Filomat 38:31 (2024), 11159–11169
https://doi.org/10.2298/FIL2431159A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a Pexider-Drygas functional equation on semigroups with an endomorphism

Youssef Aissi[a,*], Driss Zeglami[a], Aziz Mouzoun[a]

[a]*Department of Mathematics, E.N.S.A.M, Moulay ISMAIL University, B.P. 15290, Al Mansour Meknes Morocco*

Abstract. Let S be a semigroup that need not be abelian, let $(H,+)$ be a uniquely 2-divisible abelian group, and let φ be an endomorphism of S. We characterize the solutions $f,g : S \to H$ of the pexiderized version of the variant of Drygas' equation, that is,

$$f(xy) + f(\varphi(y)x) = 2f(x) + g(y),\ x, y \in S.$$

Interesting consequences of this result are presented.

1. Set up, notation and terminology

Throughout the paper, assume that S is a semigroup (a non-empty set equipped with an associative composition rule $(x, y) \mapsto xy$), $(H,+)$ is an abelian group which is uniquely 2-divisible (for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$), the maps φ and ϕ are endomorphisms of S, and σ is an involutive automorphism of S (i.e., $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma^2(x) = x$ for all $x, y \in S$). By φ^2 we means $\varphi \circ \varphi$.

A map $A : S \to H$ is said to be additive if

$$A(xy) = A(x) + A(y) \text{ for all } x, y \in S.$$

A map $Q : S \times S \to H$ is called bi-additive if it is additive in each variable.

By $\mathcal{N}(S, H, \sigma)$ we mean the set of the solutions $\theta : S \to H$ of homogeneous equation

$$\theta(xy) - \theta(\sigma(x)y) = 0,\ x, y \in S.$$

Let $f : S \to H$ be a map. f is said to be central if $f(xy) = f(yx)$ for all $x, y \in S$. The Cauchy difference C_f of the map f is defined by

$$C_f(x, y) := f(xy) - f(x) - f(y),\ x, y \in S,$$

and the maps ψ_f and A_f are defined as follow $\psi_f(x) := f(\varphi(x)x)$ for all $x \in S$ and $A_f = f - f \circ \varphi$.

2020 *Mathematics Subject Classification*. Primary 39B52.

Keywords. Drygas's Functional equation, additive function, symmetrized additive Cauchy equation, semigroup, endomorphism.
Received: 19 November 2022; Revised: 02 February 2024; Accepted: 16 July 2024
Communicated by Hemen Dutta

* Corresponding author: Youssef AISSI
Email addresses: yaissi94@gmail.com (Youssef Aissi), zeglamidriss@yahoo.fr (Driss Zeglami), mouzounposte@gmail.com (Aziz Mouzoun)

2. Introduction

In [7], Drygas dealt with a functional equation related to the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),\ x, y \in \mathbb{R}. \tag{1}$$

He generalized (1) to

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y),\ x, y \in \mathbb{R}. \tag{2}$$

The equation (2) is known in the literature as Drygas' functional equation. Many authors studied the Drygas functional equation, for example Stetkær [22], Faĭziev and Sahoo [12], Jung and Sahoo [14], Łukasik [15], Szabo [25] and Yang [26].

In [21], Stetkær determined the solutions $f : G \to \mathbb{C}$ of the functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + f(y) + f \circ \sigma(y),\ x, y \in G,$$

where $(G, +)$ is an abelian group.

In [19], Sahoo found the central solutions $f : G \to \mathbb{C}$ of the functional equation

$$f(xy) + f(\sigma_1(y)x) = 2f(x) + f(y) + f \circ \sigma_2(y),\ x, y \in G,$$

where G is a group and σ_1, σ_2 are two involutive automorphisms of G.

We refer also to the paper by Fadli et al. [11] who characterized the solutions of the variant of Drygas' functional equation, that is

$$f(xy) + f(\sigma(y)x) = 2f(x) + f(y) + f \circ \sigma(y),\ x, y \in S.$$

More details on the study of Drygas' equation can be found in [1–3, 8, 10, 17, 19, 23].

In the present paper, in terms of additive and bi-additive maps, and solutions of the symmetrized additive Cauchy equation

$$f(xy) + f(yx) = 2f(x) + 2f(y),\ x, y \in S, \tag{3}$$

we characterize the solutions $f : S \to H$ of the following functional equation

$$f(xy) + f(\varphi(y)x) = 2f(x) + g(y),\ x, y \in S, \tag{4}$$

on a semigroup S that need not be abelian. So, our main contribution is a natural extension of the works by Sabour [17], and by Akkaoui et al. [4], where they studied the solutions of Eq. (4) on abelian semigroups (φ is supposed to be involutive in [4]).

Equation (4), in case where $g = 2f \circ \phi$, becomes the following quadratic type equation

$$f(xy) + f(\varphi(y)x) = 2f(x) + 2f \circ \phi(y),\ x, y \in S, \tag{5}$$

which generalizes the variant of the quadratic functional equation

$$f(xy) + f(\varphi(y)x) = 2f(x) + 2f(y),\ x, y \in S,$$

which was studied, under the condition that φ is an involutive automorphism, by Fadli et al. [11].

Equation (4) with $g = f \circ \phi + f \circ \varphi$ becomes the following Drygas' type equation

$$f(xy) + f(\varphi(y)x) = 2f(x) + f \circ \phi(y) + f \circ \varphi(y),\ x, y \in S. \tag{6}$$

If ϕ is the identity map, then Eq.(6) becomes

$$f(xy) + f(\varphi(y)x) = 2f(x) + f(y) + f \circ \varphi(y),\ x, y \in S,$$

which was treated by Mouzoun et al. [16].

When $g = f + f \circ \phi$, Eq.(4) takes the form

$$f(xy) + f(\varphi(y)x) = 2f(x) + f(y) + f \circ \phi(y),\ x, y \in S. \tag{7}$$

This equation was solved by Sahoo [18] under the condition that φ and ϕ are two involutive automorphisms. Each of equations (5), (6) and (7) is with two endomorphisms.

Moreover, we solve the perturbed Jensen's functional equation, in which $\alpha \in H$ is a constant,

$$f(xy) + f(\varphi(y)x) = 2f(x) + \alpha,\ x \in S, \tag{8}$$

When $\alpha = 0$, Eq. (8) was treated by Fadli et al. in [9] under the condition that the endomorphism φ is surjective.

Let $E_1(f) = 0$ and $E_2(g) = 0$ be two functional equations for maps $f, g : S \to H$. The equations E_1 and E_2 are said to be strongly alien in the sense of Dhombres, if any solution $f, g : S \to H$ of

$$E_1(f) + E_2(g) = 0$$

is a solution of $E_1(f) = 0$ and $E_2(g) = 0$. This definition was introduced in [6].

As a further application of our result, we will find the solutions $f, h : S \to H$ of the functional equation

$$f(xy) + f(\varphi(y)x) + h(xy) + h(\varphi(y)x) = 2f(x) + 2h(x) + 2h(y),\ x, y \in S.$$

This allows us to show that, if additionally φ is involutive, then the variant of Jensen equation $f(xy) + f(\varphi(y)x) = 2f(x),\ x, y \in S$, and the variant of the quadratic equation $h(xy) + h(\varphi(y)x) = 2h(x) + 2h(y),\ x, y \in S$ are strongly alien in the sense of Dhombres.

We will encounter the results about Whitehead's functional equation

$$f(xyz) = f(xy) + f(xz) + f(yz) - f(x) - f(y) - f(z),\ x, y, z \in S, \tag{9}$$

and the following variant of Drygas' equation

$$f(xy) + f(\varphi(y)x) = 2f(x) + f(y) + f \circ \varphi(y),\ x, y \in S, \tag{10}$$

which were given in [24] and [16], respectively.

3. Preliminary results

We start with a crucial connection between Eq. (4) and solutions of Whitehead's functional equation (9).

Lemma 3.1. *[3, Lemma 3.1] Suppose $f, g : S \to H$ satisfy (4). Then, g is a solution of Whitehead's functional equation (9).*

In the following lemma we derive some basic properties of the solutions of (4).

Lemma 3.2. *Suppose $f, g : S \to H$ satisfy (4). Then, the following statements hold:*

1. *$f + f \circ \varphi - g$ is a constant map.*
2. *$f - f \circ \varphi$ is an additive map.*

Proof. Let $f, g : S \to H$ be a solution of (4).

1. If we replace (x, y) by $(\varphi(y), x)$ in (4), we obtain

$$f(\varphi(y)x) + f \circ \varphi(xy) = 2f \circ \varphi(y) + g(x), \; x, y \in S. \tag{11}$$

If we subtract (4) from (11) we see that

$$(f - f \circ \varphi)(xy) = 2f(x) - g(x) - 2f \circ \varphi(y) + g(y), \; x, y \in S,$$

which is equivalently

$$\begin{aligned} & (f - f \circ \varphi)(xy) - (f - f \circ \varphi)(x) - (f - f \circ \varphi)(y) \\ = \; & f(x) + f \circ \varphi(x) - g(x) - f(y) - f \circ \varphi(y) + g(y), \; x, y \in S. \end{aligned} \tag{12}$$

Consider the map $\Gamma : S \times S \to H$ defined by

$$\Gamma(x, y) := (f - f \circ \varphi)(xy) - (f - f \circ \varphi)(x) - (f - f \circ \varphi)(y) \; x, y \in S.$$

As a Cauchy difference, the map Γ satisfies the cocycle functional equation

$$\Gamma(xy, z) + \Gamma(x, y) = \Gamma(x, yz) + \Gamma(y, z), \; x, y, z \in S. \tag{13}$$

If we use (12) in (13), we obtain after some computations that

$$\begin{aligned} f(xy) \; + & \; f \circ \varphi(xy) - g(xy) + f(yz) + f \circ \varphi(yz) - g(yz) \\ = & \; 2f(y) + 2f \circ \varphi(y) - 2g(y), \; x, y, z \in S. \end{aligned} \tag{14}$$

Putting $z = x$ in (14), we get

$$\begin{aligned} & f(xy) + f \circ \varphi(xy) - g(xy) + f(yx) + f \circ \varphi(yx) - g(yx) \\ & = 2f(y) + 2f \circ \varphi(y) - 2g(y), \; x, y \in S. \end{aligned} \tag{15}$$

Since the left hand side of (15) is invariant under interchange of x and y and H is uniquely 2-divisible, we find that $f + f \circ \varphi - g$ is a constant map.

2. Since $f + f \circ \varphi - g$ is a constant map, the identity (12) becomes

$$(f - f \circ \varphi)(xy) = (f - f \circ \varphi)(x) + (f - f \circ \varphi)(y), \; x, y \in S.$$

This means that $f - f \circ \varphi$ is an additive map.

□

The following lemma plays a key role in the next section. It lists pertinent properties of the solutions of Eq. (10).

Lemma 3.3. *[3, Lemma 3.3] Suppose that $f : S \to H$ satisfies (10), that is,*

$$f(xy) + f(\varphi(y)x) = 2f(x) + f(y) + f(\varphi(y)), \; x, y \in S.$$

Then, the following statements hold

1. *$A_f := f - f \circ \varphi$ is an additive map.*
2. *f is a solution of Whitehead's functional equation (9).*
3. *$C_f : S \times S \to H$ is a bi-additive map satisfying*

$$C_f(\varphi(y), x) = -C_f(x, y) \text{ for all } x, y \in S. \tag{16}$$

4. *Define $\psi_f : S \to H$ by $\psi_f(x) := f(\varphi(x)x)$, $x \in S$. Then we have*

$$2f(x) = C_f(x, x) + \psi_f(x) + A_f(x) \text{ for all } x \in S. \tag{17}$$

5. *The map* ψ_f *is a solution of (3) such that*

$$\psi_f(xy) - \psi_f(\varphi(x)y) = C_f(x,y) - C_f(\varphi^2(x),y) + A_f(\varphi(x)x),\ x,y \in S.$$

Remark 3.4. *In Lemma 3.3, the maps* ψ_f *and* A_f *satisfy*

$$(\psi_f + A)\circ\varphi = \psi_f - A_f. \tag{18}$$

Indeed, if we replace x *by* $\varphi(x)$ *in (17) and subtract (17) from the obtained result, we get*

$$\begin{aligned} 2f(x) - 2f\circ\varphi(x) &= [C_f(x,x) - C_f(\varphi(x),\varphi(x))] + [\psi_f(x) - \psi_f\circ\varphi(x)] \\ &+ [A_f(x) - A_f\circ\varphi(x)],\ x \in S. \end{aligned}$$

By the definition of A_f *and the assumption (16) on* C_f *we conclude that*

$$2A_f(x) = \psi_f(x) - \psi_f\circ\varphi(x) + A_f(x) - A_f\circ\varphi(x),\ x \in S,$$

which is equivalently $\psi_f - \psi_f\circ\varphi = A_f + A_f\circ\varphi$ *and hence we obtain (18).*

4. Main results

The following lemma, that will be encountered in the process of solving (5), (6), (7) and (8), is inspired from [5, Lemma 4].

Lemma 4.1. *Let* $K, L : S \to H$ *be maps such that* $K(x^n) = n^2K(x)$ *and* $L(x^n) = nL(x)$ *for all* $n = 1, 2, \cdots$ *and* $x \in S$, *and let* $C \in H$ *be a constant. If*

$$K(x) + L(x) = C \text{ for all } x \in S, \tag{19}$$

then $K = L = C = 0$.

Proof. Replacing x by x^2 in (19), we get

$$4K(x) + 2L(x) = C. \tag{20}$$

Multiplying (19) by 4 and subtracting the obtained result from (20), we obtain

$$2L(x) = 3C,\ x \in S. \tag{21}$$

Replacing x by x^2 in (21), we get

$$4L(x) = 3C. \tag{22}$$

Subtracting (21) from (22), we obtain $2L \equiv 0$, which yields that $L \equiv 0$ (because H is a uniquely 2-divisible). This implies that

$$K(x) = C,\ x \in S. \tag{23}$$

Replacing x by x^3 in (23), we obtain $9K(x) = C$. Subtracting (23) from the last equality, we see that $2^3K \equiv 0$, then $K \equiv 0$ and hence $C = 0$. □

Now, we are in a position to present our main result.

Theorem 4.2. *The solutions $f, g : S \to H$ of (4) are the maps of the form*

$$f(x) = Q(x,x) + \psi(x) + A(x) + c \text{ and } g(x) = 2Q(x,x) + 2\psi(x), \; x \in S,$$

where $c \in H$ is a constant, $Q : S \times S \to H$ is a bi-additive map such that

$$Q(\varphi(y), x) = -Q(x,y) \text{ for all } x, y \in S, \tag{24}$$

$A : S \to H$ is an additive map, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that $(\psi + A) \circ \varphi = \psi - A$ and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x,y) - Q(\varphi^2(x), y) + A(\varphi(x)x), \; x \in S.$$

Proof. Let $f, g : S \to H$ be a solution of (4). From Lemma 3.2, we have $f + f \circ \varphi - g$ is a constant, say, $2c$. Then, $g = f + f \circ \varphi - 2c$ and hence the equation (4) becomes

$$f(xy) + f(\varphi(y)x) = 2f(x) + f(y) + f \circ \varphi(y) - 2c, \; x, y \in S.$$

This yields that

$$K(xy) + K(\varphi(y)x) = 2K(x) + K(y) + K \circ \varphi(y), \; x, y \in S,$$

where $K(x) := f(x) - c$ for all $x \in S$. In view of Lemma 3.3, we find with the notations $Q := \frac{1}{2}C_K$, $\psi := \frac{1}{2}\psi_K$ and $A := \frac{1}{2}A_K$, that

$$K(x) = Q(x,x) + \psi(x) + A(x), \; x \in S. \tag{25}$$

According to Lemma 3.3 and Remark 3.4, we have $Q : S \times S \to H$ is a bi-additive map such that $Q(\varphi(y), x) = -Q(x,y)$ for all $x, y \in S$, $A : S \to H$ is an additive map and $\psi : S \to H$ is an arbitrary solution of (3) such that

$$(\psi + A) \circ \varphi = \psi - A, \tag{26}$$

and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x,y) - Q(\varphi^2(x), y) + A(\varphi(x)x), \; x \in S.$$

From the definition of K and (25), we find that

$$f(x) = Q(x,x) + \psi(x) + A(x) + c, \; x \in S.$$

Since $g = f + f \circ \varphi - 2c$, we deduce, by using (26) and (24), that

$$g(x) = 2Q(x,x) + 2\psi(x), \; x \in S.$$

The proof of the converse implication is a simple calculation that we omit. □

In the following corollary, we give the central solutions of the functional equation (4) on semigroups.

Corollary 4.3. *The central solutions $f, g : S \to H$ of (4) are the maps of the form*

$$f(x) = Q(x,x) + A(x) + c \text{ and } g(x) = 2Q(x,x) + A(x) + A \circ \varphi(x), \; x \in S, \tag{27}$$

where $c \in H$ is a constant, $Q : S \times S \to H$ is a symmetric, bi-additive map such that

$$Q(x, \varphi(y)) = -Q(x,y),$$

for all $x, y \in S$, and where $A : S \to H$ is an additive map.

Proof. It is easy to check that any pair of maps of the form (27) is a central solution of (4). Conversely, we adopt the proof of Theorem 4.2. Assume that the pair $\{f, g\}$ is a central solution of (4). From Lemma 3.3 (5) and the definition of C_K and ψ_K where $K := f - c$, we deduce that C_K is symmetric and ψ_K is additive. Hence, from the proof of Theorem 4.2 we find, with

$$Q := \frac{1}{2}C_k,\ a := \frac{1}{2}\psi_K \text{ and } b := \frac{1}{2}A_K,$$

that

$$f(x) = Q(x,x) + a(x) + b(x) + c \text{ and } g(x) = 2Q(x,x) + 2a(x),\ x \in S,$$

where $c \in H$ is a constant, $Q : S \times S \to H$ is a symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x, y)$, for all $x, y \in S$, and where $a, b : S \to H$ are two additive maps such that

$$(a + b) \circ \varphi = a - b. \tag{28}$$

We put $A := a + b$. It is clear that $A : S \to H$ is an additive map. So, by using (28), we conclude that $A \circ \varphi = a - b$ and hence

$$2a = A + A \circ \varphi.$$

This implies that the pair $\{f, g\}$ has the form (27). □

As another consequence of Theorem 4.2, we describe the solutions of the quadratic type equation (5), namely,

$$f(xy) + f(\varphi(y)x) = 2f(x) + 2f \circ \phi(y),\ x, y \in S.$$

Corollary 4.4. *The solutions $f : S \to H$ of (5) are the maps of the form*

$$f(x) = Q(x,x) + \psi(x) + A(x),\ x \in S, \tag{29}$$

where $Q : S \times S \to H$ is a bi-additive map such that

$$Q(\varphi(y), x) = -Q(x, y)\ ,\ Q(\phi(x), \phi(x)) = Q(x, x)$$

for all $x, y \in S$, $A : S \to H$ is an additive map, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that

$$(\psi + A) \circ \varphi = \psi - A\ ,\ (\psi + A) \circ \phi = \psi$$

and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x, y) - Q(\varphi^2(x), y) + A(\varphi(x)x),\ x \in S.$$

Proof. It is elementary to show that any map of the form (29) is a solution of (5). Conversely, assume that $f : S \to H$ satisfies (5). Applying Theorem 4.2 with $g = 2f \circ \phi$ and the fact that H is uniquely 2-divisible, we get

$$\begin{cases} f(x) = Q(x,x) + \psi(x) + A(x) + c \\ f \circ \phi(x) = Q(x,x) + \psi(x) \end{cases},\ x \in S. \tag{30}$$

On the other hand, we have

$$f \circ \phi(x) = Q(\phi(x), \phi(x)) + \psi \circ \phi(x) + A \circ \phi(x) + c,\ x \in S. \tag{31}$$

From (30) and (31), we conclude that

$$[Q(x,x) - Q(\phi(x),\phi(x))] + [\psi(x) - \psi \circ \phi(x) - A \circ \phi(x)] = c$$

for all $x \in S$. If we use Lemma 4.1 with $C = c$

$$K(x) := Q(x,x) - Q(\phi(x),\phi(x)) \text{ and } L(x) := \psi(x) - \psi \circ \phi(x) - A \circ \phi(x),\ x \in S,$$

we obtain

$$Q(\phi(x),\phi(x)) = Q(x,x)\ ,\ (\psi + A) \circ \phi = \psi,$$

and $c = 0$. From Remark 3.4 we have $(\psi + A) \circ \varphi = \psi - A$ and hence f has the form (29). □

The next two corollaries give the general solution of (6) and (7).

Corollary 4.5. *The solutions $f : S \to H$ of (6) are the maps of the form*

$$f(x) = Q(x,x) + \psi(x) + A(x),\ x \in S,$$

where $Q : S \times S \to H$ is a bi-additive map such that

$$Q(\varphi(y),x) = -Q(x,y) \text{ and } Q(\phi(x),\phi(x)) = Q(x,x),$$

for all $x, y \in S$, $A : S \to H$ is an additive map, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that

$$(\psi + A) \circ \varphi = \psi - A\ ,\ (\psi + A) \circ \phi = \psi + A$$

and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x,y) - Q(\varphi^2(x),y) + A(\varphi(x)x),\ x \in S.$$

Proof. Applying Theorem 4.2 with $g = f \circ \phi + f \circ \varphi$, we obtain

$$\begin{aligned} 2Q(x,x) + 2\psi(x) &= f \circ \phi(x) + f \circ \varphi(x) \\ &= Q(x,x) + Q(\phi(x),\phi(x)) + \psi \circ \phi(x) + \psi \circ \varphi(x) \\ &+ A \circ \phi(x) + A \circ \varphi(x) + 2c. \end{aligned}$$

This is equivalent to

$$\begin{aligned} [Q(x,x) - Q(\phi(x),\phi(x))] &+ [2\psi(x) - \psi \circ \phi(x) - \psi \circ \varphi(x)] \\ &- [A \circ \phi(x) + A \circ \varphi(x)] = 2c. \end{aligned}$$

According to Lemma 4.1, as in the proof of Corollary 4.4, we get $2c = 0$ and hence $c = 0$, $Q(\phi(x),\phi(x)) = Q(x,x)$ for all $x \in S$ and

$$2\psi - \psi \circ \phi = \psi \circ \varphi + A \circ \phi + A \circ \varphi. \tag{32}$$

Since $(\psi + A) \circ \varphi = \psi - A$, (32) becomes $\psi - \psi \circ \phi = A \circ \phi - A$, which completes the proof of the first direction. The converse statement is easy to show. □

Corollary 4.6. *The solutions $f : S \to H$ of (7) are the maps of the form*

$$f(x) = Q(x,x) + \psi(x) + A(x),\ x \in S, \tag{33}$$

where $Q : S \times S \to H$ is a bi-additive map such that

$$Q(\varphi(y),x) = -Q(x,y) \text{ and } Q(\phi(x),\phi(x)) = Q(x,x),$$

for all $x, y \in S$, $A : S \to H$ is an additive map, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that

$$(\psi + A) \circ \varphi = \psi - A\ ,\ (\psi + A) \circ \phi = \psi - A$$

and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x, y) - Q(\varphi^2(x), y) + A(\varphi(x)x),\ x \in S.$$

Proof. We apply Theorem 4.2 with $g = f + f \circ \phi$, we deduce that

$$\begin{aligned} 2Q(x, x) + 2\psi(x) &= f(x) + f \circ \phi(x) \\ &= Q(x, x) + Q(\phi(x), \phi(x)) + \psi(x) + \psi \circ \phi(x) \\ &+ A(x) + A \circ \phi(x) + 2c, \end{aligned}$$

or equivalently

$$[Q(x, x) - Q(\phi(x), \phi(x))] + [\psi(x) - \psi \circ \phi(x) - A(x) - A \circ \phi(x)] = 2c.$$

According to Lemma 4.1, as in the proof of Corollary 4.4, we infer that $Q(\phi(x), \phi(x)) = Q(x, x)$ for all $x \in S$, $\psi - \psi \circ \phi = A + A \circ \phi$ and $c = 0$. Hence f has the form (33). Conversely, it is elementary to show that the form (33) of f is a solution of (7). □

The following corollary describes the solutions of the functional equation (8), that is

$$f(xy) + f(\varphi(y)x) = 2f(x) + \alpha,\ x, y \in S,$$

where $\alpha \in H$ is a constant.

Corollary 4.7. *Let $\alpha \in H$ be a constant.*

1. *If $\alpha \neq 0$, then the equation (8) has no solution.*
2. *If $\alpha = 0$, the solutions $f : S \to H$ of (8) are the maps of the form*

   $$f(x) = A(x) + c,\ x \in S,$$

   where $c \in H$ is a constant and $A : S \to H$ is an additive map such that $A \circ \varphi = -A$.

Proof. Let $f : S \to H$ be a solution of (8). Assume first that $\alpha \neq 0$, then by applying Theorem 4.2 with $g = \alpha$ we find, by using Lemma 4.1, that

$$2Q(x, x) = 2\psi(x) = \alpha = 0,$$

for all $x \in S$, which contradicts our assumption on α. Hence, the equation (8) has no solution for $\alpha \neq 0$. Assume now that $\alpha = 0$. If we apply Theorem 4.2 with $g = 0$, we deduce, from Lemma 4.1 and the fact that H is uniquely 2-divisible, that

$$Q(x, x) = \psi(x) = 0,\ x \in S.$$

Hence, from Theorem 4.2, we have $f(x) = A(x) + c$ for all $x \in S$, where $c \in H$ is a constant and $A : S \to H$ is an additive map such that $A \circ \varphi = -A$. □

Now we turn to study the alienation phenomenon between two linear functional equations with an endomorphism, namely Jensen's type equation

$$f(xy) + f(\varphi(y)x) = 2f(x),\ x, y \in S, \tag{34}$$

and the variant of the quadratic equation

$$h(xy) + h(\varphi(y)x) = 2h(x) + 2h(y),\ x, y \in S. \tag{35}$$

This comes from solving the functional equation

$$f(xy) + f(\varphi(y)x) + h(xy) + h(\varphi(y)x) = 2f(x) + 2h(x) + 2h(y),\ x, y \in S, \tag{36}$$

where $f, h : S \to H$ are unknown maps.

Corollary 4.8. *The solutions $f, g : S \to H$ of the functional equation (36) are the maps of the form*

$$f(x) = A(x) + c \ \text{ and } \ h(x) = Q(x, x) + \psi(x),\ x \in S, \tag{37}$$

where $c \in H$ is a constant, $Q : S \times S \to H$ is a bi-additive map such that $Q(\varphi(y), x) = -Q(x, y)$ for all $x, y \in S$, $A : S \to H$ is an additive map, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that $(\psi + A) \circ \varphi = \psi - A$ and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x, y) - Q(\varphi^2(x), y) + A(\varphi(x)x),\ x \in S.$$

Proof. We apply Theorem 4.2 with the pair (f, g) replaced by $(f + h, 2h)$. Then, we find that

$$\begin{cases} f(x) + h(x) &= Q(x, x) + \psi(x) + A(x) + c \\ 2h(x) &= 2Q(x, x) + 2\psi(x) \end{cases},\ x \in S, \tag{38}$$

where $c \in H$ is a constant, $Q : S \times S \to H$ is a bi-additive map such that $Q(\varphi(y), x) = -Q(x, y)$ for all $x, y \in S$, $A : S \to H$ is an additive map, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that $(\psi + A) \circ \varphi = \psi - A$ and

$$\psi(xy) - \psi(\varphi(x)y) = Q(x, y) - Q(\varphi^2(x), y) + A(\varphi(x)x),\ x \in S.$$

Since H is a uniquely 2-divisible abelian group, the system (38) shows that the pair f, h has the form (37). This completes the proof of the first direction. Conversely, any pair of maps of the form (37) is a solution of (36). □

Remark 4.9. *If φ is an involutive automorphism, then the Jensen type equation (34) and the quadratic type equation (35) are strongly alien in the sense of Dhombres.*

Indeed, assume that the pair $\{f, h\}$ satisfies (36). Since $\varphi : S \to S$ is an involutive automorphism, we deduce from Corollary 4.8 that

$$f(x) = A(x) + c \ \text{ and } \ h(x) = Q(x, x) + \psi(x),\ x \in S,$$

where $c \in H$ is a constant, $Q : S \times S \to H$ is a bi-additive map such that $Q(\varphi(y), x) = -Q(x, y)$ for all $x, y \in S$, $A : S \to H$ is an additive map such that $A \circ \varphi = -A$, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation (3) such that $\psi \circ \varphi = \psi$ and $\psi \in \mathcal{N}(S, H, \varphi)$. So, according to Corollary 4.7 and [11, Theorem 5.2], we conclude that f and h are solutions of (34) and (35), respectively.

The converse is obvious, and this proves that the equations (34) and (35) are strongly alien in the sense of Dhombres.

Acknowledgements We wish to express our thanks to the referees for useful comments and valuable suggestions.

References

[1] Y. Aissi, D. Zeglami, B. Fadli, *Alienation of Drygas' and Cauchy's functional equations*, Ann. Math. Sil. **35** (2021), 131–148.

[2] Y. Aissi, D. Zeglami, B. Fadli, *Alienation of Drygas', Cauchy's, Jensen's and the quadratic equations on semigroups with an involutive automorphism*, Aequationes Math., **96**(6) (2022), 1221–1232.

[3] Y. Aissi, D. Zeglami, A. Mouzoun, *A quadratic functional equation with involutive automorphisms on semigroups*, Bol. Soc. Mat. Mex. **28** (2022), 19.

[4] A. Akkaoui, M. El Fatini, B. Fadli, D. Zeglami, *Nice results about quadratic type functional equations on semigroups*, Aequationes Math. **94** (2020), 83–96.

[5] A. Akkaoui, B. Fadli, M. El Fatini, *The Drygas functional equation on abelian semigroups with endomorphisms*, Results Math. **76** (2021), 1–13.

[6] J. Dhombres, *Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. **35** (1988), 186–212.

[7] H. Drygas, *Quasi-inner products and their applications*, In: Gupta, A.K. (eds) Advances in Multivariate Statistical Analysis. Theory and Decision Library, vol 5. Springer, Dordrecht, (1987).

[8] B. R. Ebanks, P. L. Kannappan, P. K. Sahoo, *A common generalization of functional equations characterizing normed and quasi-inner-product spaces*, Can. Math. Bull. **35** (1992), 321–327.

[9] B. Fadli, S. Kabbaj, K. Sabour, , D. Zeglami, *Functional equations on semigroups with an endomorphism* Acta Math. Hungar.**150**(2) (2016), 363–371.

[10] B. Fadli, D. Zeglami, *On a class of functional equations which includes the Pexider's one*, Acta Math. Hungar. **165** (2021), 63–77.

[11] B. Fadli, D. Zeglami, S. Kabbaj, *A variant of the quadratic functional equation on semigroups*, Proyecciones (Antofagasta) **37** (2018), 45–55.

[12] V. A. Faĭziev, P. K. Sahoo, *On Drygas functional equation on groups*, Int. J. Appl. Math. Stat. **7** (2007), 59–69.

[13] V. A. Faĭziev, Sahoo, P. K., *Solution of Whitehead equation on groups*, Math. Bohemica **138**(2) (2013) 171–180.

[14] S.M. Jung, P. K. Sahoo, *Stability of a functional equation of Drygas*, Aequationes Math. **64** (2002), 263–273.

[15] R. Łukasik, *Some generalization of Cauchy's and the quadratic functional equations,* Aequationes Math., **83** (2012), 75–86.

[16] A. Mouzoun, D. Zeglami, Y. Aissi, *A variant of Drygas' equation with an endomorphism,* Moroccan J. Pure Appl. Anal. (submitted).

[17] Kh. Sabour, *A variant of Drygas' functional equation with an endomorphism*, Asia Math. **4** (2020), 63–68.

[18] P.K. Sahoo, *Yet another variant of the Drygas functional equation on groups*, Proyecciones (Antofagasta) **36** (2017), 13–27.

[19] P. K. Sahoo,Pl. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, Florida (2011).

[20] P. Sinopoulos, *Functional equations on semigroups,* Aequtiones Math. **59** (2000), 255–261.

[21] H. Stetkær, *Functional equations on abelian groups with involution*, Aequationes Math. **54** (1997), 144-172.

[22] H. Stetkær, *Functional equations on abelian groups with involution. II*, Aequationes Math. **55** (1998), 227–240.

[23] H. Stetkær, *Functional Equations on Groups*, World Scientific Publishing Co., Singapore (2013).

[24] H. Stetkær, *The kernel of the second order Cauchy difference on semigroups*, Aequationes Math. **91** (2017), 279–288.

[25] Gy. Szabo, *Some functional equations related to quadratic functions,* Glasnik Math. **38** (1983), 107–118.

[26] D. Yang, *Remarks on the stability of Drygas equation and the Pexider-quadratic equation,* Aequationes Math. **68** (2004), 108–116.