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Operators associated with adjoint Bernoulli polynomials

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Abstract. Very recently Yilmaz [18] proposed Kantorovich type operators based on adjoint Bernoulli polynomials. She considered the operators on the finite interval $[0, 1]$, but we observe here that these operators can be defined on positive real axis. We estimate moments using the technique of moment generating functions. We also obtain some direct results for these operators. Also we consider composition of these operators with well known Szász-Mirakjan operators and estimate some direct convergence estimates.

1. Introduction

Based on the adjoint Bernoulli polynomials $\{\beta_k(x)\}_{k=1}^{\infty}$ satisfying the generating function of the type [17, (8)]:

$$\frac{e^t - 1}{t} e^{xt} = \sum_{k=0}^{\infty} \beta_k(x) \frac{t^k}{k!}. \tag{1}$$

Yilmaz [18] defined the Kantorovich type operators as

$$A_n(f; x) = n \frac{e^{-nx}}{e-1} \sum_{k=0}^{\infty} \frac{\beta_k(nx)}{k!} \int_{k/n}^{(k+1)/n} f(t)dt \tag{2}$$

In case $\beta_k(nx) = (e-1)(nx)^k$, these operators reduce to the Szász-Kantorovich operators. In [18] author considered the operators defined for the interval $[0, 1]$, it may be observed that these operators holds good for $x \in [0, \infty)$. This motivated us to study these operators in more details and here we extend the studies to recent trends. We mention the readers to some important studies viz. [1], [2], [6], [7], [9] and [11]. The Kantorovich and Durrmeyer variants of some other related operators have been discussed in [8], [10], [12], [14] and [15] etc.

We estimate some direct results and show the convergence through graphically. Also, as composition of operators provide us another new operator (see [13] and references therein), we provide here some new operators by taking the composition with Szász-Mirakyan operators and obtain convergence results.

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2. Moments

Lemma 2.1. *With the notation* $\exp_\lambda(t) = e^{\lambda t}$*, for the operators (2), we have*

$$A_n(\exp_\lambda; x) = \frac{n(e^{\lambda/n} - 1)(e^{e^{\lambda/n}} - 1)e^{nx(e^{\lambda/n}-1)}}{\lambda e^{\lambda/n}(e-1)}.$$

Proof. By definition of operators A_n, we have

$$\begin{aligned} A_n(\exp_\lambda; x) &= n\frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{\beta_k(nx)}{k!}\int_{k/n}^{(k+1)/n} e^{\lambda t}dt \\ &= n\frac{e^{-nx}(e^{\lambda/n}-1)}{\lambda(e-1)}\sum_{k=0}^{\infty}\frac{\beta_k(nx)}{k!}e^{\lambda k/n}. \end{aligned}$$

Applying the generating function (1), the result is immediate. □

Lemma 2.2. *With the notation* $e_b(t) = t^b, b = 0, 1, 2, ...,$ *we have*

$$\begin{aligned} A_n(e_0; x) &= 1, \\ A_n(e_1; x) &= x + \frac{e+1}{2n(e-1)}, \\ A_n(e_2; x) &= x^2 + \frac{6enx + 4e - 1}{3(e-1)n^2}. \end{aligned}$$

Proof. By Lemma 2.1, we find that

$$A_n(e_b; x) = \left[\frac{\partial^b}{\partial\lambda^b}\frac{n(e^{\lambda/n} - 1)(e^{e^{\lambda/n}} - 1)e^{nx(e^{\lambda/n}-1)}}{\lambda e^{\lambda/n}(e-1)}\right]_{\lambda=0}.$$

By simple computation result follows. □

Lemma 2.3. *With the notation* $\mu^A_{n,b}(x) = A_n((e_1 - xe_0)^b); x), b = 0, 1, 2, ...,$ *we have*

$$\begin{aligned} \mu^A_{n,0}(x) &= 1, \\ \mu^A_{n,1}(x) &= \frac{e+1}{2n(e-1)}, \\ \mu^A_{n,2}(x) &= \frac{x}{n} + \frac{4e-1}{3(e-1)n^2}, \\ \mu^A_{n,3}(x) &= \frac{10enx + 2nx + 11e + 1}{4(e-1)n^3}, \\ \mu^A_{n,4}(x) &= \frac{15en^2x^2 - 15n^2x^2 + 55enx - 5nx + 41e - 1}{5(e-1)n^4}. \end{aligned}$$

Proof. By Lemma 2.1, we find that

$$\mu^A_{n,b}(x) = \left[\frac{\partial^b}{\partial\lambda^b}e^{-\lambda x}\frac{n(e^{\lambda/n} - 1)(e^{e^{\lambda/n}} - 1)e^{nx(e^{\lambda/n}-1)}}{\lambda e^{\lambda/n}(e-1)}\right]_{\lambda=0}.$$

By simple computation result follows. □

3. Convergence

The class of continuous and real functions f denoted by $C^*[0,\infty)$, have finite limit, for x tending to ∞. In [5] and [16], some interesting results have been studied for a sequence of operators L_n:

Theorem 3.1. *[16] For $L_n : C^*[0,\infty) \to C^*[0,\infty)$, if we denote the norms $\left\| L_n\left(\exp_s\right) - \exp_s \right\|_{[0,\infty)}$, $s = 0, -1, -2$ as α_n, β_n and γ_n respectively, which approach to zero as $n \to \infty$, then*

$$\left\| L_n f - f \right\|_{[0,\infty)} \leq \alpha_n \left\| f \right\|_{[0,\infty)} + (2+\alpha_n)\, \omega^*\left(f, (\alpha_n + \gamma_n + 2\beta_n)^{1/2}\right),$$

where $\omega^*(f,\delta) := \sup\limits_{\substack{|e^{-t}-e^{-u}|\leq\delta \\ t,u>0}} \left| f(u) - f(t) \right|$.

Theorem 3.2. *For $f \in C^*[0,\infty)$, there holds*

$$\left\| A_n f - f \right\|_{[0,\infty)} \leq 2\omega^*\left(f, (2\beta_n + \gamma_n)^{1/2}\right),$$

where

$$\beta_n = \left\| A_n \exp_{-1} - \exp_{-1} \right\|_{[0,\infty)} \to 0, \quad n \to \infty$$

and

$$\gamma_n = \left\| A_n \exp_{-2} - \exp_{-2} \right\|_{[0,\infty)} \to 0, \quad n \to \infty.$$

Proof. By Lemma 2.2, the operators A_n preserve constants, so $\alpha_n = 0$. We have to compute β_n and γ_n. By the software Maple, we have

$$A_n\left(\exp_{-1}; x\right) = e^{-x} + \frac{xe^{-x}}{2n} - \frac{(e+1)e^{-x}}{2n(e-1)} + \frac{x^2e^{-x}}{8n^2} - \frac{5(e+1)xe^{-x}}{12(e-1)n^2} + \frac{(4e-1)e^{-x}}{6(e-1)n^2} + O\left(n^{-3}\right).$$

Since

$$\sup_{x\geq 0} e^{-x}x = e^{-1},\ \ \sup_{x\geq 0} x^2e^{-x} = 4e^{-2},\ \ \sup_{x\geq 0} x^3e^{-x} = 27e^{-3},\ \ \sup_{x\geq 0} x^4e^{-x} = 256e^{-4},$$

we obtain

$$\begin{aligned} \beta_n &= \sup_{x\geq 0}\left| A_n\left(\exp_{-1}; x\right) - e^{-x} \right| \\ &\leq \frac{1}{2en} + \frac{(e+1)}{2n(e-1)} + \frac{1}{2e^2n^2} + \frac{5(e+1)}{12e(e-1)n^2} + \frac{(4e-1)}{6(e-1)n^2} + O\left(n^{-3}\right) \\ &\leq O\left(n^{-1}\right). \end{aligned}$$

Similarly,

$$A_n\left(\exp_{-2}; x\right) = e^{-2x} + \frac{2xe^{-2x}}{n} - \frac{(e+1)e^{-2x}}{n(e-1)} + \frac{2x^2e^{-2x}}{n^2} - \frac{(5e+1)2xe^{-2x}}{3(e-1)n^2} + \frac{(4e-1)2e^{-2x}}{3(e-1)n^2} + O\left(n^{-3}\right).$$

Since

$$\begin{aligned} \sup_{x\geq 0} e^{-2x} &= 1,\ \ \sup_{x\geq 0} e^{-2x}x = 0.5e^{-1}, \\ \sup_{x\geq 0} x^2e^{-2x} &= e^{-2},\ \ \sup_{x\geq 0} x^3e^{-2x} = \frac{27}{8}e^{-3},\ \ \sup_{x\geq 0} x^4e^{-2x} = 16e^{-4}, \end{aligned}$$

we get

$$\begin{aligned}\gamma_n &= \sup_{x\geq 0}\left|A_n\left(\exp_{-2};x\right)-e^{-2x}\right| \\ &\leq \frac{1}{en}+\frac{(e+1)}{n(e-1)}+\frac{2}{e^2n^2}+\frac{(5e+1)}{3e(e-1)n^2}+\frac{2(4e-1)}{3(e-1)n^2}+O\left(n^{-3}\right).\end{aligned}$$

Following Theorem 3.1, the proof of the theorem is immediate. □

Theorem 3.3. *Let f and its second derivative belong to the class $C^*[0,\infty)$, then for $x\geq 0$, there follows*

$$\begin{aligned}&\left|n\left[A_n(f;x)-f(x)\right]-\frac{(e+1)f'(x)}{2(e-1)}-\frac{xf''(x)}{2}\right| \\ \leq\ &\frac{4e-1}{6(e-1)n}\left|f''(x)\right|+\left[x+a_n(x)+\frac{4e-1}{3n(e-1)}\right]\omega^*\left(f'',\frac{1}{\sqrt{n}}\right),\end{aligned}$$

where

$$a_n(x)=n^2\left[A_n\left(\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4;x\right).\mu^A_{n,4}(x)\right]^{1/2}.$$

Proof. Applying Taylor's formula to A_n, we have

$$\left|A_n(f;x)-f(x)-\mu^A_{n,1}(x)f'(x)-\frac{\mu^A_{n,2}(x)}{2}f''(x)\right|\leq\left|A_n\left((t-x)^2h_{t,x};x\right)\right|$$

where $h_{t,x}=\frac{f''(\eta)-f''(x)}{2}$ and $x<\eta<t$. Using Lemma 2.3, we immediately have

$$\begin{aligned}&\left|n\left[A_n(f;x)-f(x)\right]-\frac{e+1}{2(e-1)}f'(x)-\frac{x}{2}f''(x)\right| \\ \leq\ &\left|n\mu^A_{n,1}(x)-\frac{e+1}{2(e-1)}\right|\left|f'(x)\right|+\frac{1}{2}\left|n\mu^A_{n,2}(x)-x\right|\left|f''(x)\right|+\left|nA_n\left((t-x)^2h_{t,x};x\right)\right| \\ \leq\ &\frac{4e-1}{6(e-1)n}\left|f''(x)\right|+\left|nA_n\left(h_{t,x}(t-x)^2;x\right)\right|.\end{aligned}$$

Next by the property used in [3, (3.1)], there holds the inequality

$$h_{t,x}\leq\left(1+\frac{\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^2}{\delta^2}\right)\omega^*(f'',\delta),\qquad \delta>0.$$

Using above inequality along with Cauchy–Schwarz inequality and selecting $\delta=n^{-1/2}$, we have

$$\begin{aligned}nA_n\left(\left|h_{t,x}\right|(t-x)^2;x\right) &\leq n\omega^*(f'',\delta)\mu^A_{n,2}(x) \\ &\quad+\frac{n}{\delta^2}\omega^*(f'',\delta)\left[\left(A_n\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4;x\right)\right]^{1/2}\left[\mu^A_{n,4}(x)\right]^{1/2} \\ &= \left[x+a_n(x)+\frac{4e-1}{3n(e-1)}\right]\omega^*\left(f'',\frac{1}{\sqrt{n}}\right),\end{aligned}$$

where

$$a_n(x)=n^2\left[\left(A_n\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4;x\right).\mu^A_{n,4}(x)\right]^{1/2}.$$

This concludes the proof of theorem. □

Remark 3.4. *By simple computation following limits hold:*

1. $\lim_{n\to\infty} n^2\mu^A_{n,4}(x) = 3x^2$,
2. $\lim_{n\to\infty} n^2\left(A_n\left(\exp_{-1}(x) - \exp_{-1}(t)\right)^4; x\right) = 3x^2e^{-4x}$.

Corollary 3.5. *Suppose f and its second derivative belong to $C^*[0,\infty)$, then immediately one obtains*

$$\lim_{n\to\infty} n\left[A_n(f;x) - f(x)\right] = \frac{e+1}{2(e-1)}f'(x) + \frac{x}{2}f''(x).$$

Theorem 3.6. *Let f and its fourth derivative belong to the class $C^*[0,\infty)$, then for $x\in[0,\infty)$, there follows*

$$\begin{aligned}
&\left|n\left[n\left[A_n(f;x)-f(x)\right] - \frac{(e+1)f'(x)}{2(e-1)} - \frac{xf''(x)}{2}\right] - \frac{(4e-1)f''(x)}{6(e-1)} - \frac{(1+5e)xf'''(x)}{12(e-1)} - \frac{x^2f^{(iv)}(x)}{8}\right|\\
\le\ &\left[\frac{55enx-5nx+41e-1}{5(e-1)n^2} + 3x^2 + b_n(x)\right]\omega^*\left(f^{(iv)}, n^{-1/2}\right)\\
&+\frac{(11e+1)}{24n(e-1)}\left|f'''(x)\right| + \left(\frac{(11e-1)x}{24n(e-1)} + \frac{(41e-1)}{120n^2(e-1)}\right)\left|f^{(iv)}(x)\right|,
\end{aligned}$$

where

$$b_n(x) = n^3\left[\left(A_n\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^8; x\right).\mu^A_{n,8}(x)\right]^{1/2}.$$

Proof. Applying the Taylor's formula to A_n, we have

$$\begin{aligned}
&\left|A_n(f;x) - f(x) - \mu^A_{n,1}(x)f'(x) - \frac{\mu^A_{n,2}(x)}{2}f''(x) - \frac{\mu^A_{n,3}(x)}{3!}f'''(x) - \frac{\mu^A_{n,4}(x)}{4!}f^{(iv)}(x)\right|\\
\le\ &\left|A_n\left((t-x)^4 g_{t,x}; x\right)\right|,
\end{aligned}$$

where $g_{t,x} = \frac{f^{(iv)}(\nu)-f^{(iv)}(x)}{2}$, $x<\nu<t$. Alternatively we can write

$$\begin{aligned}
&\left|n\left[n\left[A_n(f;x)-f(x)\right] - n\mu^A_{n,1}(x)f'(x) - n\frac{\mu^A_{n,2}(x)}{2!}f''(x)\right] - n^2\frac{\mu^A_{n,3}(x)}{3!}f'''(x) - n^2\frac{\mu^A_{n,4}(x)}{4!}f^{(iv)}(x)\right|\\
\le\ &\left|n^2A_n((t-x)^{(iv)}g_{t,x};x)\right|.
\end{aligned}$$

Now, by Lemma 2.3 we have

$$\begin{aligned}
&\left|n\left[n\left[A_n(f;x)-f(x)\right] - \frac{e+1}{2(e-1)}f'(x) - \frac{x}{2}f''(x)\right] - \frac{(4e-1)f''(x)}{6(e-1)} - \frac{(5e+1)x}{12(e-1)}f'''(x) - \frac{x^2}{8}f^{(iv)}(x)\right|\\
\le\ &\frac{(11e+1)|f''(x)|}{24n(e-1)} + \left(\frac{(11e-1)x}{24n(e-1)} + \frac{(41e-1)}{120n^2(e-1)}\right)|f^{(iv)}(x)|\\
&+\left|n^2A_n\left((t-x)^4g_{t,x};x\right)\right|.
\end{aligned}$$

Next by the property used in [3, (3.1)], we can write

$$g_{t,x} \le \left(1 + \frac{\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^4}{\eta^2}\right)\omega^*\left(f^{(iv)},\eta\right), \qquad \eta>0.$$

Using above and Cauchy–Schwarz inequality and selecting $\eta = n^{-1/2}$, we have

$$\begin{aligned} & n^2 A_n\left(\left|g_{t,x}\right|(t-x)^4;x\right) \\ \leq\ & n^2\omega^*\left(f^{(iv)},\delta\right)\mu^A_{n,4}(x) \\ & +\frac{n^2}{\eta^2}\omega^*\left(f^{iv},\delta\right)\left[\left(A_n\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^8;x\right)\right]^{1/2}\left[\mu^A_{n,8}(x)\right]^{1/2} \\ =\ & \left[\frac{55enx-5nx+41e-1}{5(e-1)n^2}+3x^2+b_n(x)\right]\omega^*\left(f^{(iv)},n^{-1/2}\right), \end{aligned}$$

where

$$b_n(x) = n^3\left[\left(A_n\left(\exp_{-1}(x)-\exp_{-1}(t)\right)^8;x\right).\mu^A_{n,8}(x)\right]^{1/2}.$$

This completes the proof of theorem. □

Corollary 3.7. *Suppose f and its fourth derivative belong to $C^*[0,\infty)$, then immediately one obtains*

$$\begin{aligned} & \lim_{n\to\infty} n\left[n\left[A_n(f;x)-f(x)\right]-\frac{(e+1)f'(x)}{2(e-1)}-\frac{xf''(x)}{2}\right] \\ =\ & \frac{(4e-1)f''(x)}{6(e-1)}+\frac{(1+5e)xf'''(x)}{12(e-1)}+\frac{x^2f^{(iv)}(x)}{8}. \end{aligned}$$

4. Graphical Comparison

In the following figures for the functions $x^4-x^3-5x^2+6x+1+e^{-3x}$ and $-2x^3e^{-x}+7xe^{-x}+e^{-7x}+7x^2-2x^4+5$ respectively, the graphs are indicated for different values of n.

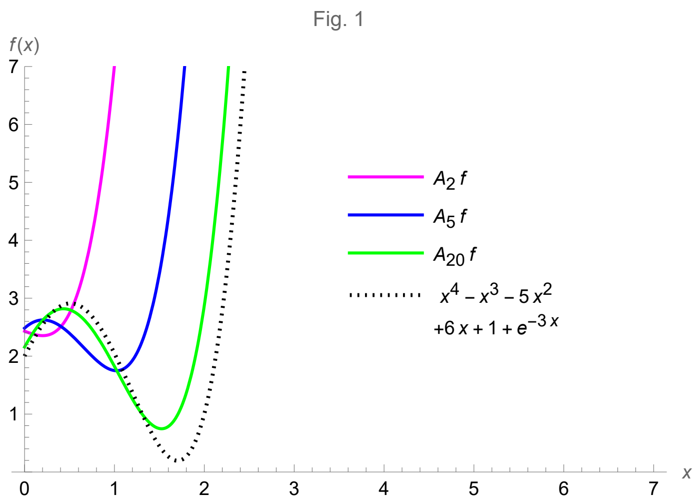

Figure 1: Convergence of the operator for the function $x^4 - x^3 - 5x^2 + 6x + 1 + e^{-3x}$ for different values of n

It is also observed from the graphs that as the value of n increases, the operator A_n converge to function more rapidly.

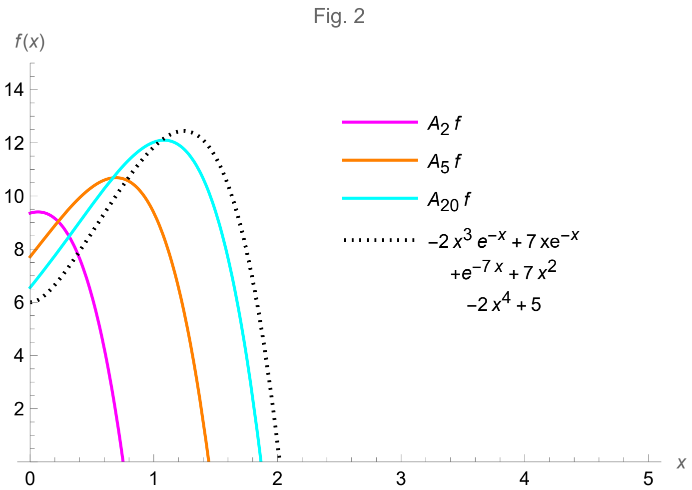

Figure 2: Convergence of the operator for the function $-2x^3e^{-x}+7xe^{-x}+e^{-7x}+7x^2-2x^4+5$ for different values of n

5. Composition $S_n \circ A_n$

The classical Szász-Mirakjan operators, which are well-known be defined by

$$S_n(f;x)=\sum_{v=0}^{\infty}e^{-nx}\frac{(nx)^v}{v!}f\left(\frac{v}{n}\right). \tag{3}$$

We consider the composition operator as

Proposition 5.1. *The concise form of the operator $S_n \circ A_n$ is given by*

$$(S_n\circ A_n)(f;x) = \frac{e^{-nx}}{e-1}\sum_{v=0}^{\infty}\frac{(nx/e)^v}{v!}\sum_{k=0}^{\infty}\frac{\beta_k(v)}{k!}\int_k^{(k+1)}f(nt)dt.$$

Proof. By definition of operators

$$\begin{aligned}(S_n\circ A_n)(f;x) &= \sum_{v=0}^{\infty}\frac{e^{-nx}(nx)^v}{v!}\frac{ne^{-v}}{e-1}\sum_{k=0}^{\infty}\frac{\beta_k(v)}{k!}\int_{k/n}^{(k+1)/n}f(t)dt\\ &= \frac{ne^{-nx}}{e-1}\sum_{v=0}^{\infty}\frac{(nx/e)^v}{v!}\sum_{k=0}^{\infty}\frac{\beta_k(v)}{k!}\int_{k/n}^{(k+1)/n}f(t)dt\\ &= \frac{e^{-nx}}{e-1}\sum_{v=0}^{\infty}\frac{(nx/e)^v}{v!}\sum_{k=0}^{\infty}\frac{\beta_k(v)}{k!}\int_k^{(k+1)}f(nt)dt.\end{aligned}$$

□

Lemma 5.2. *For the composition operators $S_n \circ A_n$, we have*

$$(S_n\circ A_n)(\exp_\lambda;x) = \frac{n(e^{\lambda/n}-1)(e^{e^{\lambda/n}}-1)}{\lambda(e-1)e^{\lambda/n}}e^{-nx(1-e^{(e^{\lambda/n}-1)})}.$$

Consequently, with the notation $\mu_{n,r}^{S\circ A}(x) = (S_n \circ A_n)((e_1 - xe_0)^b); x), b = 0, 1, 2, ...,$ *there follow:*

$$\begin{aligned}
\mu_{n,0}^{S\circ A}(x) &= 1,\\
\mu_{n,1}^{S\circ A}(x) &= \frac{e}{2n(e-1)},\\
\mu_{n,2}^{S\circ A}(x) &= \frac{2x}{n} + \frac{8e+1}{3(e-1)n^2}.
\end{aligned}$$

Proof. Using (1), we have

$$\begin{aligned}
(S_n \circ A_n)(\exp_\lambda; x) &= \frac{ne^{-nx}}{e-1}\sum_{v=0}^{\infty}\frac{(nx/e)^v}{v!}\sum_{k=0}^{\infty}\frac{\beta_k(v)}{k!}\left[\frac{e^{\lambda t}}{\lambda}\right]_{k/n}^{(k+1)/n}\\
&= \frac{n(e^{\lambda/n}-1)e^{-nx}}{\lambda(e-1)}\sum_{v=0}^{\infty}\frac{(nx/e)^v}{v!}\sum_{k=0}^{\infty}\frac{\beta_k(v)}{k!}e^{\lambda k/n}\\
&= \frac{n(e^{\lambda/n}-1)(e^{e^{\lambda/n}}-1)e^{-nx}}{\lambda(e-1)e^{\lambda/n}}\sum_{v=0}^{\infty}\frac{(nxe^{(e^{\lambda/n}-1)})^v}{v!}\\
&= \frac{n(e^{\lambda/n}-1)(e^{e^{\lambda/n}}-1)}{\lambda(e-1)e^{\lambda/n}}e^{-nx(1-e^{(e^{\lambda/n}-1)})}.
\end{aligned}$$

The other consequence holds by the methods used in Lemma 2.3, we omit the details. □

Theorem 5.3. *If* $n \in N$ *and* $f \in C_b[0,\infty)$ *(the class of bounded and continuous functions on positive real axis), then we get*

$$\left|(S_n \circ A_n f)(x) - (S_n f)(x)\right| \le 2\omega\left(f, \sqrt{\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}}\right).$$

Proof. We start as follows

$$\left|(S_n \circ A_n f)(x) - (S_n f)(x)\right| \le e^{-nx}\sum_{v\ge 0}\frac{(nx)^v}{v!}\left|(A_n f)\left(\frac{v}{n}\right) - f\left(\frac{v}{n}\right)\right|.$$

Using Lemma 2.3 in the following steps:

$$\begin{aligned}
\left|(A_n f)(x) - f(x)\right| &\le \left[1 + \frac{\mu_{n,2}^{A}(x)}{\delta^2}\right]\omega(g,\delta)\\
&= \left[1 + \frac{1}{\delta^2}\left(\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}\right)\right]\omega(f,\delta).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\left|(S_n \circ A_n f)(x) - (S_n f)(x)\right| &\le \sum_{v\ge 0} e^{-nx}\frac{(nx)^v}{v!}\left(1 + \frac{1}{\delta^2}\left(\frac{v}{n^2} + \frac{4e-1}{3(e-1)n^2}\right)\right)\omega(f,\delta)\\
&= \left(1 + \frac{1}{\delta^2}\left(\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}\right)\right)\omega(f,\delta).
\end{aligned}$$

choosing $\delta = \left(\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}\right)^{-1/2}$, the result follows. □

6. Composition $A_n \circ S_n$

We consider here the composition operator $A_n \circ S_n$.

Proposition 6.1. *The concise form of the operator is given by*

$$(A_n \circ S_n)(f;x) = \frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{\beta_k(nx)}{k!}\sum_{v=0}^{\infty}\frac{\Gamma(1+v,k)-\Gamma(1+v,k+1)}{v!}f\left(\frac{v}{n}\right),$$

where $\Gamma(a,k)$ is incomplete Gamma function.

Proof. By definition of A_n and S_n, we have

$$\begin{aligned}(A_n \circ S_n)(f;x) &= n\frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{\beta_k(nx)}{k!}\frac{1}{v!}f\left(\frac{v}{n}\right)\int_{k/n}^{(k+1)/n}\sum_{v=0}^{\infty}e^{-nt}\frac{(nt)^v}{v!}f\left(\frac{v}{n}\right)dt\\ &= \frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{\beta_k(nx)}{k!}\sum_{v=0}^{\infty}\frac{1}{v!}\int_k^{(k+1)}e^{-u}u^v du\\ &= \frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{\beta_k(nx)}{k!}\sum_{v=0}^{\infty}\frac{\Gamma(1+v,k)-\Gamma(1+v,k+1)}{v!}f\left(vn^{-1}\right).\end{aligned}$$

The proof of proposition is complete. □

Lemma 6.2. *For the composition operators $A_n \circ S_n$, we have*

$$(A_n \circ S_n)(\exp_\lambda;x) = \frac{n(e^{(e^{\lambda/n}-1)}-1)(e^{e^{(e^{\lambda/n}-1)}}-1)e^{nx(e^{(e^{\lambda/n}-1)}-1)}}{\lambda e^{(e^{\lambda/n}-1)}(e-1)}.$$

Consequently, with the notation $\mu_{n,r}^{A\circ S}(x) = (A_n \circ S_n)((e_1 - xe_0)^b);x), b = 0,1,2,...,$ one may have the following:

$$\begin{aligned}\mu_{n,0}^{A\circ S}(x) &= 1,\\ \mu_{n,1}^{A\circ S}(x) &= \frac{e}{n(e-1)},\\ \mu_{n,2}^{A\circ S}(x) &= \frac{2x}{n}+\frac{8e+1}{3(e-1)n^2}.\end{aligned}$$

Proof. Using (1), we have

$$\begin{aligned}(A_n \circ S_n)(\exp_\lambda;x) &= A_n(\exp_{n(e^{\lambda/n}-1)},x)\\ &= \frac{n(e^{(e^{\lambda/n}-1)}-1)(e^{e^{(e^{\lambda/n}-1)}}-1)e^{nx(e^{(e^{\lambda/n}-1)}-1)}}{\lambda e^{(e^{\lambda/n}-1)}(e-1)}.\end{aligned}$$

The other consequence holds by the methods used in Lemma 2.3, we omit the details. □

Theorem 6.3. *If $n \in N$ and $f \in C_b[0,\infty)$, then we get*

$$\left|(A_n \circ S_n f)(x) - (S_n f)(x)\right| \le 2\left(2 - e^{-\left(\frac{x}{n^3}+\frac{4e-1}{3(e-1)n^4}\right)^{-1/2}}\right)\omega\left(f, \sqrt{\frac{x}{n}+\frac{4e-1}{3(e-1)n^2}}\right).$$

Proof. We consider $S_n f = h$, then

$$\begin{aligned}\left|(A_n \circ S_n f)(x) - (S_n f)(x)\right| &= \left|(A_n g)(x) - g(x)\right|\\ &\le \left(1+\frac{1}{\delta^2}\left(\frac{x}{n}+\frac{4e-1}{3(e-1)n^2}\right)\right)\omega(S_n f,\delta).\end{aligned}$$

Also, by [4, Ex. (D)], we have $\omega(S_n f, \delta) \leq (2 - e^{-n\delta})\omega(f, \delta)$, thus

$$\left|(A_n \circ S_n f)(x) - (S_n f)(x)\right| \leq (2 - e^{-n\delta})\left(1 + \frac{1}{\delta^2}\left(\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}\right)\right)\omega(f, \delta),$$

choosing $\delta = \left(\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}\right)^{-1/2}$, the result follows. □

Corollary 6.4. *Under the assumptions of Theorems 5.3 and 6.3, we have*

$$\left|(A_n \circ S_n f)(x) - (S_n \circ A_n f)(x)\right| \leq 2\left(1 - e^{-\left(\frac{x}{n^3} + \frac{4e-1}{3(e-1)n^4}\right)^{-1/2}}\right)\omega\left(f, \sqrt{\frac{x}{n} + \frac{4e-1}{3(e-1)n^2}}\right).$$

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability statement in the manuscript

This manuscript has no associated data.

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