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Fractal-fractional reverse Minkowski type generalizations and related results

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Abstract. Fractional inequalities have recently gained prominence and have been the focus of numerous research studies. The effects of various sorts of inequalities have been analyzed by identifying revolutionary methodological approaches and implementations. This research determined reverse Minkowski type inequalities using fractal-fractional integral operators, such as the convolution of the index kernel, the exponential decay, and the generalized Mittag-Leffler kernel with fractal derivative. Moreover, we obtain several inequalities as particular cases of the main outcomes and their respective generalizations. We carried out a variety of inequalities using the fractal-fractional formulation strategy and achieved several interesting results in terms of (i) varying fractional order and fixing fractal-dimension, (ii) varying fractional-order and fixing fractal-dimension, and (iii) varying both fractional-order and fractal-dimension, respectively.

1. Introduction

Fractional calculus (FC), which is defined as the advancement or modification of conventional derivatives and integrals to non-integer order instances, has received considerable scholarly emphasis in recent times. Numerous scientific and other physical processes involve fractional derivatives, including ideological, aquifer, electrostatics, financing, and hydrodynamics, to highlight a few [1, 2, 18].

In 2019, an African professor, Abdon Atangana [7], pioneered a new type of derivative called fractal-fractional derivatives in the last few decades, which integrates the two key points of fractional and fractal calculus. He also expanded on the noteworthy fractal-fractional integral in the Caputo [29], Caputo-Fabrizo [19], and Atangana-Baleanu [8] contexts. These formulations are established by converging power-law, exponential-law, and generalized Mittag-Leffler-law sort kernels with fractal derivatives. The fractional-order and the fractal-dimension are the two constituents of fractal-fractional formulations. Differential equations (DEs) with the fractal-fractional derivative convert the order and dimension of the assumed scheme into a different structure or fractional-order system. This capability makes us able to generalize ordinary DEs to generalized structures to any attempt at derivatives and dimensions. However, the primary goal of identifying these derivatives is to investigate nonlocal BVPs/IVPs in existence that exhibits fractal behaviour. Several intellectuals identified numerous outcomes and constructed a few fractal-fractional frameworks that show good simulation studies for portraying mathematical formations in this general area. For example, Goméz-Aguilar et al. [5] investigated the spread of malaria by incorporating fractal-fractional

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operators. Rashid et al. [32] presented a novel fractal-fractional order model for the comprehension of an oscillatory and complex analysis of the human liver via a generalized Mittag-Leffler kernel.

In the past few years, inequalities have become increasingly important in all areas of mathematics. The notable character of inequalities in the growth and enlargement of mathematics is well known. In several areas of science and technology, mathematical inequalities have effectively spread their impact, and they are now recognized and imparted as some of the most useful mathematical disciplines. Information theory, economics, engineering, and other fields have benefited from their use [4, 11, 30].

Inequalities and their associated speculations have experienced significant growth, resulting in the creation of numerous innovative and generalized configurations of inequalities. Hardy's inequality, Minkowski inequality, Jensen inequality, and Hermite-Hadamard inequality, for example, [12, 21, 27], are several of the most well-known inequalities among researchers because of their diversified implementations in scientific disciplines. Scientific researchers are now extremely engaged in generalized inequalities, which encompass many of the variants above in one pattern or another. In recent past extension of inequalities on fractal sets is topic of supreme inters for many reproachers. Several useful inequalities have been constructed on fractal sets (see [13, 26, 33, 34, 38]) and references therein.

However, to our knowledge, the reverse Minkowski and associated variants with the assistance of fractal-fractional operators have not yet been calculated. This is the main motivation for this study. The famous Minkowski integral inequality is as follows:

Let $\mathfrak{p} \geq 1, 0 < \int_{\vartheta_1}^{\vartheta_2} \Upsilon^{\mathfrak{p}}(\varkappa)d\varkappa < \infty$ and $0 < \int_{\vartheta_1}^{\vartheta_2} \mathfrak{g}^{\mathfrak{p}}(\varkappa)d\varkappa < \infty$. Then

$$\left(\int_{\vartheta_1}^{\vartheta_2} (\Upsilon(\varkappa) + \mathfrak{g}(\varkappa))^{\mathfrak{p}} d\varkappa\right) \leq \left(\int_{\vartheta_1}^{\vartheta_2} \Upsilon^{\mathfrak{p}}(\varkappa)d\varkappa\right)^{\frac{1}{\mathfrak{p}}} + \left(\int_{\vartheta_1}^{\vartheta_2} \mathfrak{g}^{\mathfrak{p}}(\varkappa)d\varkappa\right)^{\frac{1}{\mathfrak{p}}}. \tag{1}$$

The reverse Minkowski and Hardy integral inequalities were studied in [10, 14, 15] along with several generalizations and improvisations. However, the first fractional technique to comply with reverse Minkowski inequalities were given in [20]. Anber et al. [6] proposed a recent proposal for some fractional integral inequalities within the framework of Riemann-Liouville fractional integral. By considering Katugampola's fractional techniques, the researchers in [36] investigated a few Minkowski inequalities and other variations. Many researchers have concentrated on finding the unique version of the reverse Minkowski inequality for generalized fractional conformable integrals using Hadamard fractional integral operators and generalized proportional fractional integral operators [28, 31, 37]. Some recent literature on fractal-fractional inequalities along with their applications can be observed in [16, 17, 22, 39] and references therein.

Fractional calculus, an extension of classical calculus, is frequently utilised in the sciences, especially engineering. In the majority of areas of applied sciences, classical calculus provides an outstanding tool for modelling and elucidating several vital dynamic processes. In order to combine and generalise integer-order differentiation with n-fold integration, the fractional calculus hypothesis was established. Fractional analysis is a part of the applied sciences. Numerous findings based on fractional models have been reported in a variety of scientific fields [23].

By offering more intricately connected answers to practical issues, the fractional operators of integral and derivative assist to deepen the connections between mathematics and other disciplines. Fractional integral and derivative operators have evolved over time [3, 24]. In their review article "Fractional Calculus in the Sky," [9] D. Baleanu and R.P. Agarwal, two esteemed professors, provide the most recent compact review of fractional calculus.

Motivated by the above literature, we intend to demonstrate the more general version of reverse Minkowski and associated variants with the aid of fractal-fractional operators. Considering the rigorous fractional operators, several new theorems and corollaries are derived. Inequalities of this type and their different versions are created when we have $\varpi = 1$ and $\varrho = 1$ in the obtained results. Meanwhile, the existing results are depicted in the form of remarks. We hope that this invigorating idea will also open new avenues in convexity, invexity, and inequality theory.

The structure of the article is as follows: Section 2 demonstrates the preliminary concepts of fractal-fractional derivative and integral operators. Section 3, subdivided into three sections, demonstrates the reverse Minkowski and related variants utilizing fractal-fractional integral operator having a power law kernel, an exponential decay kernel, and the generalized Mittag-Leffler kernel. In Section 4, we give some further new reverse fractional Minkowski type inequalities. Finally in Section 5, we give concluding remarks.

2. Preliminaries

Reviewing some fundamental concepts related to fractal-fractional operators is important before moving on to the mathematical description. The brief explanations of fractal-fractional calculus in relation to various operators are provided below:

Definition 2.1. [7] *We say that the fractal-fractional operator of $\varkappa(\kappa)$ having power law kernel in terms of Riemann-Liouville (RL) can be described as follows:*

$$^{FFP}\mathbb{D}_{0,\kappa}^{\varpi,\varrho}\varkappa(\kappa) = \frac{1}{\Gamma(\mathfrak{r}-\varpi)}\frac{d}{d\kappa^{\varrho}}\int_0^{\kappa}[\kappa-\tau]^{\mathfrak{r}-\varpi-1}\varkappa(\tau)d\tau, \tag{2}$$

where $\frac{d\varkappa(\tau)}{d\tau^{\varrho}} = \lim_{\kappa\to\tau}\frac{\varkappa(\kappa)-\varkappa(\tau)}{\kappa^{\varrho}-\kappa^{\tau}}$, $\mathfrak{r}-1<\varpi$ *and* $\varrho \le \mathfrak{r} \in \mathbb{N}$.

Definition 2.2. [7] *We say that the fractal-fractional operator of $\varkappa(\kappa)$ having exponential kernel in terms of RL can be described as follows:*

$$^{FFE}\mathbb{D}_{0,\kappa}^{\varpi,\varrho}\varkappa(\kappa) = \frac{\varkappa(\varpi)}{(1-\varpi)}\frac{d}{d\kappa^{\varrho}}\int_0^{\kappa}exp\left(\frac{\varpi}{(1-\varpi)}[\kappa-\tau]\right)\varkappa(\tau)d\tau, \tag{3}$$

such that $\mathfrak{q}(0)=\mathfrak{q}(1)=1$ *having* $\varpi>0$ *and* $\varrho\le\mathfrak{r}\in\mathbb{N}$.

Definition 2.3. [7] *We say that the fractal-fractional operator of $\varkappa(\kappa)$ having Mittag-Leffler kernel in terms of RL can be described as follows:*

$$^{FFM}\mathbb{D}_{0,\kappa}^{\varpi,\varrho}\varkappa(\kappa) = \frac{AB(\varpi)}{(1-\varpi)}\frac{d}{d\kappa^{\varrho}}\int_0^{\kappa}E_{\varpi}\left(\frac{\varpi}{(1-\varpi)}[\kappa-\tau]\right)\varkappa(\tau)d\tau, \tag{4}$$

such that $AB(\varpi)=1-\varpi+\frac{\varpi}{\Gamma(\varpi)}$ *having* $\varpi>0$ *and* $\varrho\le 1\in\mathbb{N}$.

Definition 2.4. [7] *The respective fractal-fractional integral version of* (2) *is stated as follows:*

$$^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\varkappa(\kappa) = \frac{\varrho}{\Gamma(\varpi)}\int_0^{\kappa}[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}\varkappa(\tau)d\tau.$$

Definition 2.5. [7] *The respective fractal-fractional integral version of* (3) *is stated as follows:*

$$^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\varkappa(\kappa) = \frac{\varrho\varpi}{\mathfrak{q}(\varpi)}\int_0^{\kappa}\tau^{\varrho-1}\varkappa(\tau)d\tau + \frac{\varrho(1-\varpi)\kappa^{\varrho-1}\varkappa(\kappa)}{\mathfrak{q}(\varpi)}.$$

Definition 2.6. [7] *The respective fractal-fractional integral version of* (4) *is stated as follows:*

$$^{FFM}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\varkappa(\kappa) = \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\varkappa(\kappa) + \int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\varkappa(\tau)d\tau.$$

3. Fractal-Fractional reverse Minkowski and associated variants

First, we will discuss the reverse Minkowski and related inequalities via the Caputo, Caputo-Fabrizo, and Atangana-Baleanu-Caputo senses.

3.1. Fractal-Fractional operator in the Caputo sense

Our first result is the following theorem.

Theorem 3.1. *Let $\nu > 0$ and $\mathfrak{p} \geq 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\mu(\kappa) < \infty$ and ${}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}_+^*$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq A\left[{}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \tag{5}$$

where

$$A = \frac{\theta(1+\varpi) + (\theta+1)}{(1+\varpi)(1+\theta)}. \tag{6}$$

Proof. From the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we have

$$\mu(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)(\mu(\kappa)+\nu(\kappa)). \tag{7}$$

Taking the $\mathfrak{p}th$ power on both sides of equation (7), we get

$$\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{8}$$

Replace $\kappa = \tau$ in (8) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we obtain

$$\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\mathfrak{p}}(\tau) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}. \tag{9}$$

Integrating both sides of (9) with respect to τ, we have

$$\int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau. \tag{10}$$

This implies

$${}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}{}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{11}$$

Taking $\frac{1}{\mathfrak{p}}th$ power on both sides (11), we have

$$\left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \left(\frac{\theta}{\theta+1}\right)\left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}. \tag{12}$$

On the other hand, by using the condition $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)}$, we directly get

$$\nu(\kappa) \leq \left(\frac{1}{\varpi+1}\right)(\mu(\kappa)+\nu(\kappa)). \tag{13}$$

Taking the $\mathfrak{p}th$ power on both sides of (13), then we have

$$\nu^{\mathfrak{p}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{14}$$

Replace $\kappa = \tau$ in (14) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we obtain

$$\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\nu^{\mathfrak{p}}(\tau) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}. \tag{15}$$

Integrating both sides of (15) with respect to τ, we have

$$\int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\nu^{\mathfrak{p}}(\tau)d\tau \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}} \int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau. \tag{16}$$

This implies

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}} {}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{17}$$

Taking $\frac{1}{\mathfrak{p}}th$ power on both sides (17), we have

$$\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \left(\frac{1}{\varpi+1}\right)\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}. \tag{18}$$

By (12) and (18), we have the required inequality

$$\left({}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \mathrm{A}\left[{}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}. \tag{19}$$

This completes the proof. □

Corollary 3.2. *Let* $\nu > 0$ *and* $\mathfrak{p} \geq 1$. *Let* $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ *be two positive functions in* $[0, \infty)$ *such that* ${}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) < \infty$ *and* ${}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) < \infty$ *for all* $\kappa > \vartheta_1$. *If* $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ *for some* $\varpi, \theta \in \mathbb{R}^*_+$ *and all* $\kappa \in [\vartheta_1, \vartheta_2]$, *then*

$$\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq A\left[{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \tag{20}$$

where A is defined by (6).

Remark 3.3. *(i) If* $\varrho = 1$, *then Theorem 3.1 simplifies to the result proposed by Dahmani [20].*

(ii) If $\varrho = \varpi = 1$, *then Theorem 3.1 simplifies to the result proposed by Set et al. [35].*

Theorem 3.4. *Let* $\nu > 0$ *and* $\mathfrak{p}, \mathfrak{q} > 1$ *with* $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. *Let* $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ *be two positive functions in* $[0, \infty)$ *such that* ${}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu(\kappa) < \infty$ *and* ${}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu(\kappa) < \infty$ *for all* $\kappa > \vartheta_1$. *If* $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ *for some* $\varpi, \theta \in \mathbb{R}^*_+$ *and all* $\kappa \in [\vartheta_1, \vartheta_2]$, *then*

$$\left({}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu(\kappa)\right)^{\frac{1}{\mathfrak{p}}}\left({}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu(\kappa)\right)^{\frac{1}{\mathfrak{q}}} \leq \left(\frac{\theta}{\varpi}\right)^{\frac{1}{\mathfrak{pq}}}\left[{}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa))\right]. \tag{21}$$

Proof. Using the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we have

$$\mu^{\frac{1}{\mathfrak{q}}}(\kappa) \leq \theta^{\frac{1}{\mathfrak{q}}}\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{22}$$

Multiplying by $\mu^{\frac{1}{\mathfrak{p}}}(\kappa)$ on both sides of (22), we get

$$\mu(\kappa) \leq \mu^{\frac{1}{\mathfrak{p}}}(\kappa)\theta^{\frac{1}{\mathfrak{q}}}\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{23}$$

Replace $\kappa = \tau$ in (23) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we obtain

$$\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau) \leq \theta^{\frac{1}{\mathfrak{q}}}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau). \tag{24}$$

Integrating both sides of (24) with respect to τ, we have

$$\int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau)d\tau \leq \theta^{\frac{1}{\mathfrak{q}}} \int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau)d\tau. \tag{25}$$

This implies

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) \leq \theta^{\frac{1}{\mathfrak{q}}}\,{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}[\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa)]. \tag{26}$$

Taking $\frac{1}{\mathfrak{p}}th$ power on both sides (26), we get

$$\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \theta^{\frac{1}{\mathfrak{pq}}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\left[\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa)\right]\right)^{\frac{1}{\mathfrak{p}}}. \tag{27}$$

Using the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we obtain

$$\nu^{\frac{1}{\mathfrak{p}}}(\kappa) \leq \varpi^{-\frac{1}{\mathfrak{p}}}\mu^{\frac{1}{\mathfrak{p}}}(\kappa). \tag{28}$$

Multiplying by $\nu^{\frac{1}{\mathfrak{q}}}(\kappa)$ on both sides of (28), we have

$$\nu(\kappa) \leq \nu^{\frac{1}{\mathfrak{q}}}(\kappa)\varpi^{-\frac{1}{\mathfrak{p}}}\mu^{\frac{1}{\mathfrak{p}}}(\kappa). \tag{29}$$

Replace $\kappa = \tau$ in (29) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we have

$$\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\nu(\tau) \leq \varpi^{-\frac{1}{\mathfrak{p}}}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau). \tag{30}$$

Integrating both sides of (30) with respect to τ, we get

$$\int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\nu(\tau)d\tau \leq \varpi^{-\frac{1}{\mathfrak{p}}}\int_0^\kappa \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau)d\tau. \tag{31}$$

This implies

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) \leq \varpi^{-\frac{1}{\mathfrak{p}}}\,{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}[\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa)]. \tag{32}$$

Taking $\frac{1}{\mathfrak{q}}th$ power on both sides (32), we have

$$\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa)\right)^{\frac{1}{\mathfrak{q}}} \leq \varpi^{-\frac{1}{\mathfrak{pq}}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\left[\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa)\right]\right)^{\frac{1}{\mathfrak{q}}}. \tag{33}$$

By (27) and (33), we have the required inequality. This completes the proof. □

Corollary 3.5. *Let $\nu > 0$ and $\mathfrak{p}, \mathfrak{q} > 1$ with $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) < \infty$ and ${}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\right)^{\frac{1}{\mathfrak{p}}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa)\right)^{\frac{1}{\mathfrak{q}}} \leq \left(\frac{\theta}{\varpi}\right)^{\frac{1}{\mathfrak{pq}}}\left[{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa))\right]. \tag{34}$$

Remark 3.6. *(i) If $\varrho = 1$, then Theorem 3.4 simplifies to the result proposed by Dahmani [20].*

(ii) If $\varrho = \varpi = 1$, then Theorem 3.4 simplifies to the result proposed by Set et al. [35].

Theorem 3.7. *Let $\nu > 0$ and $\mathfrak{p}, \mathfrak{q} > 1$ with $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\mu(\kappa) < \infty$ and ${}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \le \frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$ for some $\varpi, \theta \in \mathbb{R}_+^*$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$ {}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa) + \nu(\kappa)) \le \Phi\, {}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu^{\mathfrak{p}}(\kappa) + \nu^{\mathfrak{p}}(\kappa)) + \Psi\, {}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu^{\mathfrak{q}}(\kappa) + \nu^{\mathfrak{q}}(\kappa)), \tag{35} $$

where

$$ \Phi = \frac{2^{\mathfrak{p}-1}\theta^{\mathfrak{p}}}{\mathfrak{p}(\theta + 1)^{\mathfrak{p}}} \tag{36} $$

and

$$ \Psi = \frac{2^{\mathfrak{q}-1}}{\mathfrak{q}(\varpi + 1)^{\mathfrak{q}}}. \tag{37} $$

Proof. From the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$, we have

$$ \mu(\kappa) \le \left(\frac{\theta}{\theta + 1}\right)(\mu(\kappa) + \nu(\kappa)) \tag{38} $$

Taking the $\mathfrak{p}th$ power on both sides of equation (38), we have

$$ \mu^{\mathfrak{p}}(\kappa) \le \left(\frac{\theta}{\theta + 1}\right)^{\mathfrak{p}} (\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{39} $$

Replace $\kappa = \tau$ in equation (39) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we have

$$ \frac{\varrho[\kappa - \tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\mathfrak{p}}(\tau) \le \left(\frac{\theta}{\theta + 1}\right)^{\mathfrak{p}} \frac{\varrho[\kappa - \tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau) + \nu(\tau))^{\mathfrak{p}}. \tag{40} $$

Integrating both sides of e(40) with respect to τ, we have

$$ \int_0^\kappa \frac{\varrho[\kappa - \tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \le \left(\frac{\theta}{\theta + 1}\right)^{\mathfrak{p}} \int_0^\kappa \frac{\varrho[\kappa - \tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau) + \nu(\tau))^{\mathfrak{p}}d\tau. \tag{41} $$

This implies

$$ {}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}\mu^{\mathfrak{p}}(\kappa) \le \left(\frac{\theta}{\theta + 1}\right)^{\mathfrak{p}} {}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{42} $$

Multiplying $\frac{1}{\mathfrak{p}}$ on both sides (42), we have

$$ \frac{1}{\mathfrak{p}}\left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}\mu^{\mathfrak{p}}(\kappa)\right) \le \frac{1}{\mathfrak{p}}\left(\frac{\theta}{\theta + 1}\right)^{\mathfrak{p}} \left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}\right). \tag{43} $$

On the other hand, by using the condition $0 < \varpi \le \frac{\mu(\kappa)}{\nu(\kappa)}$, we directly get

$$ \nu(\kappa) \le \left(\frac{1}{\varpi + 1}\right)(\mu(\kappa) + \nu(\kappa)). \tag{44} $$

Taking the $\mathfrak{q}th$ power on both sides of (44), we have

$$ \nu^{\mathfrak{q}}(\kappa) \le \left(\frac{1}{\varpi + 1}\right)^{\mathfrak{q}} (\mu(\kappa) + \nu(\kappa))^{\mathfrak{q}}. \tag{45} $$

Replace $\kappa = \tau$ in (45) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we have

$$\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\nu^{\mathfrak{q}}(\tau) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{q}}. \tag{46}$$

Integrating both sides of (46) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\nu^{\mathfrak{q}}(\tau)d\tau \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \int_0^{\kappa} \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{q}}d\tau. \tag{47}$$

This implies

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{q}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} {}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}. \tag{48}$$

Multiplying $\frac{1}{\mathfrak{q}}$ on both sides (48), we have

$$\frac{1}{\mathfrak{q}}({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)) \leq \frac{1}{\mathfrak{q}}\left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} ({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}). \tag{49}$$

By (43) and (49), we have the required inequality

$$\begin{aligned} &\frac{1}{\mathfrak{p}}({}^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu^{\mathfrak{q}}(\kappa)) + \frac{1}{\mathfrak{q}}({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)) \\ &\leq \frac{1}{\mathfrak{p}}\left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right) + \frac{1}{\mathfrak{q}}\left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}\right). \end{aligned} \tag{50}$$

To complete our proof, we have to use Young's inequality,

$$\mu(\kappa))\nu(\kappa)) \leq \frac{\mu^{\mathfrak{p}}(\kappa))}{\mathfrak{p}} + \frac{\nu^{\mathfrak{q}}(\kappa))}{\mathfrak{q}} \tag{51}$$

Replace $\kappa = \tau$ in (51) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$ then we have

$$\mu(\tau)\nu(\tau) \leq \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\tau)}{\mathfrak{p}} + \frac{\nu^{\mathfrak{q}}(\tau)}{\mathfrak{q}}\right]. \tag{52}$$

Integrating both sides of (52) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau)\nu(\tau)d\tau \leq \int_0^{\kappa} \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\tau)}{\mathfrak{p}} + \frac{\nu^{\mathfrak{q}}(\tau)}{\mathfrak{q}}\right]d\tau. \tag{53}$$

This implies

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\nu(\kappa) \leq \frac{1}{\mathfrak{p}}{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa) + \frac{1}{\mathfrak{q}}{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{q}}(\kappa). \tag{54}$$

Using (50) and (54), we have

$$\begin{aligned} &{}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\nu(\kappa) \\ &\leq \frac{1}{\mathfrak{p}}\left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right) + \frac{1}{\mathfrak{q}}\left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}}\left({}^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}\right). \end{aligned} \tag{55}$$

Using the inequality

$$(\mu(\kappa)+\nu(\kappa))^{\mathfrak{r}} \leq 2^{\mathfrak{r}-1}(\mu^{\mathfrak{r}}(\kappa)+\nu^{\mathfrak{r}}(\kappa)) \quad \mu,\nu \geq 0,\ \mathfrak{r} > 0. \tag{56}$$

Replace $\kappa = \tau$ with $\mathfrak{r} = \mathfrak{p}$ in (56) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we have

$$\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu^{\mathfrak{p}}(\tau)+\nu^{\mathfrak{p}}(\tau)). \tag{57}$$

Integrating both sides of (57) with respect to τ, we have

$$\begin{aligned}&\int_0^{\kappa}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau\\ &\leq 2^{\mathfrak{p}-1}\int_0^{\kappa}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}(\mu^{\mathfrak{p}}(\tau)+\nu^{\mathfrak{p}}(\tau))d\tau.\end{aligned} \tag{58}$$

This implies

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1\,FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa)). \tag{59}$$

Repeating the same process with $\mathfrak{r} = \mathfrak{q}$, we get

$$^{FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}} \leq 2^{\mathfrak{q}-1\,FFP}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\mathfrak{q}}(\kappa)+\nu^{\mathfrak{q}}(\kappa)). \tag{60}$$

Substituting (59) and (60) into (55), this completes the proof. □

Theorem 3.8. *Let $\nu > 0$ and $\mathfrak{p} \geq 1$. Let $\mu, \nu \in C_{\nu}[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that $^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu(\kappa) < \infty$ and $^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\frac{1}{\theta}\left(^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu(\kappa)\nu(\kappa))\right) \leq^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^2 \leq \frac{1}{\varpi}\left(^{FFP}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu(\kappa)\nu(\kappa))\right). \tag{61}$$

Proof. Using the condition, we have

$$0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta, \tag{62}$$

we conclude that

$$(1+\varpi)\nu(\kappa) \leq (\mu(\kappa)+\nu(\kappa)) \leq (1+\theta)\nu(\kappa) \tag{63}$$

and

$$\frac{1+\theta}{\theta}\mu(\kappa) \leq (\mu(\kappa)+\nu(\kappa)) \leq \frac{1+\varpi}{\varpi}\mu(\kappa). \tag{64}$$

By (63) and (64), we have

$$\frac{1}{\theta}(\mu(\kappa)\nu(\kappa)) \leq \frac{(\mu(\kappa)+\nu(\kappa))^2}{(1+\varpi)(1+\theta)} \leq \frac{1}{\varpi}(\mu(\kappa)\nu(\kappa)). \tag{65}$$

Replace $\kappa = \tau$ in (65) and multiply by $\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}$, we have

$$\begin{aligned}\frac{1}{\theta}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau)\nu(\tau) &\leq \frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)}\\ &\leq \frac{1}{\varpi}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau)\nu(\tau).\end{aligned} \tag{66}$$

Integrating both sides of (66) with respect to τ, we have

$$\frac{1}{\theta}\int_0^{\kappa}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau)\nu(\tau)d\tau \le \int_0^{\kappa}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)}d\tau \le \frac{1}{\varpi}\int_0^{\kappa}\frac{\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{\Gamma(\varpi)}\mu(\tau)\nu(\tau)d\tau. \tag{67}$$

This implies

$$\frac{1}{\theta}\left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)\nu(\kappa))\right) \le {}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)+\nu(\kappa))^2 \le \frac{1}{\varpi}\left({}^{FFP}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)\nu(\kappa))\right).$$

Therefore, the proof is completed. □

3.2. *Fractal-Fractional operator in the Caputo-Fabrizio sense*

New Minkowski type inequalities utilizing fractal-fractional integral operator pertaining exponential kernel is presented in this section.

Theorem 3.9. *Let $\nu > 0$ and $\mathfrak{p} \ge 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\mu(\kappa) < \infty$ and ${}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \le \frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$ for some $\varpi, \theta \in \mathbb{R}_+^*$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\left({}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \le A\left[{}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \tag{68}$$

where A is defined by (6).

Proof. From the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$, we have

$$\mu(\kappa) \le \left(\frac{\theta}{\theta+1}\right)(\mu(\kappa)+\nu(\kappa)). \tag{69}$$

Taking the $\mathfrak{p}th$ power on both sides of equation (69), we have

$$\mu^{\mathfrak{p}}(\kappa) \le \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{70}$$

Multiplying (70) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathrm{q}(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathrm{q}(\varpi)}\mu^{\mathfrak{p}}(\kappa) \le \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathrm{q}(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{71}$$

Replace $\kappa = \tau$ in (70) and multiply by $\frac{\varpi\varrho\tau^{\varrho-1}}{\mathrm{q}(\varpi)}$, we have

$$\frac{\varpi\varrho\tau^{\varrho-1}}{\mathrm{q}(\varpi)}\mu^{\mathfrak{p}}(\tau) \le \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{\varpi\varrho\tau^{\varrho-1}}{\mathrm{q}(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}. \tag{72}$$

Integrating both sides of (72) with respect to τ, we have

$$\int_0^{\kappa}\frac{\varpi\varrho\tau^{\varrho-1}}{\mathrm{q}(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \le \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\int_0^{\kappa}\frac{\varpi\varrho\tau^{\varrho-1}}{\mathrm{q}(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau. \tag{73}$$

The proof can be followed similarly as Theorem 3.1. □

Theorem 3.10. *Let $\nu > 0$ and $\mathfrak{p}, \mathfrak{q} > 1$ with $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu(\kappa) < \infty$ and ${}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \le \frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\left({}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \left({}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu(\kappa)\right)^{\frac{1}{\mathfrak{q}}} \le \left(\frac{\theta}{\varpi}\right)^{\frac{1}{\mathfrak{p}\mathfrak{q}}} \left[{}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa))\right]. \tag{74}$$

Proof. Using the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$, we have

$$\mu^{\frac{1}{\mathfrak{q}}}(\kappa) \le \theta^{\frac{1}{\mathfrak{q}}}\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{75}$$

Multiplying by $\mu^{\frac{1}{\mathfrak{p}}}(\kappa)$ on both sides of (75), we have

$$\mu(\kappa) \le \mu^{\frac{1}{\mathfrak{p}}}(\kappa)\theta^{\frac{1}{\mathfrak{q}}}\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{76}$$

Multiplying (76) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathfrak{q}(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu(\kappa) \le \theta^{\frac{1}{\mathfrak{q}}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{77}$$

Replace $\kappa = \tau$ in (76) and multiply by $\frac{\varpi\varrho\tau^{\varrho-1}}{\mathfrak{q}(\varpi)}$, we have

$$\frac{\varpi\varrho\tau^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu(\tau) \le \theta^{\frac{1}{\mathfrak{q}}}\frac{\varpi\varrho\tau^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau). \tag{78}$$

Integrating both sides of (78) with respect to τ, we have

$$\int_0^\kappa \frac{\varpi\varrho\tau^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu(\tau)d\tau \le \theta^{\frac{1}{\mathfrak{q}}}\int_0^\kappa \frac{\varpi\varrho\tau^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau)d\tau. \tag{79}$$

The proof can be followed similarly as Theorem 3.4. □

Theorem 3.11. *Let $\nu > 0$ and $\mathfrak{p}, \mathfrak{q} > 1$ with $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\mu(\kappa) < \infty$ and ${}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \le \frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$${}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu(\kappa) + \nu(\kappa)) \le \Phi\, {}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu^{\mathfrak{p}}(\kappa) + \nu^{\mathfrak{p}}(\kappa)) + \Psi\, {}^{FFE}\mathbb{J}^{\varpi,\varrho}_{0,\kappa}(\mu^{\mathfrak{q}}(\kappa) + \nu^{\mathfrak{q}}(\kappa)), \tag{80}$$

where Φ and Ψ are defined by (36) and (37), respectively.

Proof. From the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \le \theta$, we have

$$\mu(\kappa) \le \left(\frac{\theta}{\theta+1}\right)(\mu(\kappa) + \nu(\kappa)) \tag{81}$$

Taking the $\mathfrak{p}th$ power on both sides of (81), we have

$$\mu^{\mathfrak{p}}(\kappa) \le \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{82}$$

Multiplying (82) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathfrak{q}(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathfrak{q}(\varpi)}\mu^{\mathfrak{p}}(\kappa) \le \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{\mathfrak{q}(\varpi)}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{83}$$

Replace $\kappa = \tau$ in (82) and multiply by $\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}$, we have

$$\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\mu^{\mathfrak{p}}(\tau) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}. \tag{84}$$

Integrating both sides of (84) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \int_0^{\kappa} \frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau. \tag{85}$$

The proof can be followed similarly as Theorem 3.7. □

Theorem 3.12. *Let $\nu > 0$ and $\mathfrak{p} \geq 1$. Let $\mu, \nu \in C_{\nu}[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\mu(\kappa) < \infty$ and ${}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}_+^*$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\frac{1}{\theta}\left({}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)\nu(\kappa))\right) \leq {}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)+\nu(\kappa))^2 \leq \frac{1}{\varpi}\left({}^{FFE}\mathbb{J}_{0,\kappa}^{\varpi,\varrho}(\mu(\kappa)\nu(\kappa))\right). \tag{86}$$

Proof. Using the condition, we have

$$0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta, \tag{87}$$

we conclude that

$$(1+\varpi)\nu(\kappa) \leq (\mu(\kappa)+\nu(\kappa)) \leq (1+\theta)\nu(\kappa), \tag{88}$$

and

$$\frac{1+\theta}{\theta}\mu(\kappa) \leq (\mu(\kappa)+\nu(\kappa)) \leq \frac{1+\varpi}{\varpi}\mu(\kappa). \tag{89}$$

By (88) and (89), we have

$$\frac{1}{\theta}(\mu(\kappa)\nu(\kappa)) \leq \frac{(\mu(\kappa)+\nu(\kappa))^2}{(1+\varpi)(1+\theta)} \leq \frac{1}{\varpi}(\mu(\kappa)\nu(\kappa)). \tag{90}$$

Multiplying (90) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{q(\varpi)}$, we have

$$\begin{aligned}\frac{1}{\theta}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{q(\varpi)}\mu(\kappa)\nu(\kappa) &\leq \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{q(\varpi)}\frac{(\mu(\kappa)+\nu(\kappa))^2}{(1+\varpi)(1+\theta)} \\ &\leq \frac{1}{\varpi}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{q(\varpi)}\mu(\kappa)\nu(\kappa).\end{aligned} \tag{91}$$

Replace $\kappa = \tau$ in (90) and multiply by $\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}$, we have

$$\begin{aligned}\frac{1}{\theta}\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\mu(\tau)\nu(\tau) &\leq \frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)} \\ &\leq \frac{1}{\varpi}\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\mu(\tau)\nu(\tau).\end{aligned} \tag{92}$$

Integrating both sides of (92) with respect to τ, we have

$$\begin{aligned}\frac{1}{\theta}\int_0^{\kappa}\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\mu(\tau)\nu(\tau)d\tau &\leq \int_0^{\kappa}\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)}d\tau \\ &\leq \frac{1}{\varpi}\int_0^{\kappa}\frac{\varpi\varrho\tau^{\varrho-1}}{q(\varpi)}\mu(\tau)\nu(\tau)d\tau.\end{aligned} \tag{93}$$

The proof can be followed similarly as Theorem 3.8. □

3.3. Fractal-Fractional operator in the Atangana-Baleanu-Caputo sense

Theorem 3.13. *Let $\nu > 0$ and $\mathfrak{p} \geq 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) < \infty$ and ${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq A\left[{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\left(\mu^{\mathfrak{p}}(\kappa) + \nu^{\mathfrak{p}}(\kappa)\right)\right]^{\frac{1}{\mathfrak{p}}}, \tag{94}$$

where A is defined by (6).

Proof. From the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we have

$$\mu(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)(\mu(\kappa) + \nu(\kappa)). \tag{95}$$

Taking the $\mathfrak{p}th$ power on both sides of (95), then we have

$$\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{96}$$

Multiplying (96) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, then we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{97}$$

Replace $\kappa = \tau$ in (96) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau) + \nu(\tau))^{\mathfrak{p}}. \tag{98}$$

Integrating both sides of (98) with respect to τ, we have

$$\int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau) + \nu(\tau))^{\mathfrak{p}}d\tau. \tag{99}$$

Add (99) and (97), we have

$$\begin{aligned}
&\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\kappa) + \int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \\
&\leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}} \\
&\quad + \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau) + \nu(\tau))^{\mathfrak{p}}d\tau.
\end{aligned}$$

This implies

$${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}. \tag{100}$$

Taking $\frac{1}{\mathfrak{p}}th$ power on both sides (100), we have

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \left(\frac{\theta}{\theta+1}\right)\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa) + \nu(\kappa))^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}. \tag{101}$$

On the other hand, by using the condition $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)}$, we directly get

$$\nu(\kappa) \leq \left(\frac{1}{\varpi+1}\right)(\mu(\kappa)+\nu(\kappa)). \tag{102}$$

Taking the $\mathfrak{p}th$ power on both sides of (102), we have

$$\nu^{\mathfrak{p}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{103}$$

Multiplying (103) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{p}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{104}$$

Replace $\kappa = \tau$ in (103) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{p}}(\tau) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}. \tag{105}$$

Integrating both sides of (105) with respect to τ, we have

$$\int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{p}}(\tau)d\tau \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}\int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau. \tag{106}$$

Add (106) and (104), we have

$$\begin{aligned}
&\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{p}}(\kappa) + \int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \\
&\leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\nu(\kappa)+\nu(\kappa))^{\mathfrak{p}} \\
&\quad + \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}\int_0^{\kappa}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau.
\end{aligned}$$

This implies

$$^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{p}}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{107}$$

Taking $\frac{1}{\mathfrak{p}}th$ power on both sides (107), we have

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \left(\frac{1}{\varpi+1}\right)({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}. \tag{108}$$

By (101) and (108), we have the required inequality

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \mathrm{A}\left[{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa))\right]^{\frac{1}{\mathfrak{p}}}. \tag{109}$$

This completes the proof. □

Remark 3.14. *(i) If we set $\varpi = 1 = \varrho$ in Theorem 3.13, then we obtain [10, Theorem 1.2].*

(ii) If we set $\varrho = 1$ in Theorem 3.13, then we get

$$\left({}^{AB}_{0}I^{\varpi}_{\kappa}\mu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} + \left({}^{AB}_{0}I^{\varpi}_{\kappa}\nu^{\mathfrak{p}}(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq A\left[{}^{AB}_{0}I^{\varpi}_{\kappa}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa))\right]^{\frac{1}{\mathfrak{p}}},$$

which is proved by Khan et al. in [25], when $\vartheta_1 \to 0$.

4. Other type of Inequalities

In this section we will present some further reverse Minkowski type inequalities for fractal-fractional integral operators.

Theorem 4.1. *Let $\nu > 0$ and $\mathfrak{p}, \mathfrak{q} > 1$ with $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) < \infty$ and ${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa)\right)^{\frac{1}{\mathfrak{q}}} \leq \left(\frac{\theta}{\varpi}\right)^{\frac{1}{\mathfrak{pq}}} \left[{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa))\right]. \tag{110}$$

Proof. Using the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we have

$$\mu^{\frac{1}{\mathfrak{q}}}(\kappa) \leq \theta^{\frac{1}{\mathfrak{q}}}\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{111}$$

Multiplying by $\mu^{\frac{1}{\mathfrak{p}}}(\kappa)$ on both sides of equation (111), we have

$$\mu(\kappa) \leq \mu^{\frac{1}{\mathfrak{p}}}(\kappa)\theta^{\frac{1}{\mathfrak{q}}}\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{112}$$

Multiplying (112) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa) \leq \theta^{\frac{1}{\mathfrak{q}}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa). \tag{113}$$

Replace $\kappa = \tau$ in (112) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau) \leq \theta^{\frac{1}{\mathfrak{q}}}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau). \tag{114}$$

Integrating both sides of (114) with respect to τ, we have

$$\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)d\tau \leq \theta^{\frac{1}{\mathfrak{q}}}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau)d\tau. \tag{115}$$

Add (115) and (113), we have

$$\begin{aligned}&\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa) + \int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)d\tau \\ &\leq \theta^{\frac{1}{\mathfrak{q}}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa) + \theta^{\frac{1}{\mathfrak{q}}}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{\mathfrak{p}}}(\tau)\nu^{\frac{1}{\mathfrak{q}}}(\tau)d\tau.\end{aligned}$$

This implies

$${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) \leq \theta^{\frac{1}{\mathfrak{q}}}\,{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\left[\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa)\right]. \tag{116}$$

Taking $\frac{1}{\mathfrak{p}}th$ power on both sides (116), we have

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\right)^{\frac{1}{\mathfrak{p}}} \leq \theta^{\frac{1}{\mathfrak{pq}}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\left[\mu^{\frac{1}{\mathfrak{p}}}(\kappa)\nu^{\frac{1}{\mathfrak{q}}}(\kappa)\right]\right)^{\frac{1}{\mathfrak{p}}}. \tag{117}$$

Using the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we have

$$\nu^{\frac{1}{\mathfrak{p}}}(\kappa) \leq \varpi^{-\frac{1}{\mathfrak{p}}}\mu^{\frac{1}{\mathfrak{p}}}(\kappa). \tag{118}$$

Multiplying by $\nu^{\frac{1}{q}}(\kappa)$ on both sides of (118), we have

$$\nu(\kappa) \leq \nu^{\frac{1}{q}}(\kappa)\varpi^{-\frac{1}{p}}\mu^{\frac{1}{p}}(\kappa). \tag{119}$$

Multiplying (119) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\nu(\kappa) \leq \varpi^{-\frac{1}{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{p}}(\kappa)\nu^{\frac{1}{q}}(\kappa). \tag{120}$$

Replace $\kappa = \tau$ in (119) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu(\tau) \leq \varpi^{-\frac{1}{p}}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{p}}(\tau)\nu^{\frac{1}{q}}(\tau). \tag{121}$$

Integrating both sides of (121) with respect to τ, we have

$$\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu(\tau)d\tau \leq \varpi^{-\frac{1}{p}}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{p}}(\tau)\nu^{\frac{1}{q}}(\tau)d\tau. \tag{122}$$

Add (122) and (120), we have

$$\begin{aligned}&\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\nu(\kappa) + \int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu(\tau)d\tau\\ &\leq \varpi^{-\frac{1}{p}}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{p}}(\kappa)\nu^{\frac{1}{q}}(\kappa) + \varpi^{-\frac{1}{p}}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\frac{1}{p}}(\tau)\nu^{\frac{1}{q}}(\tau)d\tau.\end{aligned}$$

This implies

$$^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) \leq \varpi^{-\frac{1}{p}}\,{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\left[\mu^{\frac{1}{p}}(\kappa)\nu^{\frac{1}{q}}(\kappa)\right]. \tag{123}$$

Taking $\frac{1}{q}th$ power on both sides (123), we have

$$\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa)\right)^{\frac{1}{q}} \leq \varpi^{-\frac{1}{pq}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\left[\mu^{\frac{1}{p}}(\kappa)\nu^{\frac{1}{q}}(\kappa)\right]\right)^{\frac{1}{q}}. \tag{124}$$

By equations (117) and (124), we have the required inequality. □

Remark 4.2. *If we set $\varrho = 1$ in Theorem 4.1, then we get*

$$\left({}^{AB}_0I^{\varpi}_{\kappa}\mu(\kappa)\right)^{\frac{1}{p}}\left({}^{AB}_0I^{\varpi}_{\kappa}\nu(\kappa)\right)^{\frac{1}{q}} \leq \left(\frac{\theta}{\varpi}\right)^{\frac{1}{pq}}\left[{}^{AB}_0I^{\varpi}_{\kappa}(\mu^{\frac{1}{p}}(\kappa)\nu^{\frac{1}{q}}(\kappa))\right], \tag{125}$$

which is proved by Khan et al. in [25], when $\vartheta_1 \to 0$.

Theorem 4.3. *Let $\nu > 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa) < \infty$ and ${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}^*_+$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa)) \leq \Phi\,{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{p}(\kappa)+\nu^{p}(\kappa)) + \Psi\,{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{q}(\kappa)+\nu^{q}(\kappa)), \tag{126}$$

where Φ and Ψ are defined by (36) *and* (37)*, respectively.*

Proof. From the given condition $\frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$, we have

$$\mu(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)(\mu(\kappa)+\nu(\kappa)) \tag{127}$$

Taking the $\mathfrak{p}th$ power on both sides of (127), then we have

$$\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} (\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{128}$$

Multiplying (128) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{129}$$

Replace $\kappa=\tau$ in (128) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}. \tag{130}$$

Integrating both sides of (130) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau. \tag{131}$$

Add (131) and (129), we have

$$\begin{aligned}
&\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\kappa) + \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \\
&\leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}} \\
&\quad + \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau.
\end{aligned}$$

This implies

$$^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa) \leq \left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} {}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}. \tag{132}$$

Multiplying $\frac{1}{\mathfrak{p}}$ on both sides (132), then we have

$$\frac{1}{\mathfrak{p}}({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)) \leq \frac{1}{\mathfrak{p}}\left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} ({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}). \tag{133}$$

On the other hand, by using the condition $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)}$, we directly get

$$\nu(\kappa) \leq \left(\frac{1}{\varpi+1}\right)(\mu(\kappa)+\nu(\kappa)). \tag{134}$$

Taking the $\mathfrak{q}th$ power on both sides of (134), we have

$$\nu^{\mathfrak{q}}(\kappa) \leq \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} (\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}. \tag{135}$$

Multiplying (135) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{q}}(\kappa) \le \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}. \tag{136}$$

Replace $\kappa = \tau$ in (135) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{q}}(\tau) \le \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{q}}. \tag{137}$$

Integrating both sides of (137) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{q}}(\tau)d\tau \le \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{q}}d\tau. \tag{138}$$

Add (138) and (136), we have

$$\begin{aligned}
&\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\nu^{\mathfrak{q}}(\kappa) + \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu^{\mathfrak{p}}(\tau)d\tau \\
&\le \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\nu(\kappa)+\nu(\kappa))^{\mathfrak{q}} \\
&\quad + \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{q}}d\tau.
\end{aligned} \tag{139}$$

This implies

$${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{q}}(\kappa) \le \left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} {}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}. \tag{140}$$

Multiplying $\frac{1}{\mathfrak{q}}$ on both sides (140), we have

$$\frac{1}{\mathfrak{q}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right) \le \frac{1}{\mathfrak{q}}\left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} \left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}\right). \tag{141}$$

By (133) and (141), we have the required inequality

$$\begin{aligned}
\frac{1}{\mathfrak{p}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{q}}(\kappa)\right) + \frac{1}{\mathfrak{q}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{p}}(\kappa)\right) \le \frac{1}{\mathfrak{p}}\left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}} &\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right) \\
+ \frac{1}{\mathfrak{q}}\left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}} &\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}\right).
\end{aligned} \tag{142}$$

To complete our proof, we have to use Young's inequality

$$\mu(\kappa))\nu(\kappa)) \le \frac{\mu^{\mathfrak{p}}(\kappa))}{\mathfrak{p}} + \frac{\nu^{\mathfrak{q}}(\kappa))}{\mathfrak{q}} \tag{143}$$

Multiplying (143) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa)\nu(\kappa) \le \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\kappa)}{\mathfrak{p}} + \frac{\nu^{\mathfrak{q}}(\kappa)}{\mathfrak{q}}\right]. \tag{144}$$

Replace $\kappa = \tau$ in (143) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau) \le \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\tau)}{\mathfrak{p}} + \frac{\nu^{\mathfrak{q}}(\tau)}{\mathfrak{q}}\right]. \tag{145}$$

Integrating both sides of (145) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau)d\tau \leq \int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\tau)}{\mathfrak{p}}+\frac{\nu^{\mathfrak{q}}(\tau)}{\mathfrak{q}}\right]d\tau. \tag{146}$$

Add (146) and (144), we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa)\nu(\kappa)+\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau)d\tau$$
$$\leq \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\kappa)}{\mathfrak{p}}+\frac{\nu^{\mathfrak{q}}(\kappa)}{\mathfrak{q}}\right]+\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\left[\frac{\mu^{\mathfrak{p}}(\tau)}{\mathfrak{p}}+\frac{\nu^{\mathfrak{q}}(\tau)}{\mathfrak{q}}\right]d\tau.$$

This implies

$${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\nu(\kappa) \leq \frac{1}{\mathfrak{p}}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu^{\mathfrak{p}}(\kappa)+\frac{1}{\mathfrak{q}}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\nu^{\mathfrak{q}}(\kappa). \tag{147}$$

Using (142) and (147), we have

$$\begin{aligned}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}\mu(\kappa)\nu(\kappa) \leq{}& \frac{1}{\mathfrak{p}}\left(\frac{\theta}{\theta+1}\right)^{\mathfrak{p}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}\right)\\ &+\frac{1}{\mathfrak{q}}\left(\frac{1}{\varpi+1}\right)^{\mathfrak{q}}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}}\right).\end{aligned} \tag{148}$$

Using the inequality

$$(\mu(\kappa)+\nu(\kappa))^{\mathfrak{r}} \leq 2^{\mathfrak{r}-1}(\mu^{\mathfrak{r}}(\kappa)+\nu^{\mathfrak{r}}(\kappa)), \quad \mu,\nu \geq 0,\ \mathfrak{r}>0. \tag{149}$$

With $\mathfrak{r}=\mathfrak{p}$ and multiplying (149) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa)). \tag{150}$$

Replace $\kappa=\tau$ with $\mathfrak{r}=\mathfrak{p}$ in (149) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu^{\mathfrak{p}}(\tau)+\nu^{\mathfrak{p}}(\tau)). \tag{151}$$

Integrating both sides of (151) with respect to τ, we have

$$\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau \leq 2^{\mathfrak{p}-1}\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu^{\mathfrak{p}}(\tau)+\nu^{\mathfrak{p}}(\tau))d\tau. \tag{152}$$

Add (152) and (150), we have

$$\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}}+\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu(\tau)+\nu(\tau))^{\mathfrak{p}}d\tau$$
$$\leq 2^{\mathfrak{p}-1}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa))+2^{\mathfrak{p}-1}\int_0^{\kappa} \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}(\mu^{\mathfrak{p}}(\tau)+\nu^{\mathfrak{p}}(\tau))d\tau.$$

This implies

$${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa)). \tag{153}$$

Repeating the same process with $\mathfrak{r}=\mathfrak{q}$, we get

$${}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^{\mathfrak{q}} \leq 2^{\mathfrak{q}-1}{}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu^{\mathfrak{q}}(\kappa)+\nu^{\mathfrak{q}}(\kappa)). \tag{154}$$

Substituting by (153) and (154) into (148), this completes the proof. □

Remark 4.4. *If we set $\varrho = 1$ in Theorem 4.3, then we get*

$$ {}_{0}^{AB}I_{\kappa}^{\varpi}(\mu(\kappa)+\nu(\kappa)) \leq \Phi\, {}_{0}^{AB}I_{\kappa}^{\varpi}(\mu^{\mathfrak{p}}(\kappa)+\nu^{\mathfrak{p}}(\kappa)) + \Psi\, {}_{0}^{AB}I_{\kappa}^{\varpi}(\mu^{\mathfrak{q}}(\kappa)+\nu^{\mathfrak{q}}(\kappa)), \tag{155}$$

which is proved by Khan et al. in [25], when $\vartheta_1 \to 0$.

Theorem 4.5. *Let $\nu > 0$ and $\mathfrak{p} \geq 1$. Let $\mu, \nu \in C_\nu[\vartheta_1, \vartheta_2]$ be two positive functions in $[0, \infty)$ such that ${}^{FFM}\mathbb{J}_{0,\kappa}^{\varpi}\mu(\kappa) < \infty$ and ${}^{FFM}\mathbb{J}_{0,\kappa}^{\varpi}\nu(\kappa) < \infty$ for all $\kappa > \vartheta_1$. If $0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta$ for some $\varpi, \theta \in \mathbb{R}_+^*$ and all $\kappa \in [\vartheta_1, \vartheta_2]$, then*

$$\frac{1}{\theta}\left({}^{FFM}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa)\nu(\kappa))\right) \leq {}^{FFM}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa)+\nu(\kappa))^2 \leq \frac{1}{\varpi}\left({}^{FFM}\mathbb{J}_{0,\kappa}^{\varpi}(\mu(\kappa)\nu(\kappa))\right). \tag{156}$$

Proof. Using the condition, we have

$$0 < \varpi \leq \frac{\mu(\kappa)}{\nu(\kappa)} \leq \theta, \tag{157}$$

we conclude that

$$(1+\varpi)\nu(\kappa) \leq (\mu(\kappa)+\nu(\kappa)) \leq (1+\theta)\nu(\kappa) \tag{158}$$

and

$$\frac{1+\theta}{\theta}\mu(\kappa) \leq (\mu(\kappa)+\nu(\kappa)) \leq \frac{1+\varpi}{\varpi}\mu(\kappa). \tag{159}$$

By (158) and (159), we have

$$\frac{1}{\theta}(\mu(\kappa)\nu(\kappa)) \leq \frac{(\mu(\kappa)+\nu(\kappa))^2}{(1+\varpi)(1+\theta)} \leq \frac{1}{\varpi}(\mu(\kappa)\nu(\kappa)). \tag{160}$$

Multiplying (160) on both sides by $\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}$, we have

$$\frac{1}{\theta}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa)\nu(\kappa) \leq \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\frac{(\mu(\kappa)+\nu(\kappa))^2}{(1+\varpi)(1+\theta)} \leq \frac{1}{\varpi}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa)\nu(\kappa). \tag{161}$$

Replace $\kappa = \tau$ in (160) and multiply by $\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}$, we have

$$\begin{aligned}\frac{1}{\theta}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau) &\leq \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)} \\ &\leq \frac{1}{\varpi}\frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau).\end{aligned} \tag{162}$$

Integrating both sides of (162) with respect to τ, we have

$$\begin{aligned}\frac{1}{\theta}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau)d\tau &\leq \int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)}d\tau \\ &\leq \frac{1}{\varpi}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau)d\tau.\end{aligned} \tag{163}$$

Add (146) and (161), we have

$$\begin{aligned}&\frac{1}{\theta}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa)\nu(\kappa) + \frac{1}{\theta}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau)d\tau \\ &\leq \frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\frac{(\mu(\kappa)+\nu(\kappa))^2}{(1+\varpi)(1+\theta)} + \int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\frac{(\mu(\tau)+\nu(\tau))^2}{(1+\varpi)(1+\theta)}d\tau \\ &\quad + \frac{1}{\varpi}\frac{(1-\varpi)\varrho\kappa^{\varrho-1}}{AB(\varpi)}\mu(\kappa)\nu(\kappa) + \frac{1}{\theta}\int_0^\kappa \frac{\varpi\varrho[\kappa-\tau]^{\varpi-1}\tau^{\varrho-1}}{AB(\varpi)}\mu(\tau)\nu(\tau)d\tau.\end{aligned}$$

This implies

$$\frac{1}{\theta}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)\nu(\kappa))\right) \leq^{FFM} \mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)+\nu(\kappa))^2 \leq \frac{1}{\varpi}\left({}^{FFM}\mathbb{J}^{\varpi}_{0,\kappa}(\mu(\kappa)\nu(\kappa))\right).$$

Therefore, the proof is completed. □

Remark 4.6. *If we set $\varrho = 1$ in Theorem 4.5, then we get*

$$\frac{1}{\theta}\left({}^{AB}_{0}I^{\varpi}_{\kappa}(\mu(\kappa)\nu(\kappa))\right) \leq {}^{AB}_{0}I^{\varpi}_{\kappa}(\mu(\kappa)+\nu(\kappa))^2 \leq \frac{1}{\varpi}\left({}^{AB}_{0}I^{\varpi}_{\kappa}(\mu(\kappa)\nu(\kappa))\right), \tag{164}$$

which is proved by Khan et al. in [25], when $\vartheta_1 \to 0$.

5. Conclusion

In this article, we fruitfully formed the development of mathematical inequalities for reverse Minkowski and associated inequalities utilizing fractal-fractional integral operators. Furthermore, we accomplished the desired findings by employing the Caputo, Caputo-Fabrizio and Atangana-Baleanu approaches. Moreover, we addressed the previous findings by incorporating various fractional-order derivatives and fractal dimension values. Several new generalizations are provided in fractal-dimensions when fractional-order is assumed to be 1. It would be interesting to apply these insights to the fractal-fractional derivative/integral operators as well as other convexities. Numerous depictions were employed to explain the outcomes in order to illustrate the significant characteristics of the fractal-fractional inequalities under concern. We truly think that our addition of new notions and concepts will be the subject of extensive research in the fascinating fields of numerical analysis and inequality theory. The efficacious and constructive implementation of the approach is examined and acknowledged in order to establish that it can be applied to variety of other useful fractional operators that originate in fractional calculus, particularly to new fractal-fractional integral operators.

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