0\},\cdot)$ such that if $d \in \mathrm{D}$, then $\{x \in \mathbb{N}\backslash\{0\} \mid \mathrm{l}(x) = \mathrm{l}(d)\} \subseteq \mathrm{D}$. Let $A \subseteq \mathbb{N}\backslash\{0\}$. Denote by $\mathrm{L}(A) = \{\mathrm{l}(a) \mid a \in A\}$. We will say that a numerical semigroup S is a digital numerical semigroup if there is a digital semigroup D such that $S = \mathrm{L}(\mathrm{D}) \cup \{0\}$. In this work we show that $\mathscr{D} = \{S \mid S \text{ is a digital numerical semigroup}\}$ is a Frobenius variety, $\mathscr{D}(\mathrm{Frob}{=}F) = \{S \in \mathscr{D} \mid \mathrm{F}(S) = F\}$ is a covariety and $\mathscr{D}(\mathrm{mult} = m) = \{S \in \mathscr{D} \mid \mathrm{m}(S) = m\}$ is a Frobenius pseudo-variety. As a consequence we present some algorithms to compute $\mathscr{D}(\mathrm{Frob}{=}F)$, $\mathscr{D}(\mathrm{mult} = m)$ and $\mathscr{D}(\mathrm{gen} = g) = \{S \in \mathscr{D} \mid \mathrm{g}(S) = g\}$.

If $X \subseteq \mathbb{N}\backslash\{0\}$, we denote by $\mathscr{D}[X]$ the smallest element of $\mathscr{D}$ containing X. If $S = \mathscr{D}[X]$, then we will say that X is a $\mathscr{D}$-system of generators of S. We will prove that if $S \in \mathscr{D}$, then S admits a unique minimal $\mathscr{D}$-system of generators, denoted by $\mathscr{D}\mathrm{msg}(S)$. The cardinality of $\mathscr{D}\mathrm{msg}(S)$ is called the $\mathscr{D}$-rank of S. We solve the Frobenius problem to elements of $\mathscr{D}$ with $\mathscr{D}$-rank equal to 1. Moreover, we present an algorithmic procedure to calculate all the elements of $\mathscr{D}$ with fixed $\mathscr{D}$-rank.

1. Introduction

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N} = \{z \in \mathbb{Z} \mid z \geq 0\}$. A *submonoid* of $(\mathbb{N},+)$ is a subset of $\mathbb{N}$ which is closed under addition and contains the element 0. A *numerical semigroup* is a submonoid S of $(\mathbb{N},+)$ such that $\mathbb{N}\backslash S = \{x \in \mathbb{N} \mid x \notin S\}$ has finitely many elements.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S\backslash\{0\})$, $\mathrm{F}(S) = \max\{z \in \mathbb{Z} \mid z \notin S\}$ and $\mathrm{g}(S) = \sharp(\mathbb{N}\backslash S)$ (where $\sharp X$ denotes the cardinality of a set X) are three important invariants of S, called the *multiplicity*, the *Frobenius number* and the *genus* of S, respectively.

Given A a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A. That is, $\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\}, \{a_1,\ldots,a_n\} \subseteq A \text{ and } \{\lambda_1,\ldots,\lambda_n\} \subseteq \mathbb{N}\}$. In [14, Lema 2.1]) it is shown that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If M is a submonoid of $(\mathbb{N},+)$ and $M = \langle A\rangle$, then we say that A is a *system of generators* of M. Moreover, if $M \neq \langle B\rangle$ for all $B \subsetneq A$, then we will say that A is a *minimal system of generators* of M. In [14, Corollary 2.8]

2020 *Mathematics Subject Classification*. Primary 20M14; Secondary 11D07.
Keywords. Digital numerical semigroup, Frobenius number, genus, multiplicity, algorithm, covariety, Frobenius variety, Frobenius pseudo-variety.
Received: 19 June 2023; Revised: 30 January 2024; Accepted: 28 August 2024
Communicated by Marko Petković
Research supported partially supported by Junta de Andalucía groups FQM-298 and FQM-343, Proyecto de Excelencia de la Junta de Andalucía ProyExcel_00868, Proyecto de investigación del Plan Propio–UCA 2022-2023 (PR2022-011) and Proyecto de investigación del Plan Propio–UCA 2022-2023 (PR2022-004).
* Corresponding author: M.A. Moreno-Frías
Email addresses: mariangeles.moreno@uca.es (M.A. Moreno-Frías), jrosales@ugr.es (J.C. Rosales)

it is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which in addition it is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M. The cardinality of $\mathrm{msg}(M)$ is called the *embedding dimension* of M and will be denoted by $\mathrm{e}(M)$. In [14, Proposition 2.10] it is shown that $\mathrm{e}(S) \le \mathrm{m}(S)$.

Let n be a positive integer. The number of digits in base ten system of n is denoted by $\mathrm{l}(n)$. For instance, $\mathrm{l}(2335) = 4$.

A *digital semigroup* is a subsemigroup D of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in \mathrm{D}$, then $\{x \in \mathbb{N}\backslash\{0\} \mid \mathrm{l}(x) = \mathrm{l}(d)\} \subseteq \mathrm{D}$.

Let $A \subseteq \mathbb{N}\backslash\{0\}$. Denote by $\mathrm{L}(A) = \{\mathrm{l}(a) \mid a \in A\}$. In [13, Proposition 2] it is shown that if D is a digital semigroup, then $\mathrm{L(D)} \cup \{0\}$ is a numerical semigroup.

This fact, leads to give the following definition. We will say that a numerical semigroup S is *a digital numerical semigroup* if there is a digital semigroup D such that $S = \mathrm{L(D)} \cup \{0\}$.

In [13, Theorema 4] the following characterization of a digital numerical semigroup is shown. A numerical semigroup S is digital if and only if $a + b - 1 \in S$ for all $\{a, b\} \subseteq S\backslash\{0\}$. We denote by $\mathscr{D} = \{S \mid S \text{ is a digital numerical semigroup}\}$.

The Frobenius problem (see [7] for a nice state of art on this problem) focuses on finding explicit formulas to calculate the Frobenius number and the genus of a numerical semigroup from its minimal system of generators. The problem was solved in [15] for numerical semigroups with embedding dimension two. Nowadays, the problem is still open in the case of numerical semigroups with embedding dimension greater than or equal to three. Furthemore, in this case the problem of computing the Frobenius number of a general numerical semigroup becomes NP-hard (see [8]).

The contents of this paper are organized as follows. In Section 2, we will see that if $X \subseteq \mathbb{N}\backslash\{0\}$, then there exists the smallest element of $\mathscr{D}$ containing X. This semigroup will be denoted by $\mathscr{D}[X]$. If $S = \mathscr{D}[X]$, then we will say that X is a *$\mathscr{D}$-system of generators* of S. We will prove that if $S \in \mathscr{D}$, then S admits a unique minimal $\mathscr{D}$-system of generators, denoted by $\mathscr{D}\mathrm{msg}(S)$. The cardinality of $\mathscr{D}\mathrm{msg}(S)$ is called the *$\mathscr{D}$-rank* of S and it will be denoted by $\mathscr{D}\mathrm{rank}\,(S)$. We finish Section 2, solving the Frobenius problem to elements of $\mathscr{D}$ with $\mathscr{D}$-rank equal to 1.

In Section 3, we prove that $\mathscr{D}(\mathrm{Frob}{=}F) = \{S \in \mathscr{D} \mid \mathrm{F}(S) = F\}$ is a covariety. This fact enables us, by using the results of [6], to present an algorithm which computes the set $\mathscr{D}(\mathrm{Frob}{=}F)$.

In Section 4, we prove that $\mathscr{D}$ is a Frobenius variety. This fact allow us, by applying the results of [11] and [13] to present an algorithm for obtaining the set $\mathscr{D}(\mathrm{gen}{=}g) = \{S \in \mathscr{D} \mid \mathrm{g}(S) = g\}$.

In Section 5, we show that $\mathscr{D}(\mathrm{mult} = m) = \{S \in \mathscr{D} \mid \mathrm{m}(S) = m\}$ is a Frobenius pseudo-variety. This fact allows, by using the results of [10], to present an algorithm to calculates the set $\mathscr{D}(\mathrm{mult} = m)$.

Finally, in Section 6, we describe an algorithm procedure to build the elements of $\mathscr{D}$ with a fixed $\mathscr{D}$-rank.

Throughout this paper, some examples are given to illustrate the results. The computation of these examples are performed by using the GAP (see [5]) package `numericalsgps` ([3]). As well as we have implemented some gap functions. For instance we have implemented the function `IsDigital` which allows us to know if a numerical semigroup is a digital numerical semigroup. Also, by using the function `DF`, it is possible to calculate all the digital numerical semigroups with given Frobenius number.

2. $\mathscr{D}$-system of generators

Let $\mathscr{D} = \{S \mid S \text{ is a digital numerical semigroup}\}$. From [13, Lemma 11], we can deduce the below result.

Lemma 2.1. *Whit the above notation, if $\{S, T\} \subseteq \mathscr{D}$, then $S \cap T \in \mathscr{D}$.*

It is known that if S is a numerical semigroup, then $\mathbb{N}\backslash S$ is finite. As an immediate consequence of this fact, we have the following result.

Lemma 2.2. *If S is a numerical semigroup, then $\{T \mid T$ is a numerical semigroup and $S \subseteq T\}$ is a finite set.*

Lemma 2.3. *Let X be a nonempty subset of $\mathbb{N}\backslash\{0\}$ and $\mathscr{D}(X) = \{S \in \mathscr{D} \mid X \subseteq S\}$. Then $\mathscr{D}(X)$ is a nonempty and finite set.*

Proof. It is clear that $\{0, \min(X), \rightarrow\} \in \mathscr{D}$, where the symbol $\rightarrow$ means that every integer greater than $\min(X)$ belongs to the set; and $X \subseteq \{0, \min(X), \rightarrow\}$. Thus $\mathscr{D}(X) \neq \emptyset$.

If $x \in X, S \in \mathscr{D}$ and $X \subseteq S$, then $\langle x, 2x-1\rangle \subseteq S$. Therefore, $\mathscr{D}(X) \subseteq \{S \mid S$ is a numerical semigroup and $\langle x, 2x-1\rangle \subseteq S\}\}$. By applying Lemma 2.2, we deduce that $\mathscr{D}(X)$ is finite. □

If X is a nonempty set of $\mathbb{N}\backslash\{0\}$, then we denote by $\mathscr{D}[X]$ the intersection of all the elements of $\mathscr{D}(X)$. By applying Lemma 2.1 and 2.3, we obtain the following result.

Proposition 2.4. *If X is a nonempty subset of $\mathbb{N}\backslash\{0\}$, then $\mathscr{D}[X]$ is the smallest element of $\mathscr{D}(X)$.*

If $S = \mathscr{D}[X]$, then we say that X is a *$\mathscr{D}$-system of generators* of S. Besides, if $S \neq \mathscr{D}[Y]$ for all $Y \subsetneq X$, then X is a *minimal $\mathscr{D}$-system of generators* of S. From [13, Proposition 22], we obtain the following proposition.

Proposition 2.5. *Every digital numerical semigroup admits a unique minimal $\mathscr{D}$-system of generators.*

Let $S \in \mathscr{D}$. Then the minimal $\mathscr{D}$-system of generators of S will be denoted by $\mathscr{D}\mathrm{msg}(S)$. The cardinality of $\mathscr{D}\mathrm{msg}(S)$ is called the *$\mathscr{D}$-rank* of S and we denote it by $\mathscr{D}\mathrm{rank}\,(S)$.

By combining Proposition 24 with Corollary 25 from [13] we obtain the following.

Proposition 2.6. *Let $S \in \mathscr{D}$ such that $S \neq \mathbb{N}$ and $x \in S$. The following conditions are equivalent.*

1) $x \in \mathscr{D}\mathrm{msg}(S)$.
2) $S\backslash\{x\} \in \mathscr{D}$.
3) $x \in \mathrm{msg}(S)$ *and* $x + 1 \in (\mathbb{N}\backslash S) \cup \mathrm{msg}(S)$.

Let $A_1, A_2, \cdots, A_n$ be nonempty subsets of $\mathbb{Z}$. We denote by $A_1 + A_2 + \cdots + A_n = \{a_1 + a_2 + \cdots + a_n \mid a_i \in A_i, 1 \leq i \leq n\}$.

A characterization of the elements of $\mathscr{D}$ in terms of their minimal system of generators can be deduced from [13, Proposition 28].

Proposition 2.7. *Let S be a numerical semigroup. Then $S \in \mathscr{D}$ if and only if* $\mathrm{msg}(S) + \mathrm{msg}(S) + \{-1\} \subseteq S$.

In the following example we will illustrate as the Propositions 2.7 and 2.6 can be used to compute the minimal $\mathscr{D}$-system of generators of an element of $\mathscr{D}$.

Example 2.8. *Let $S = \langle 4, 7, 10, 13\rangle$. From Proposition 2.7, easily follows that $S \in \mathscr{D}$, $S\backslash\{4\} \in \mathscr{D}$, $S\backslash\{7\} \notin \mathscr{D}$, $S\backslash\{10\} \notin \mathscr{D}$ and $S\backslash\{13\} \notin \mathscr{D}$. Therefore, by using Proposition 2.6, we conclude that* $\mathscr{D}\mathrm{msg}(S) = \{4\}$.

We can use the GAP order, `IsDigital` which has been implemented by us, to obtain the previous results:

```
gap> IsDigital([4,7,10,13]);
true
gap> IsDigital([7,8,10,11,12,13]);
true
gap> IsDigital([4,10,11,13]);
false
gap> IsDigital([4,7,13,17]);
false
gap> IsDigital([4,7,10]);
false
```

By using Proposition 2.6, it is not hard to prove the following characterization of digital numerical semigroups with rank equal to one.

Proposition 2.9. *Let $S \in \mathscr{D}$. Then the following hold:*

1) $S\backslash\{\mathrm{m}(S)\} \in \mathscr{D}$.

2) $1 \leq \mathscr{D}\mathrm{rank}\,(S) \leq \mathrm{e}(S)$.

3) $\mathscr{D}\mathrm{rank}\,(S) = 1$ *if and only if* $\mathscr{D}\mathrm{msg}(S) = \{\mathrm{m}(S)\}$.

The next proposition can be deduced from [13, Theorem 31].

Proposition 2.10. *Let* $X = \{x_1, \cdots, x_p\} \subseteq \mathbb{N}\backslash\{0\}$. *Then*

$$\mathscr{D}[X] = \{\lambda_1 x_1 + \cdots + \lambda_p x_p - r \mid \{\lambda_1, \cdots, \lambda_p, r\} \subseteq \mathbb{N} \text{ and } r < \lambda_1 + \cdots + \lambda_p\}.$$

Let S be a numerical semigroup and $n \in S\backslash\{0\}$. The Apéry set of n in S (named so in honour of [1]) is defined as $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$.

The following result is deduced from [14, Lemma 2.4].

Lemma 2.11. *If S is a numerical semigroup and $n \in S\backslash\{0\}$, then* $\mathrm{Ap}(S, n)$ *is a set with cardinality n. Moreover,* $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, *where $w(i)$ is the least element of S congruent with i modulo n, for all* $i \in \{0, \ldots, n-1\}$.

In [14, Proposition 2.12] appears the following result, which allows us to know the Frobenius number and the genus of a numerical semigroup from the Apéry set.

Lemma 2.12. *Let S be a numerical semigroup and let $n \in S\backslash\{0\}$. Then*

1) $\mathrm{F}(S) = (\max \mathrm{Ap}(S, n)) - n$,

2) $\mathrm{g}(S) = \dfrac{1}{n}\left(\displaystyle\sum_{w \in \mathrm{Ap}(S,n)} w\right) - \dfrac{n-1}{2}$.

Recall that the embedding dimension of a numerical semigroup does not exceed the multiplicity of the numerical semigroup. We say that a numerical semigroup S has *maximal embedding dimension* if $\mathrm{e}(S) = \mathrm{m}(S)$.

The solution of Frobenius problem for digital numerical semigroup with $\mathscr{D}$-rank equal to one is shown in the following proposition.

Proposition 2.13. *Let $m \in \mathbb{N}\backslash\{0, 1\}$, then the following conditions hold:*

1) $\mathscr{D}[\{m\}] = \{am - r \mid \{a, r\} \subseteq \mathbb{N} \text{ and } r < a\} \cup \{0\}$.

2) $\mathrm{Ap}\,(\mathscr{D}[\{m\}], m) = \{0, 2m-1, 3m-2, \cdots, m \cdot m - (m-1)\}$.

3) $\mathrm{F}\,(\mathscr{D}[\{m\}]) = (m-1)^2$.

4) $\mathrm{g}\,(\mathscr{D}[\{m\}]) = \dfrac{m(m-1)}{2}$.

5) $\mathscr{D}[\{m\}]$ *is a numerical semigroup having maximal embedding dimension.*

Proof. *1)* It is an immediate consequence from Proposition 2.10.

2) It is enough to observe that for every $i \in \{1, \cdots, m-1\}$ it is verified that $(i+1)m - i \in S$ and $(i+1)m - i - m = im - i \notin S$.

3) and *4)* are an immediate consequence from *2)* and Lemma 2.12.

5) The reader can check that $\{m, 2m-1, 3m-2, \cdots, m \cdot m - (m-1)\} = \mathrm{msg}(\mathscr{D}[\{m\}])$. Therefore, $\mathrm{e}\,(\mathscr{D}[\{m\}]) = m = \mathrm{m}\,(\mathscr{D}[\{m\}])$ and so $\mathscr{D}[\{m\}]$ is a numerical semigroup with maximal embedding dimension. □

The content of the previous proposition is illustrated in the next example.

Example 2.14. *By Proposition 2.13, we know that* $\mathrm{Ap}\,(\mathscr{D}[\{5\}], 5) = \{0, 9, 13, 17, 21\}$, $\mathrm{F}\,(\mathscr{D}[\{5\}]) = (5-1)^2 = 16$ *and* $\mathrm{g}\,(\mathscr{D}[\{5\}]) = \dfrac{5 \cdot 4}{2} = 10$. *Moreover, we also know that* $\mathscr{D}[\{5\}] = \langle 5, 9, 13, 17, 21 \rangle$ *is a numerical semigroup having maximal embedding dimension.*

3. Digital numerical semigroups with given Frobenius number

Throughout this section, F denotes a positive integer. Our main goal here will be to show an algorithm to compute the set $\mathscr{D}(\mathrm{Frob}{=}F) = \{S \in \mathscr{D} \mid \mathrm{F}(S) = F\}$. In order to show this algorithm we need to recall some notation and results. The following definition appears in [6].

A *covariety* is a nonempty family $\mathscr{C}$ of numerical semigroups that fulfills the following conditions:

1) There exists the minimum of $\mathscr{C}$, with respect to set inclusion.
2) If $\{S, T\} \subseteq \mathscr{C}$, then $S \cap T \in \mathscr{C}$.
3) If $S \in \mathscr{C}$ and $S \neq \min(\mathscr{C})$, then $S \backslash \{\mathrm{m}(S)\} \in \mathscr{C}$.

The next result is straightforward to prove.

Lemma 3.1. *Let S and T be numerical semigroups and $x \in S$. Then the following conditions hold:*

1. *$S \cap T$ is a numerical semigroup and* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.
2. *$S \backslash \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*
3. $\mathrm{m}(S) = \min(\mathrm{msg}(S))$.

By applying Lemmas 2.1 and 3.1; and Proposition 2.9, we obtain the following result.

Proposition 3.2. *With the above notation, $\mathscr{D}(Frob{=}F)$ is a covariety and $\Delta(F) = \{0, F+1, \rightarrow\}$ is its minimum.*

A *graph* G is a pair (V, E) where V is a nonempty set and E is a subset of $\{(u, v) \in V \times V \mid u \neq v\}$. The elements of V and E are called *vertices* and *edges*, respectively.

A *path (of length n)* connecting the vertices x and y of G, is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

A graph G is *a tree* if there exists a vertex r (known as *the root* of G) such that for any other vertex x of G there exists a unique path connecting x and r. If (u, v) is an edge of the tree G, we say that u is a *child* of v.

Define the graph $\mathrm{G}(\mathscr{D}(\mathrm{Frob}{=}F))$ in the following way:

- the set of vertices of $\mathrm{G}(\mathscr{D}(\mathrm{Frob}{=}F))$ is $\mathscr{D}(\mathrm{Frob}{=}F)$,
- $(S, T) \in \mathscr{D}(\mathrm{Frob}{=}F) \times \mathscr{D}(\mathrm{Frob}{=}F)$ is an edge of $\mathrm{G}(\mathscr{D}(\mathrm{Frob}{=}F))$ if and only if $T = S \backslash \{\mathrm{m}(S)\}$.

As a consequence of Propositions 3.2 and [6, Proposition 2.6], we have the following result.

Proposition 3.3. *Under the standing notation,* $\mathrm{G}(\mathscr{D}(Frob{=}F))$ *is a tree with root* $\Delta(F)$.

Note that a tree can be built recurrently starting from the root and connecting, through an edge, the vertices already built with their children. Hence, it is very interesting to characterize the children of an arbitrary vertex of the tree $\mathrm{G}(\mathscr{D}(\mathrm{Frob}{=}F))$. For this reason, next we are going to introduce some necessary concepts and results for the development of the work.

Following the terminology introduced in [12], an integer z is a *pseudo-Frobenius number* of S if $z \notin S$ and $z + s \in S$ for all $s \in S \backslash \{0\}$. We denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is an important invariant of S (see [4] and [2]) called the *type* of S, denoted by $\mathrm{t}(S)$.

Given a numerical semigroup S, denote by $\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \in S\}$. The elements of $\mathrm{SG}(S)$ will be called the *special gaps* of S. The following result is Proposition 4.33 from [14].

Lemma 3.4. *Let S be a numerical semigroup and $x \in \mathbb{N} \backslash S$. Then $x \in \mathrm{SG}(S)$ if and only if $S \cup \{x\}$ is a numerical semigroup.*

As a consequence of Proposition 3.2 and [6, Proposition 2.9], we have a characterization of the children of $S \in \mathscr{D}(\mathrm{Frob}{=}F)$.

Proposition 3.5. *If $S \in \mathscr{D}(Frob{=}F)$, then the set formed by the children of S in the tree* $\mathrm{G}(\mathscr{D}(Frob{=}F))$ *is* $\{S \cup \{x\} \mid x \in \mathrm{SG}(S), x < \mathrm{m}(S)$ *and* $S \cup \{x\} \in \mathscr{D}(Frob{=}F)\}$.

Lemma 3.6. *Let $S \in \mathscr{D}(\text{Frob}=F)$, $x \in \mathrm{SG}(S)$ and $A = \{a \in \mathrm{msg}(S) \mid a < F\}$. Then $S \cup \{x\} \in \mathscr{D}(\text{Frob}=F)$ if and only if $x \neq F$ and $\{x-1\} + (A \cup \{x\}) \subseteq S$.*

Proof. Necessity. If $S \cup \{x\} \in \mathscr{D}(\text{Frob}=F)$, then $F \notin S \cup \{x\}$ and so $x \neq F$. As $x \in (S \cup \{x\})\backslash\{0\}$, then $x + x - 1 \in S$. If $a \in A$, then $a \in (S \cup \{x\})\backslash\{0\}$. Thus $a + x - 1 \in S$.

Sufficiency. We will prove that if $\{a, b\} \subseteq (S \cup \{x\})\backslash\{0\}$, then $a + b - 1 \in S \cup \{x\}$. We distinguish four cases:

1) If $a > F$, then $a + b - 1 > F$ and so $a + b - 1 \in S \cup \{x\}$.
2) If $a = b = x$, then $a + b - 1 = x + x - 1 \in S \subseteq S \cup \{x\}$.
3) If $a = x$ and $b \neq x$, then by 1) we can suppose that $b < F$. Thus there exists $\alpha \in A$ and $s \in S$ such that $b = \alpha + s$. Consequently, $a + b - 1 = x + \alpha - 1 + s \in S \subseteq S \cup \{x\}$.
4) If $a \neq x$ and $b \neq x$, then $a + b - 1 \in S \cup \{x\}$.

□

As a consequence from Proposition 3.5 and Lemma 3.6, we can enounce the next result.

Proposition 3.7. *If $S \in \mathscr{D}(\text{Frob}=F)$, then the set formed by the children of S in the tree $\mathrm{G}(\mathscr{D}(\text{Frob}=F))$ is $\{S \cup \{x\} \mid x \in \mathrm{SG}(S),\ x \neq F, x < \mathrm{m}(S)$ and $\{x-1\} + (\{a \in \mathrm{msg}(S) \mid a < F\} \cup \{x\}) \subseteq S\}$.*

Remark 3.8.
1. *In [6, Remark 3.5] it is shown that if S is a numerical semigroup and we know $\mathrm{Ap}(S, n)$ for some $n \in S\backslash\{0\}$, then we can easily compute $\mathrm{SG}(S)$.*
2. *In [6, Remark 3.8] it is shown that if S is a numerical semigroup and we know $\mathrm{Ap}(S, n)$ for some $n \in S\backslash\{0\}$, it is trivial to compute $\mathrm{Ap}(S \cup \{x\}, n)$ for all $x \in \mathrm{SG}(S)$.*
3. *Obviously, if S is a numerical semigroup with Frobenius number F, then $\{a \in \mathrm{msg}(S) \mid a < F\} = \{w \in \mathrm{Ap}(S, F+1) \mid w < F$ and $w - w' \notin \mathrm{Ap}(S, F+1)$ for all $w' \in \mathrm{Ap}(S, F+1)\backslash\{0, w\}\}$.*

We already have all the necessary tools to present the announced algorithm.

Algorithm 3.9.

Input: A positive integer F.
Output: $\mathscr{D}(\text{Frob}=F)$.

(1) $\mathscr{D}(\text{Frob}=F) = \{\Delta(F)\}$, $B = \{\Delta(F)\}$ and $\mathrm{Ap}(\Delta(F), F+1) = \{0, F+2, \cdots, 2F+1\}$.
(2) For every $S \in B$ compute $\theta(S) = \{x \in \mathrm{SG}(S) \mid x < \mathrm{m}(S), x \neq F$ and $\{x-1\} + (\{a \in \mathrm{msg}(S) \mid a < F\} \cup \{x\}) \subseteq S\}$.
(3) If $\bigcup_{S \in B} \theta(S) = \emptyset$, then return $\mathscr{D}(\text{Frob}=F)$.
(4) $C = \bigcup_{S \in B} \{S \cup \{x\} \mid x \in \theta(S)\}$.
(5) $\mathscr{D}(\text{Frob}=F) = \mathscr{D}(\text{Frob}=F) \cup C$ and $B = C$.
(6) Compute $\mathrm{Ap}(S, F+1)$ for every $S \in B$ and go to Step (2).

In the next example, we show how the previous algorithm works.

Example 3.10. *We are going to compute $\mathscr{D}(\text{Frob}=9)$, by using Algorithm 3.9.*

- $\mathscr{D}(\text{Frob}=9) = \{\Delta(9)\}$, $B = \{\Delta(9)\}$ *and* $\mathrm{Ap}(\Delta(9), 10) = \{0, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.
- $\theta(\Delta(9)) = \{6, 7, 8\}$ *and* $C = \{\Delta(9) \cup \{6\}, \Delta(9) \cup \{7\}, \Delta(9) \cup \{8\}\}$.
- $\mathscr{D}(\text{Frob}=9) = \{\Delta(9), \Delta(9)\cup\{6\}, \Delta(9)\cup\{7\}, \Delta(9)\cup\{8\}\}$, $B = \{\Delta(9)\cup\{6\}, \Delta(9)\cup\{7\}, \Delta(9)\cup\{8\}\}$, $\mathrm{Ap}(\Delta(9)\cup\{6\}, 10) = \{0, 6, 11, 12, 13, 14, 15, 17, 18, 19\}$, $\mathrm{Ap}(\Delta(9)\cup\{7\}, 10) = \{0, 7, 11, 12, 13, 14, 15, 16, 18, 19\}$ *and* $\mathrm{Ap}(\Delta(9)\cup\{8\}, 10) = \{0, 8, 11, 12, 13, 14, 15, 16, 17, 19\}$.
- $\theta(\Delta(9) \cup \{6\}) = \emptyset$, $\theta(\Delta(9) \cup \{7\}) = \{6\}$, $\theta(\Delta(9) \cup \{8\}) = \{6, 7\}$ *and* $C = \{\Delta(9) \cup \{6, 7\}, \Delta(9) \cup \{6, 8\}, \Delta(9) \cup \{7, 8\}\}$.

- $\mathscr{D}(Frob=9) = \{\Delta(9), \Delta(9)\cup\{6\}, \Delta(9)\cup\{7\}, \Delta(9)\cup\{8\}, \Delta(9)\cup\{6,7\}, \Delta(9)\cup\{6,8\}, \Delta(9)\cup\{7,8\}\}$, $B = \{\Delta(9)\cup\{6,7\}, \Delta(9)\cup\{6,8\}, \Delta(9)\cup\{7,8\}\}$, $\mathrm{Ap}(\Delta(9)\cup\{6,7\},10) = \{0,6,7,11,12,13,14,15,18,19\}$, $\mathrm{Ap}(\Delta(9)\cup\{6,8\},10) = \{0,6,8,11,12,13,14,15,17,19\}$ *and* $\mathrm{Ap}(\Delta(9)\cup\{7,8\},10) = \{0,7,8,11,12, 13,14,15,16,19\}$.

- $\theta(\Delta(9)\cup\{6,7\}) = \emptyset$, $\theta(\Delta(9)\cup\{6,8\}) = \emptyset$, $\theta(\Delta(9)\cup\{7,8\}) = \{6,4\}$ *and and* $C = \{\Delta(9)\cup\{6,7,8\}, \Delta(9)\cup\{4,7,8\}\}$.

- $\mathscr{D}(Frob=9) = \{\Delta(9), \Delta(9)\cup\{6\}, \Delta(9)\cup\{7\}, \Delta(9)\cup\{8\}, \Delta(9)\cup\{6,7\}, \Delta(9)\cup\{6,8\}, \Delta(9)\cup\{7,8\}, \Delta(9)\cup\{6,7,8\}, \Delta(9)\cup\{4,7,8\}\}$, $B = \{\Delta(9)\cup\{6,7,8\}, \Delta(9)\cup\{4,7,8\}\}$, $\mathrm{Ap}(\Delta(9)\cup\{6,7,8\},10) = \{0,6,7,8,11,12,13,14,15,19\}$, $\mathrm{Ap}(\Delta(9)\cup\{4,7,8\},10) = \{0,4,7,8,11,12,13,15,16,19\}$.

- $\theta(\Delta(9)\cup\{6,7,8\}) = \emptyset$ *and* $\theta(\Delta(9)\cup\{4,7,8\}) = \emptyset$.

- *The Algorithm 3.9 returns* $\mathscr{D}(Frob=9) = \{\Delta(9), \Delta(9)\cup\{6\}, \Delta(9)\cup\{7\}, \Delta(9)\cup\{8\}, \Delta(9)\cup\{6,7\}, \Delta(9)\cup\{6,8\}, \Delta(9)\cup\{7,8\}, \Delta(9)\cup\{6,7,8\}, \Delta(9)\cup\{4,7,8\}\}$.

These computations can be obtained by the order `DF`, which we have implemented in GAP.

```
gap> DF(9);
[ [ 10 .. 19 ], [ 6, 10, 11, 13, 14, 15 ],
[ 7, 10, 11, 12, 13, 15, 16 ], [ 8, 10, 11, 12, 13, 14, 15, 17 ],
[ 6, 7, 10, 11, 15 ], [ 6, 8, 10, 11, 13, 15 ],
[ 7, 8, 10, 11, 12, 13 ], [ 4, 7, 10, 13 ], [ 6, 7, 8, 10, 11 ] ]
```

4. Digital numerical semigroups with given genus

Throughout this section, g will denote a positive integer and we denote by $\mathscr{D}(\mathrm{gen}=g) = \{S \in \mathscr{D} \mid \mathrm{g}(S) = g\}$. Our next aim is to show an algorithm which computes $\mathscr{D}(\mathrm{gen}=g)$. For this reason, we need to recall some concepts and results.

A *Frobenius variety* (see [11]) is a nonempty family $\mathscr{V}$ of numerical semigroups fulfilling the following conditions:

1) If $\{S,T\} \subseteq \mathscr{V}$, then $S \cap T \in \mathscr{V}$.
2) If $S \in \mathscr{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathscr{V}$.

The following result is [13, Proposition 12].

Proposition 4.1. *With the above notation, the set* $\mathscr{D}$ *is a Frobenius variety.*

In a similar way to the construction of the graph $\mathrm{G}(\mathscr{D}(\mathrm{Frob}=F))$ made in Section 3, next we define the graph $\mathrm{G}(\mathscr{D})$:

- $\mathscr{D}$ is its set of vertices.
- $(S,T) \in \mathscr{D}\times\mathscr{D}$ is an edge if and only if $T = S\cup\{\mathrm{F}(S)\}$.

The following result is [13, Theorem 13].

Proposition 4.2. *The graph* $\mathrm{G}(\mathscr{D})$ *is a tree with root* $\mathbb{N}$. *Moreover, the set formed by the children of* $S \in \mathscr{D}$ *in the tree* $\mathrm{G}(\mathscr{D})$ *is* $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S), x > \mathrm{F}(S)$ *and* $S\backslash\{x\} \in \mathscr{D}\}$.

The following result is Corollary 15 from [13].

Proposition 4.3. *Let* $S \in \mathscr{D}$, $S \neq \mathbb{N}$ *and* $x \in \mathrm{msg}(S)$ *such that* $x > \mathrm{F}(S)$. *Then* $S\backslash\{x\} \in \mathscr{D}$ *if and only if* $x+1 \in \mathrm{msg}(S)$.

As an immediate consequence of Propositions 4.2 and 4.3 we have the next result.

Corollary 4.4. *Let $S \in \mathscr{D}$ such that $S \neq \mathbb{N}$. Then the set formed by the children of S in the tree* $\mathrm{G}(\mathscr{D})$ *is* $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S), x > \mathrm{F}(S)$ *and* $x+1 \in \mathrm{msg}(S)\}$.

Let G be a tree rooted and let v one of its vertices. We define the *depth* of v, denoted by $\mathrm{d}(v)$, as the length of the unique path connecting v with the root.

If $k \in \mathbb{N}$, we denote by $\mathrm{N}(G,k) = \{x \in V \mid \mathrm{d}(v) = k\}$. The *height* of G is $\mathrm{h}(G) = \max\{k \in \mathbb{N} \mid \mathrm{N}(G,k) \neq \emptyset\}$.

The following result has an immediate proof.

Lemma 4.5. *If $n \in \mathbb{N}$, then the following conditions hold:*

1. $\mathrm{N}(G(\mathscr{D}), n) = \{S \in \mathscr{D} \mid \mathrm{g}(S) = n\}$.
2. $\mathrm{N}(G(\mathscr{D}), n+1) = \{S \mid S$ *is a child of an element of* $\mathrm{N}(G(\mathscr{D}), n)$ *in the tree* $\mathrm{G}(\mathscr{D})\}$.

We realize the following.

Remark 4.6. *Let us observe that:*

- $\langle 2,3\rangle$ *is the unique child of* $\mathbb{N}$ *in the tree* $\mathrm{G}(\mathscr{D})$.
- $\mathscr{D}(gen{=}g) \neq \emptyset$ *because* $\{0, g+1, \rightarrow\} \in \mathscr{D}(gen{=}g)$.

The previous results allow us to present the announced algorithm at the beginning of this section.

Algorithm 4.7.

Input: A positive integer g.
Output: $\mathscr{D}(\mathrm{gen}{=}g)$.

(1) $A = \{\langle 2,3\rangle\}$, $i = 1$.
(2) If $i = g$, return A.
(3) For all $S \in A$, compute $\alpha(S) = \{x \in \mathrm{msg}(S) \mid x > \mathrm{F}(S)$ and $x+1 \in \mathrm{msg}(S)\}$.
(4) $A = \bigcup_{S \in A} \{S\backslash\{x\} \mid x \in \alpha(S)\}$, $i = i+1$ and go Step (2).

Observe that in Algorithm 4.7, if we know $\mathrm{msg}(S)$ and $x \in \mathrm{msg}(S)$ such that $x > \mathrm{F}(S)$, we have to calculate $\mathrm{msg}(S\backslash\{x\})$. To help this computation, we can use the next result that appears in [9, Corollary 18].

Lemma 4.8. *Let S be a numerical semigroup and $x \in \mathrm{msg}(S)$ such that $x > \mathrm{F}(S)$ and $x \neq \mathrm{m}(S)$. Then*

$$\mathrm{msg}(S\backslash\{x\}) = \begin{cases} \mathrm{msg}(S)\backslash\{x\} & \text{if } x + \mathrm{m}(S) - y \in S \text{ for some } y \in \mathrm{msg}(S)\backslash\{x, \mathrm{m}(S)\}, \\ (\mathrm{msg}(S)\backslash\{x\}) \cup \{x + \mathrm{m}(S)\} & \text{otherwise.} \end{cases}$$

We illustrate this algorithm with an example.

Example 4.9. *We are going to compute $\mathscr{D}(gen{=}5)$ by using Algorithm 4.7.*

- $A = \{\langle 2,3\rangle\}$, $i = 1$.
- $\alpha(\langle 2,3\rangle) = \{2\}$.
- $A = \{\langle 3,4,5\rangle\}$, $i = 2$.
- $\alpha(\langle 3,4,5\rangle) = \{3,4\}$.
- $A = \{\langle 4,5,6,7\rangle, \langle 3,5,7\rangle\}$, $i = 3$.

- $\alpha(\langle 4,5,6,7\rangle) = \{4,5,6\}$ *and* $\alpha(\langle 3,5,7\rangle) = \emptyset$.
- $A = \{\langle 5,6,7,8,9\rangle, \langle 4,6,7,9\rangle, \langle 4,5,7\rangle\}$, $i = 4$.
- $\alpha(\langle 5,6,7,8,9\rangle) = \{5,6,7,8\}$, $\alpha(\langle 4,6,7,9\rangle) = \{6\}$ *and* $\alpha(\langle 4,5,7\rangle) = \emptyset$.
- $A = \{\langle 6,7,8,9,10,11\rangle, \langle 5,7,8,9,11\rangle, \langle 5,6,8,9\rangle, \langle 5,6,7,9\rangle, \langle 4,7,9,10\rangle\}$, $i = 5$.
- *The Algorithm 4.7 returns*

$$\mathscr{D}(gen{=}5) = \{\langle 6,7,8,9,10,11\rangle, \langle 5,7,8,9,11\rangle, \langle 5,6,8,9\rangle, \langle 5,6,7,9\rangle, \langle 4,7,9,10\rangle\}.$$

Next, we presente the GAP code which we are implemented. It is important to emphasize that our GAP code does not require to make calls to other libraries or GAP packages. This makes our method more versatile and suitable to be implemented in other programming languages.

```
alpha:= function(S)
local C,i,s,F;
s:=NumericalSemigroup(S);
F:=FrobeniusNumber(s);
C:=[];
for i in [1..Length(S)-1] do
if S[i]>F  and \in(S[i]+1,S)=true then
Add(C,S[i]);
fi;
i:=i+1;
od;
return C;
end;;


Dg:= function(g)
local i,A,C,j,al,sj,y,sr,msg;
i:=1;
A:=[[2,3]];
while i<=g-1 do  C:=[];
for j in [1..Length(A)]  do
al:=alpha(A[j]);
sj:=NumericalSemigroup(A[j]);
for y in [1..Length(al)] do
sr:=RemoveMinimalGeneratorFromNumericalSemigroup(al[y],sj);
msg:=MinimalGenerators(sr);
Add(C,msg);
od;
od;
A:=C;
i:=i+1;
od;
return A;
end;;
```

By using the previous code, the computations of Example 4.9 can be made by using the following GAP order:

```
gap> Dg(5);
[ [ 6 .. 11 ], [ 5, 7, 8, 9, 11 ], [ 5, 6, 8, 9 ], [ 5, 6, 7, 9 ],
[ 4, 7, 9, 10 ] ]
```

5. Digital numerical semigroups with fixed multiplicity

Along this section m denotes an integer greater than or equal to 2. We denote by $\mathscr{D}(\text{mult}{=}m) = \{S \in \mathscr{D} \mid \text{m}(S) = m\}$. Our main goal will be to show an algorithm to compute $\mathscr{D}(\text{mult}{=}m)$. For this reason, we need to introduce some concepts and results.

A *Frobenius pseudo-variety* (see [10]) is a nonempty family $\mathscr{P}$ of numerical semigroups verifying the following conditions:

1) $\mathscr{P}$ has a maximum.
2) If $\{S, T\} \subseteq \mathscr{P}$, then $S \cap T \in \mathscr{P}$.
3) If $S \in \mathscr{P}$ and $S \neq \max(\mathscr{P})$, then $S \cup \{\text{F}(S)\} \in \mathscr{P}$.

It is easy to prove from the above definitions the following proposition.

Proposition 5.1. *With the above notation, $\mathscr{D}(\text{mult}{=}m)$ is a Frobenius pseudo-variety and $\Delta(m-1) = \{0, m, \rightarrow\}$ is its maximum.*

In a similar way to the construction of the graphs $\text{G}(\mathscr{D}(\text{Frob}{=}F))$ and $\text{G}(\mathscr{D})$ presented in Section 3 and Section 1, respectively, next we define the graph $\text{G}(\mathscr{D}(\text{mult}{=}m))$:

- $\mathscr{D}(\text{mult}{=}m)$ is its set of vertices.
- $(S, T) \in \mathscr{D}(\text{mult}{=}m) \times \mathscr{D}(\text{mult}{=}m)$ is an edge if and only if $T = S \cup \{\text{F}(S)\}$.

The following result can be deduced from [10, Theorem 3].

Proposition 5.2. *The graph $\text{G}(\mathscr{D}(\text{mult}{=}m))$ is a tree rooted in $\Delta(m-1)$. Moreover, the set formed by the children of $S \in \mathscr{D}(\text{mult}{=}m)$ in the tree $\text{G}(\mathscr{D}(\text{mult}{=}m))$ is $\{S\backslash\{x\} \mid x \in \text{msg}(S), x > \text{F}(S)$ and $S\backslash\{x\} \in \mathscr{D}(\text{mult}{=}m)\}$.*

As a consequence from Propositions 4.3 and 5.2, we have the following result.

Corollary 5.3. *Let $S \in \mathscr{D}(\text{mult}{=}m)$. Then the set formed by the children of S in the tree $\text{G}(\mathscr{D}(\text{mult}{=}m))$ is $\{S\backslash\{x\} \mid x \in \text{msg}(S), x > \text{F}(S), x \neq m$ and $x+1 \in \text{msg}(S)\}$.*

Next we can present the algorithm stated at the beginning of this section.

Algorithm 5.4.

Input: An integer greater than or equal to 2.
Output: $\mathscr{D}(\text{mult}{=}m)$.

(1) $A = \{\Delta(m-1)\}$ and $B = \{\Delta(m-1)\}$.
(2) For every $S \in B$ compute $\beta(S) = \{x \in \text{msg}(S) \mid x > \text{F}(S), x+1 \in \text{msg}(S)$ and $x \neq m\}$.
(3) If $\bigcup_{S \in B} \beta(B) = \emptyset$, then return A.
(4) $C = \bigcup_{S \in B} \{S\backslash\{x\} \mid x \in \beta(B)\}$.
(5) $A = A \cup C$, $B = C$ and go Step (2).

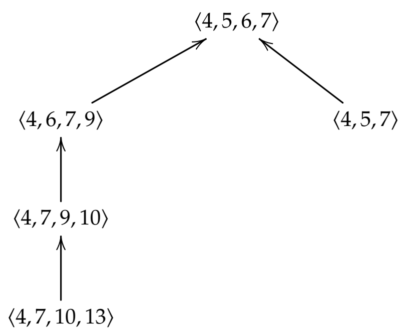
We illustrate this algorithm with an example.

Example 5.5. *We are going to compute $\mathscr{D}(mult{=}4)$ by using Algorithm 5.4.*

- $A = \{\langle 4,5,6,7\rangle\}$ *and* $B = \{\langle 4,5,6,7\rangle\}$.
- $\beta(\langle 4,5,6,7\rangle) = \{5,6\}$.
- $C = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle\}$.
- $A = \{\langle 4,5,6,7\rangle, \langle 4,6,7,9\rangle, \langle 4,5,7\rangle\}$ *and* $B = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle\}$.
- $\beta(\langle 4,6,7,9\rangle) = \{6\}$ *and* $\beta(\langle 4,5,7\rangle) = \emptyset$.
- $C = \{\langle 4,7,9,10\rangle\}$.
- $A = \{\langle 4,5,6,7\rangle, \langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,7,9,10\rangle\}$ *and* $B = \{\langle 4,7,9,10\rangle\}$.
- $\beta(\langle 4,7,9,10\rangle) = \{9\}$.
- $C = \{\langle 4,7,10,13\rangle\}$.
- $A = \{\langle 4,5,6,7\rangle, \langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,7,9,10\rangle, \langle 4,7,10,13\rangle\}$ *and* $B = \{\langle 4,7,10,13\rangle\}$.
- $\beta(\langle 4,7,10,13\rangle) = \emptyset$.
- *The Algorithm returns*

$$\mathscr{D}(mult{=}4) = \{\langle 4,5,6,7\rangle, \langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,7,9,10\rangle, \langle 4,7,10,13\rangle\}.$$

The tree $\mathscr{D}$(mult=4) is shown below.



Also observe that $\mathscr{D}[\{4\}] = \langle 4,7,10,13\rangle$.
By using Proposition 2.13 is not hard to prove the following proposition.

Proposition 5.6. *With the above notaticon, we have:*

1. $\mathscr{D}[\{m\}]$ *is the minimum of* $\mathscr{D}(mult{=}m)$.
2. $\{\mathrm{g}(S) \mid S \in \mathscr{D}(mult{=}m)\} = \{m-1, m, \cdots, \frac{m(m-1)}{2}\}$.
3. $\mathrm{G}(\mathscr{D}(mult{=}m))$ *is a tree with height equal to* $\frac{(m-1)(m-2)}{2}$.

6. Digital numerical semigroups with given $\mathscr{D}$-rank

In this section, p will denote an integer greater than or equal to 2. Our main goal will be to present an algorithmic procedure to obtain all the digital numerical semigroups with $\mathscr{D}$-rank equal to p.

Proposition 6.1. *Let $X = \{x_1, \cdots, x_p\}$ be a nonempty subset of $\mathbb{N}\backslash\{0,1\}$ and $T = \langle x_1, x_1 - 1, \cdots, x_p, x_p - 1\rangle$. Then $\mathscr{D}[X] = (X + T) \cup \{0\}$.*

Proof.
- If $a \in X+T$, then there exists $i \in \{1, \cdots, p\}$ such that $a \in \{x_i\}+T$. Therefore, there is $\{\lambda_1, \mu_1, \cdots, \lambda_p, \mu_p\} \subseteq \mathbb{N}$ such that $a = x_i + \lambda_1 x_1 + \mu_1(x_1 - 1) + \cdots + \lambda_p x_p + \mu_p(x_p - 1) = (\lambda_1 + \mu_1)x_1 + \cdots + (\lambda_i + \mu_i + 1)x_i + \cdots + (\lambda_p + \mu_p)x_p - (\mu_1 + \cdots + \mu_p)$. By applying Proposition 2.10 we assert that $a \in \mathscr{D}[X]$.
- If $a \in \mathscr{D}[X]\backslash\{0\}$, then by Proposition 2.10, we know that there is $\{\lambda_1, \cdots, \lambda_p, \mu\} \subseteq \mathbb{N}$ such that $a = \lambda_1 x_1 + \cdots + \lambda_p x_p - \mu$ being $\mu < \lambda_1 + \cdots + \lambda_p$.

  Let $\{(\alpha_1, \beta_1), \cdots, (\alpha_p, \beta_p)\} \subseteq \mathbb{N}^2$ such that $\alpha_i + \beta_i = \lambda_i$ for all $i \in \{1, \cdots, p\}$ and $\beta_1 + \cdots + \beta_p = \mu$. Then $a = \alpha_1 x_1 + \cdots + \alpha_p x_p + \beta_1 x_1 + \cdots + \beta_p x_p - (\beta_1 + \cdots + \beta_p)$. As $\mu < \lambda_1 + \cdots + \lambda_p$, then we deduce that there is $i \in \{1, \cdots, p\}$ such that $\alpha_i \neq 0$. Then $a = x_i + \alpha_1 x_1 + \beta_1(x_1 - 1) + \cdots + (\alpha_i - 1)x_i + \beta_i(x_i - 1) + \cdots + \alpha_p x_p + \beta_p(x_p - 1) \in \{x_i\} + T$. Therefore, $a \in X + T$.

  □

From Proposition 4.1 and [11, Lemma 16], we deduce the following upshot.

Lemma 6.2. *Let $X \subseteq \mathbb{N}\backslash\{0,1\}$ and $S = \mathscr{D}[X]$. Then X is a mininimal $\mathscr{D}$-system of generators of S if and only if $x \notin \mathscr{D}[X\backslash\{x\}]$ for all $x \in X$.*

Notice that if $2 \in X$, then $\mathscr{D}[X] = \langle 2, 3\rangle$.

Proposition 6.3. *Let $x_1, x_2, \cdots, x_p$ be integers such that $3 \leq x_1 < x_2 < \cdots < x_p$ and $X = \{x_1, x_2, \cdots, x_p\}$. Then X is a minimal $\mathscr{D}$-system of generators of $\mathscr{D}[X]$ if and only if $x_{i+1} \notin \{x_1, x_2, \cdots, x_i\} + \langle x_1, x_1 - 1, \cdots, x_i, x_i - 1\rangle$ for all $i \in \{1, \cdots, p-1\}$.*

Proof. By Lemma 6.2, we know that X is the minimal $\mathscr{D}$-system of generators of $\mathscr{D}[X]$ if and only if $x_{i+1} \notin \mathscr{D}[X\backslash\{x_{i+1}\}]$ for all $i \in \{1, \cdots, p-1\}$.

By applying now, Proposition 6.1, we easily deduce that $x_{i+1} \notin \mathscr{D}[X\backslash\{x_{i+1}\}]$ if and only if $x_{i+1} \notin \{x_1, \cdots, x_i\} + \langle x_1, x_1 - 1, \cdots, x_i, x_i - 1\rangle$. □

In Example 6.5, we will see that the previous result can be viewed as a procedure to construct a digital numerical semigroup with $\mathscr{D}$-rank equal to p.

The following proposition can be deduced from [13, Corollary 25].

Proposition 6.4. *Let S be a digital numerical semigroup. Then $\mathscr{D}\mathrm{msg}(S) = \{x \in \mathrm{msg}(S) \mid x+1 \in (\mathbb{N}\backslash S) \cup \mathrm{msg}(S)\}$.*

In the next example, by using Proposition 6.3 we are going to build a digital numerical semigroup with $\mathscr{D}$-rank equal to 3 and by applying Proposition 6.4, we will calculate its mininimal $\mathscr{D}$-system of generators.

Example 6.5. *Let $x_1 = 5$ and $x_2 \notin \{5\} + \langle 4, 5\rangle$. For instance, $x_2 = 7$. Take now $x_3 \notin \{5, 7\} + \langle 4, 5, 6, 7\rangle$. Let $x_3 = 8$. Then by applying Proposition 6.3, we have $S = (\{5, 7, 8\} + \langle 4, 5, 6, 7, 8\rangle) \cup \{0\} = (\{5, 7, 8\} + \langle 4, 5, 6, 7\rangle) \cup \{0\} = \langle 5, 7, 8, 9, 11\rangle$ is a digital numerical semigroup with $\mathscr{D}\mathrm{rank}(S) = 3$. Note that by Proposition 6.4, we have $\mathscr{D}\mathrm{msg}(S) = \{5, 7, 8\}$.*

Acknowledgements. The authors would like to thank the referees for their useful comments and suggestions that helped to improve this work.

References

[1] R. Apéry, *Sur les branches superlinéaires des courbes algébriques*, C.R. Acad. Sci. Paris **222** (1946), 1198–2000.

[2] V. Barucci, D.E. Dobbs, M. Fontana, *Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analitycally Irreducible Local Domains*, Memoirs Amer. Math. Soc. **598** (1997).

[3] M. Delgado, P.A. García-Sánchez, J. Morais, *NumericalSgps, A package for numerical semigroups*, Version 1.3.1 (2022), (Refereed GAP package), `https://gap-packages.github.io/numericalsgps`.

[4] R. Fröberg, G. Gottlieb, R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35** (1987), 63–83.

[5] The GAP group, GAP – Groups, Algorithms, and Programming, Version 4.12.2, 2022, `https://www.gap-system.org`.

[6] M.A. Moreno-Frías, J.C. Rosales, *The covariety of numerical semigroups with fixed Frobenius number*, J. Algebr. Comb. **60** (2024), 555–568.

[7] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, London, 2005.

[8] J.L. Ramírez Alfonsín, *Complexity of the Frobenius problem*, Combinatorica **16** (1996), 143–147.

[9] A.M. Robles-Pérez, J.C. Rosales, *The numerical semigroup of phrases' lengths in a simple alphabet*, Scient. World. J. (2013), Article ID 459024 (9 pages).

[10] A.M. Robles-Pérez, J. C. Rosales, *Frobenius pseudo-varieties in numerical semigroups*, Ann. Mat. Pura Appl. **194** (2015), 275-287.

[11] J.C. Rosales, *Families of numerical semigroups closed under finite intersections and for the Frobenius number*, Houst. J. Mathem. **34** (2008), 339–348.

[12] J.C. Rosales, M.B. Branco, *Numerical Semigroups that can be expressed as an intersection of symmetric numerical semigroups*, J. Pure Appl. Algebra **171** (2002), 303–314.

[13] J.C. Rosales, M. B. Branco, D. Torrão, *Sets of positve integers closed under product and the number of decimal digits*, J. Number Theory **147** (2015), 1–13.

[14] J.C. Rosales and P.A. García-Sánchez, *Numerical Semigroups*, Developments in Mathematics, Vol. 20, Springer, New York, 2009.

[15] J.J. Sylvester, *Mathematical question with their solutions*, Educational Times **41** (1884), 21.