Filomat 38:31 (2024), 10851–10865
https://doi.org/10.2298/FIL2431851K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a new type of Szász-Chlodowsky operators in terms of 2D Appell Polynomials

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Abstract. The main goal of this research is to obtain several approximation properties of Kantorovich variant Szász-Chlodowsky operators based on 2D Appell Polynomials. We provide a connection between weighted $\mathcal{B}$-statistical convergence (WBSC) and the rate of WBSC concepts and approximation theory. We also prove a Voronovskaja theorem related to WBSC. Finally, we consider certain examples and discuss the advantages of proposed operators via certain numerical results.

1. Introduction

From the 19th century until today, many mathematicians who have been researching, especially on the theory of approximations, have the following question in mind. Can we approximate the functions that are difficult to work on, that is, functions difficult in terms of applying mathematical operations, with the easier and simpler sequence of polynomials, and how can we provide this approximate in the best way? From this point of view, Bernstein [19] gave the simplest proof of Weierstrass's famous approximation theorem via a sequence of polynomials and this proof has been a source of light for many researchers (see [3–5, 8–16, 23–25, 29, 33, 35, 40–42, 45–53, 58] and references therein).

In [6], Appell proposed and characterised a polynomial sequence that appears various application areas, such as engineering, probability theory, and statistics, analytic number theory and approximation theory in pure mathematics and also provide certain solutions for specified problems for fluid mechanics and heat equation in physics.

Around two decades ago, Bretti [21] established for any $\rho \geq 2$, the bivariate Appell polynomials (called 2D Appell Polynomials) $\Gamma_d^{(\rho)}(x;a)$ concerning the following generating function

$$\mathcal{A}(t)e^{xt+at^{\rho}} = \sum_{d=0}^{\infty} \Gamma_d^{(\rho)}(x;a)\frac{t^d}{d!}, \quad x,a \geq 0;\ t \neq 0$$

2020 *Mathematics Subject Classification*. 41A10, 41A25, 41A36.
Keywords. 2D Appell Polynomials, Szász operators, Chlodowsky operators, weighted statistical convergence, Voronovskaja type theorem
Received: 05 July 2024 Revised: 22 August 2024; Accepted: 30 August 2024
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where an analytic function $\mathcal{A}(t)$ has the formal power series of the following version:

$$\mathcal{A}(t) = \sum_{h=0}^{\infty} \frac{\Omega_h}{h!} t^h, \quad \mathcal{A}(0) \neq 0.$$

Based on the generating function expansion of Appell polynomials, Jakimovski and Leviatan [32] proposed a new version of linear positive operators on an unbounded interval as:

$$U_c(\eta; x) = \frac{1}{e^{cx}\gamma(1)} \sum_{b=0}^{\infty} p_b(cx)\eta(\frac{b}{c}), \tag{1}$$

where, $\gamma(t)e^{ct} = \sum\limits_{b=0}^{\infty} p_b(t)c^b$ is generating function with Appell polynomials $p_b(t) \geq 0$ and $\gamma(v) = \sum\limits_{c=0}^{\infty} a_c v^c$, $|v| < U$, $U > 1$ and $\gamma(1) \neq 0$.

In [60], Wood established some variation-diminishing properties and also the state of convergence of operators (1) in the complex domain. Asymptotic expansion for the derivatives of the operators (1) were obtained by Abel and Ivan [1]. A modification of the operators (1) is constructed by Ciupa [27], and estimation of the degree of approximation and also proof of Voronovskaya type theorem are given by him. Further, in 2014, Büyükyazıcı et al. [22] considered a generalization of the operators (1) and studied some direct theorems also convergence in a weighted space of functions on $[0, \infty)$ for these operators. Ari and Serenbay [7] considered Kantorovich-type generalization of Chlodowsky [26] version of operators (1) and showed the convergence of these operators in a weighted space of functions on $[0, \infty)$. In special case for $\gamma(1) = 1$ in (1), one has $p_b(nx) = \frac{(nx)^b}{b!}$ and it turns out to the following well-known Szász [57] operators, which is associated with the poisson distribution:

$$L_c(\eta; x) = \sum_{b=0}^{\infty} e^{-cx} \frac{(cx)^b}{b!} \eta(\frac{b}{c}). \tag{2}$$

In the literature, to estimate better approximation results, one may get with the aid of generating function of Brenke type polnomials [59], Boas-Buck polynomials [28, 31, 55], Charlier polynomials [36], Gould-Hopper polynomials [17, 43] and Sheffer polynomials [20, 56].

Very recently, Ali and Kadak [2] constructed and studied the following Chlodowsky type of linear positive operators (2) including 2D Appell polynomials

$$L_{c,\Gamma}(\eta; x) = \frac{e^{-\frac{c}{q_c}x - a}}{\mathcal{A}(1)} \sum_{b=0}^{\infty} \frac{\Gamma_b^{(\rho)}(\frac{c}{q_c}x, a)}{b!} \eta(\frac{b}{c}q_c), \ x \in [0, \infty), \tag{3}$$

where, $a \geq 0$ and q_c is a positive increasing sequence with the assumptions:

$$\lim_{c\to\infty} q_c = \infty, \quad \lim_{c\to\infty} \frac{q_c}{c} = 0. \tag{4}$$

They obtained the order of approximation in respect of the ordinary modulus of continuity, examined weighted $\mathcal{B}$-statistical and summability properties. Also, they presented several numerical and graphical convergence results in order to show the accuracy of operators (3).

Taking courage from the all above-mentioned papers, in this work, we aim to investigate several approximation properties of the following Kantorovich [37] type of operators (3)

$$\mathcal{F}_{c,\Gamma}(\eta; x) = \frac{c}{q_c} \frac{e^{-\frac{c}{q_c}x - a}}{\mathcal{A}(1)} \sum_{b=0}^{\infty} \frac{\Gamma_b^{(\rho)}(\frac{c}{q_c}x, a)}{b!} \int_{\frac{b}{c}q_c}^{\frac{(b+1)}{c}q_c} \eta(u)du, \ x \in [0, \infty), \tag{5}$$

where, $a \geq 0$, $\Gamma_b^{(\rho)}$ are the 2D Appell polynomials and the sequences q_c is defined as in (4). It is easy to check that operators $\mathcal{F}_{c,\Gamma}$ is linear and positive. Note that, in case $a = 0$, operators given by (5) reduce to the Kantorovich type generalization of Jakimovski-Leviatan operators constructed by Ari and Serenbay [7].

Outline of the work is as follows: In section 2, we evaluate needed moments and give the main Bohman-Korovkin theorem for operators (5). In section 3, we provide a connection between WBSC, rate of WBSC, and approximation of (5). We also proof a Voronovskaja theorem related to WBSC in this section. In the last section, we consider certain examples and discuss advantages of (5) via the numerical results.

2. Preliminaries

In this section, firstly we compute some moments and central moments of (5). Let $e_u(t) = t^u$ be the test functions for $u = 0, \cdots, 4$. Then, we give the main Bohman-Korovkin theorem for operators $\mathcal{F}_{c,\Gamma}$.

Lemma 2.1. *The operators $\mathcal{F}_{c,\Gamma}$ defined by* (5) *satisfy*

$$\mathcal{F}_{c,\Gamma}(e_0;x) = 1, \tag{6}$$

$$\mathcal{F}_{c,\Gamma}(e_1;x) = x + \frac{q_c}{c}\left(a\rho + \frac{1}{2} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\right). \tag{7}$$

$$\begin{aligned}\mathcal{F}_{c,\Gamma}(e_2;x) &= x^2 + \frac{q_c}{c}x\left(2 + 2a\rho + \frac{2\mathcal{A}'(1)}{\mathcal{A}(1)}\right)\\ &+\frac{q_c^2}{c^2}\left(\frac{1}{3} + a^2\rho^2 + a\rho^2 + a\rho + \frac{2\mathcal{A}'(1) + \mathcal{A}''(1) + 2a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right),\end{aligned} \tag{8}$$

$$\begin{aligned}\mathcal{F}_{c,\Gamma}(e_3;x) &= x^3 + \frac{q_c}{c}x^2\left(\frac{9}{2} + 3a\rho + \frac{3\mathcal{A}'(1)}{\mathcal{A}(1)}\right)\\ &+\frac{q_c^2}{c^2}x\left(\frac{7}{2} + 9a\rho + 3a^2\rho^2 + 3a\rho(\rho-1) + \frac{9\mathcal{A}'(1) + 6a\rho\mathcal{A}'(1) + 3\mathcal{A}''(1)}{\mathcal{A}(1)}\right)\\ &+\frac{q_c^3}{c^3}\left(\nu_{a,r}^{(1)} + \frac{\mathcal{A}'''(1) + 4\mathcal{A}''(1) + 3\mathcal{A}'(1) + 3a\rho\mathcal{A}''(1)}{\mathcal{A}(1)}\right.\\ &\left.+\frac{8a\rho\mathcal{A}'(1) + 3a\rho(\rho-1)\mathcal{A}'(1) + 3a^2\rho^2\mathcal{A}'(1)}{\mathcal{A}(1)}\right),\end{aligned} \tag{9}$$

$$\begin{aligned}\mathcal{F}_{c,\Gamma}(e_4;x) &= x^4 + \frac{q_c}{c}x^3\left(8 + 4a\rho + \frac{4\mathcal{A}'(1)}{\mathcal{A}(1)}\right) + \frac{q_c^2}{c^2}x^2\left(\nu_{a,r}^{(2)} + \frac{24\mathcal{A}'(1) + 12a\rho\mathcal{A}'(1) + 6\mathcal{A}''(1)}{\mathcal{A}(1)}\right)\\ &+\frac{q_c^3}{c^3}x\left(\nu_{a,r}^{(3)} + \frac{4\mathcal{A}'''(1) + 24\mathcal{A}''(1) + 30\mathcal{A}'(1) + 12a\rho\mathcal{A}''(1)}{\mathcal{A}(1)}\right.\\ &\left.+\frac{12a^2\rho^2\mathcal{A}'(1) + 48a\rho\mathcal{A}'(1) + 12a\rho(\rho-1)\mathcal{A}'(1)}{\mathcal{A}(1)}\right)\\ &+\frac{q_c^4}{c^4}\left(\nu_{a,r}^{(4)} + \frac{A^{(iv)}(1) + 8\mathcal{A}'''(1) + 15\mathcal{A}''(1) + 6\mathcal{A}'(1)}{\mathcal{A}(1)}\right.\\ &+\frac{4a\rho\mathcal{A}'''(1) + 6a^2\rho^2\mathcal{A}''(1) + 6a\rho(\rho-1)\mathcal{A}''(1)}{\mathcal{A}(1)}\\ &+\frac{+24a\rho\mathcal{A}''(1) + 4a^3\rho^3\mathcal{A}'(1) + 24a\rho(\rho-1)\mathcal{A}'(1)}{\mathcal{A}(1)}\\ &+\frac{4a\rho(\rho-1)(\rho-2)\mathcal{A}'(1) + 12a^2\rho^2(\rho-1)\mathcal{A}'(1)}{\mathcal{A}(1)}\\ &\left.+\frac{24a^2\rho^2\mathcal{A}'(1) + 24a\rho(\rho-1)\mathcal{A}'(1) + 24a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right),\end{aligned} \tag{10}$$

where

$$
\begin{aligned}
\nu_{a,r}^{(1)} &= \frac{1}{4}+3a\rho+4a\rho(\rho-1)+4a^2\rho^2+3a^2\rho^2(\rho-1)+a\rho(\rho-1)(\rho-2)+a^3\rho^3,\\
\nu_{a,r}^{(2)} &= 15+18a\rho+6a^2\rho^2+6a\rho^2,\\
\nu_{a,r}^{(3)} &= 6+30a\rho+12a^2\rho^2(\rho-1)+24a\rho(\rho-1)+4a\rho(\rho-1)(\rho-2)+24a^2\rho^2+4a^3\rho^3,\\
\nu_{a,r}^{(4)} &= \frac{1}{5}+6a\rho+15a^2\rho^2+13a\rho(\rho-1)+8a^3\rho^3+2r(\rho-1)+8a\rho(\rho-1)(\rho-2)\\
&\quad +24a^2\rho^2(\rho-1)+6a^3\rho^3(\rho-1)+a\rho(\rho-1)(\rho-2)(\rho-3)+3a^2\rho^2(\rho-1)(2\rho-3)\\
&\quad ++a^2\rho^2(\rho-1)(\rho-2)+a^4\rho^4.
\end{aligned}
$$

The following results are obtained by the linearity of (5) and Lemma 2.1:

$$
\mathcal{F}_{c,\Gamma}((e_1-x);x) = \frac{q_c}{c}\left(a\rho+\frac{1}{2}+\frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\right). \tag{11}
$$

$$
\mathcal{F}_{c,\Gamma}((e_1-x)^2;x) = \frac{q_c}{c}x+\frac{q_c^2}{c^2}\left(\frac{1}{3}+a^2\rho^2+a\rho^2+a\rho+\frac{2\mathcal{A}'(1)+\mathcal{A}''(1)+2a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right). \tag{12}
$$

$$
\begin{aligned}
\mathcal{F}_{c,\Gamma}((e_1-x)^4;x) &= 3\frac{q_c^2}{c^2}x^2+\frac{q_c^3}{c^3}x\left(5-10a\rho+18a\rho^2-16a^2\rho^2+\frac{8\mathcal{A}''(1)+18\mathcal{A}'(1)+16a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right)\\
&\quad +\frac{q_c^4}{c^4}\left(\nu_{a,r}^{(4)}+\frac{A^{(iv)}(1)+8\mathcal{A}'''(1)+15\mathcal{A}''(1)+6\mathcal{A}'(1)}{\mathcal{A}(1)}\right.\\
&\quad +\frac{4a\rho\mathcal{A}'''(1)+6a^2\rho^2\mathcal{A}''(1)+6a\rho(\rho-1)\mathcal{A}''(1)}{\mathcal{A}(1)}\\
&\quad +\frac{24a\rho\mathcal{A}''(1)+4a^3\rho^3\mathcal{A}'(1)+24a\rho(\rho-1)\mathcal{A}'(1)}{\mathcal{A}(1)}\\
&\quad +\frac{4a\rho(\rho-1)(\rho-2)\mathcal{A}'(1)+12a^2\rho^2(\rho-1)\mathcal{A}'(1)}{\mathcal{A}(1)}\\
&\quad \left.+\frac{24a^2\rho^2\mathcal{A}'(1)+24a\rho(\rho-1)\mathcal{A}'(1)+24a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right).
\end{aligned} \tag{13}
$$

Next, we introduce the Korovkin type approximation theorem. As it is known, the space $C_B[0,\infty)$ denotes the all continuous and bounded functions on $[0,\infty)$ and it is endowed with the sup-norm for a function η as below:

$$
\left\|\eta\right\|_{[0,\infty)} = \sup_{x\in[0,\infty)}\left|\eta(x)\right|.
$$

Theorem 2.2. *If* $\eta \in C_B[0,\infty)$ *then*

$$
\lim_{c\to\infty}\mathcal{F}_{c,\Gamma}(\eta;x) = \eta(x),
$$

uniformly on $[0,\infty)$.

Proof. According to the Bohman-Korovkin theorem [39], we have to show that

$$
\lim_{c\to\infty}\sup_{x\in[0,\infty)}\left|\mathcal{F}_{c,\Gamma}(e_s;x)-x^s\right| = 0, \text{ for } s=0,1,2.
$$

One can get the above expression immediately with the help of Lemma 2.1, hence we omit the details. □

3. A Connection with a summability method

Here, we provide a connection between summability methods and the approximation of the proposed operators.

The $\mathcal{B}$-summability was defined in [38]. Suppose that $\mathcal{B} = (\mathcal{B}_\vartheta)$ is a sequence of infinite matrices with $\mathcal{B}_\vartheta = (\tau_{cm}(\vartheta))$. Then $x \in \ell_\infty$ is said to be $\mathcal{B}$-summable to the value $\mathcal{B}-\lim x$, if $\lim_{c\to\infty}(\mathcal{B}_\vartheta x)_c = \mathcal{B}-\lim x$ uniformly for $\vartheta = 0, 1, 2, \cdots$.

Definition 3.1 ([54] and [18]). *The method $\mathcal{B} = (\mathcal{B}_\vartheta)$ is regular if and only if*

(a) $\|\mathcal{B}\| = \sup_{c,\vartheta} \sum_m |\tau_{cm}(\vartheta)| < \infty$;

(b) $\lim_{c\to\infty} \tau_{cm}(\vartheta) = 0$ *uniformly in* ϑ *for each* $m \in \mathbb{N}$;

(c) $\lim_{c\to\infty} \sum_m \tau_{cm}(\vartheta) = 1$ *uniformly in* ϑ.

Definition 3.2. *By $\mathcal{R}^+$ we denote the set of all regular methods $\mathcal{B}$ with $\tau_{cm}(\vartheta) \geqq 0$ for all c, m and ϑ. For a given $\mathcal{B} \in \mathcal{R}^+$ regular non-negative summability matrix, $x = (x_m)$ is said to be $\mathcal{B}$-statistically convergent to the number ℓ, if for every $\epsilon > 0$,*

$$\lim_{c\to\infty} \sum_{m:|x_m-\ell|\geqq\epsilon} \tau_{cm}(\vartheta) \to 0$$

uniformly in ϑ as $c \to \infty$.

Definition 3.3 ([44]). Let $\mathcal{B} = (\mathcal{B}_\vartheta)_{\vartheta\in\mathbb{N}} \in \mathcal{R}^+$. Further, let $w = (w_m)$ be a sequence of nonnegative numbers with $w_0 > 0$ and $W_c = \sum_{m=0}^{c} w_m \to \infty$ as $c \to \infty$. A sequence $x = (x_c)$ is said to be WBSC to the number ℓ if, for every $\epsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{W_j} \sum_{c=0}^{j} w_c \sum_{m:|x_m-\ell|\geqq\epsilon} \tau_{cm}(\vartheta) = 0 \text{ uniformly in } \vartheta.$$

It is denoted by $\left[\mathrm{stat}_{\mathcal{B}}, w_c\right] - \lim x = \ell$.

Theorem 3.4. *Assuming $\eta \in C[0,\Delta]$ and $\mathcal{B} \in \mathcal{R}^+$ we have*

$$[\mathrm{stat}_{\mathcal{B}}, w_c] - \lim_{c\to\infty} \|\mathcal{F}_{c,\Gamma}(\eta, x) - \eta\|_{C[0,\Delta]} = 0.$$

Proof. Assume that $\eta \in C[0,\Delta]$ and $x \in [0,\Delta]$ be fixed. Considering Korovkin theorem, it is sufficient to verify the following relation:

$$[\mathrm{stat}_{\mathcal{B}}, w_c] - \lim_{c\to\infty} \|\mathcal{F}_{c,\Gamma}(e_s; x) - e_s\|_{C[0,\Delta]} = 0,$$

where $e_s(x) = z^s$, $x \in [0,\Delta]$ and $s = 0, 1, 2$. Using the moments results, we get the following relation

$$[\mathrm{stat}_{\mathcal{B}}, w_c] - \lim_{c\to\infty} \|\mathcal{F}_{c,\Gamma}(e_0; x) - e_0\|_{C[0,\Delta]} = 0. \tag{14}$$

Again, by the moments of proposed operators for $s = 1$, one has

$$\sup_{x\in[0,\Delta]} \left|\mathcal{F}_{c,\Gamma}(e_1; x) - e_1(x)\right| = \sup_{x\in[0,\Delta]} \left| x + \frac{q_c}{c}\left(a\rho + \frac{1}{2} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\right) - x\right| = \frac{q_c}{c}\left(a\rho + \frac{1}{2} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\right).$$

Selecting a number $\varepsilon > 0$ for a given $\varepsilon' > 0$ so that $\varepsilon < \varepsilon'$ we define the sets

$$\mathcal{J} := \left\{c \in \mathbb{N} : \left\|\mathcal{F}_{c,\Gamma}(e_1; x) - e_1\right\| \geqq \varepsilon'\right\},$$
$$\mathcal{J}_1 := \left\{c \in \mathbb{N} : \frac{q_c}{c}\left(a\rho + \frac{1}{2} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)}\right) \geqq \varepsilon' - \varepsilon\right\}.$$

So we get

$$\frac{1}{W_j}\sum_{c=0}^{j} w_c \sum_{m\in\mathcal{J}} \tau_{cm}(\vartheta) \le \frac{1}{W_j}\sum_{c=0}^{j} w_c \sum_{m\in\mathcal{J}_1} \tau_{cm}(\vartheta).$$

Taking limit to the last inequality when $j \to \infty$ we attain

$$[\text{stat}_{\mathcal{B}}, w_c] - \lim_{c\to\infty} \|\mathcal{F}_{c,\Gamma}(e_1;x) - e_1\|_{C[0,\Delta]} = 0. \tag{15}$$

Now for $s = 2$, let $a \geq 0$, $\rho \geq 2$ and $g(a,\rho) = 1/3 + a^2\rho^2 + a\rho^2 + a\rho$. By definition of proposed operators and Lemma 2.1 one obtains

$$\sup_{x\in[0,\Delta]} \left|\mathcal{F}_{c,\Gamma}(e_2;x) - e_2(x)\right| \le \left|\frac{q_c}{c}x\left(2 + 2a\rho + \frac{2\mathcal{A}'(1)}{\mathcal{A}(1)}\right) + \frac{q_c^2}{c^2}\left(g(a,\rho) + \frac{2\mathcal{A}'(1) + \mathcal{A}''(1) + 2a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right)\right|.$$

Selecting a number $\varepsilon > 0$ for a given $\varepsilon' > 0$ so that $\varepsilon < \varepsilon'$ we define the sets

$$\mathcal{P} := \left\{c \in \mathbb{N} : \left\|\mathcal{F}_{c,\Gamma}(e_2;x) - e_2\right\| \geqq \varepsilon'\right\},$$

$$\mathcal{P}_1 := \left\{c \in \mathbb{N} : \frac{q_c}{c}x\left(2 + 2a\rho + \frac{2\mathcal{A}'(1)}{\mathcal{A}(1)}\right) \geqq \frac{\varepsilon' - \varepsilon}{2}\right\},$$

$$\mathcal{P}_2 := \left\{c \in \mathbb{N} : \frac{q_c^2}{c^2}\left(g(a,\rho) + \frac{2\mathcal{A}'(1) + \mathcal{A}''(1) + 2a\rho\mathcal{A}'(1)}{\mathcal{A}(1)}\right) \geqq \frac{\varepsilon' - \varepsilon}{2}\right\}.$$

Then the inclusion $\mathcal{P} \subset \mathcal{P}_1 \cup \mathcal{P}_2$ holds true and implies

$$\frac{1}{W_j}\sum_{c=0}^{j} w_c \sum_{m\in\mathcal{P}} \tau_{cm}(\vartheta) \le \frac{1}{W_j}\sum_{c=0}^{j} w_c \sum_{m\in\mathcal{P}_1} \tau_{cm}(\vartheta) + \frac{1}{W_j}\sum_{c=0}^{j} w_c \sum_{m\in\mathcal{P}_2} \tau_{cm}(\vartheta).$$

In conclusion, using the same technique as above,

$$[\text{stat}_{\mathcal{B}}, w_c] - \lim_{c\to\infty} \|\mathcal{F}_{c,\Gamma}(e_2;x) - e_2\|_{C[0,\Delta]} = 0. \tag{16}$$

We attain the desired result by considering (14), (15) and (16) together. □

Definition 3.5 ([34]). Let $\mathcal{B} \in \mathcal{R}_w^+$. A sequence $x = (x_c)$ is statistically weighted $\mathcal{B}$-summable to ℓ if, for each $\epsilon > 0$,

$$\lim_{s} \frac{1}{s}\left|\left\{j \leqq s : \left|\frac{1}{W_j}\sum_{c=0}^{j} w_c \sum_{m=1}^{\infty} x_c\tau_{cm}(\vartheta) - \ell\right| \geqq \epsilon\right\}\right| = 0 \text{ uniformly in } \vartheta.$$

It is denoted as $\overline{N}_{\mathcal{B}}(\text{stat}) - \lim x = \ell$ *for this case.*

Corollary 3.6. *Assuming $\mathcal{B}$ in $\mathcal{R}^+$ and η in $C[0,\Delta]$ we have*

$$\overline{N}_{\mathcal{B}}(stat) - \lim \|\mathcal{F}_{c,\Gamma}(\eta, x) - \eta\|_{C[0,\Delta]} = 0.$$

Proof. The proof is based on Theorem 3.4 and [34, Theorem 7]. □

Definition 3.7 ([34]). Let $\mathcal{B} \in \mathcal{R}^+$. Suppose that (ϕ_m) is a positive non-decreasing sequence. A sequence $x = (x_m)$ is said to be WBSC to ℓ with the rate $o\big(\phi_m\big)$ if, for any $\epsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{\phi_j W_j} \sum_{c=0}^{j} w_c \sum_{m:|x_m-\ell|\geqq\epsilon} \tau_{cm}(\vartheta) = 0 \quad \text{uniformly in } \vartheta.$$

In this case, we denote it by $x_m - \ell = \big[\text{stat}_{\mathcal{B}}, w_c\big] - o\big(\phi_m\big)$.

Theorem 3.8. *Let $(\phi_c)_{c\in\mathbb{N}}$ positive non-decreasing sequence and $\mathcal{B} \in \mathcal{R}^+$. Suppose that the condition*

$$\omega(\eta;\delta_c) = \big[\text{stat}_{\mathcal{B}}, w_c\big] - o\big(\phi_c\big) \ \text{ on } \ [0,\Delta],$$

where $\delta_c := \|\mathcal{F}_{c,\Gamma}(\mu;x)\|^{1/2}_{C[0,\Delta]}$ with $\mu(x) = (t-x)^2$, $t \in [0,\Delta]$, is satisfied. Then

$$\|\mathcal{F}_{c,\Gamma}(\eta) - \eta\|_{C[0,\Delta]} = \big[\text{stat}_{\mathcal{B}}, w_c\big] - o\big(\phi_c\big),$$

for every $\eta \in C[0,\Delta]$.

Proof. Let $\Delta \in \mathbb{R}$ be fixed and $\eta \in C[0,\Delta]$ and $x \in [0,\Delta]$. Since $\mathcal{F}_{c,\Gamma}$ is monotone and linear, we get

$$\begin{aligned}
|\mathcal{F}_{c,\Gamma}(\eta(t);x) - \eta(x)| &\le |\mathcal{F}_{c,\Gamma}\Big(|\eta(t)-\eta(x)|;x\Big) + T\,|\mathcal{F}_{c,\Gamma}(e_0;x) - e_0| \\
&\le \omega(\eta,y)\mathcal{F}_{c,\Gamma}\Big(\frac{|t-x|}{y} + 1;x\Big) \\
&= \omega(\eta,y)\Big\{\mathcal{F}_{c,\Gamma}(e_0;x) + \frac{1}{y^2}\mathcal{F}_{c,\Gamma}(\mu;x)\Big\},
\end{aligned} \tag{17}$$

where $T = \sup_{x\in[0,\Delta]}|\eta(x)|$. Implementing supremum over $x \in [0,\Delta]$ on each sides of (17) and selecting $y = \delta_c = \|\mathcal{F}_{c,\Gamma}(\mu;x)\|^{1/2}_{C[0,\Delta]} y = \delta_c$, we get

$$\|\mathcal{F}_{c,\Gamma}(\eta) - \eta\|_{C[0,\Delta]} \le \omega(\eta,\delta_c)\Big\{1 + \frac{1}{\delta_c^2}\|\mathcal{F}_{c,\Gamma}(\mu;x)\|_{C[0,\Delta]}\Big\} = 2\omega(\eta,\delta_c).$$

For a given $\epsilon > 0$, we define the sets:

$$\mathcal{U} = \{c : \|\mathcal{F}_{c,\Gamma}(\eta) - \eta\|_{C[0,\Delta]} \ge \epsilon\},$$
$$\mathcal{U}_1 = \Big\{c : \omega(\eta,\delta_c) \ge \frac{\epsilon}{2}\Big\}.$$

Subsequently, one can easily verify the relation

$$\frac{1}{\phi_c}\sum_{m\in\mathcal{U}} \tau_{cm}(\vartheta) \quad \le \quad \frac{1}{\phi_c}\sum_{m\in\mathcal{U}_1} \tau_{cm}(\vartheta).$$

This result, combined with the assumption, yields

$$\|\mathcal{F}_{c,\Gamma}(\eta) - \eta\|_{C[0,\Delta]} = \big[\text{stat}_{\mathcal{B}}, w_c\big] - o\big(\phi_c\big),$$

which completes the proof. □

As a preliminary to the next theorem, we assume that $C^2[0,\Delta]$ is the space of all functions $\eta \in C[0,\Delta]$ such that $\eta', \mu'' \in C[0,\Delta]$.

Theorem 3.9. *Let $\mathcal{B} = (\mathcal{B}_\vartheta)_{\vartheta\in\mathbb{N}} \in \mathcal{R}^+$. Let $\eta \in C[0,\Delta]$, $x \in [0,\Delta]$ and $\frac{d\eta(x)}{dx}$, $\frac{d^2\eta(x)}{dx^2}$ exist. Then*

$$[\text{stat}_{\mathcal{B}}, w_c] - \lim_{c\to\infty}\left\{\frac{c\mathcal{F}_{c,\Gamma}(\eta,x)}{q_c} - \frac{c\eta(x)}{q_c}\right\} = \left\{\frac{\mathcal{A}'(1)}{\mathcal{A}(1)} + a\rho + \frac{1}{2}\right\}\frac{d\eta(x)}{dx} + \frac{x}{2}\frac{d^2\eta(x)}{dx^2}$$

uniformly in ϑ. If $\eta \in C^2[0,\Delta]$, convergence is still uniform in $x \in [0,\Delta]$.

Proof. Let $\eta \in C^2[0,\Delta]$ and $x \in [0,\Delta]$ be fixed. Considering Taylor's expansion with Peano's form of reminder we attain

$$\eta(t) - \eta(x) = (t-x)\frac{d\eta(x)}{dx} + \frac{1}{2}(t-x)^2\frac{d^2\eta(x)}{dx^2} + (t-x)^2\,\xi_x(t), \tag{18}$$

where $\xi_x(t)$ is the remainder term so that $\xi_x(t) \in C[0,\Delta]$ and $\xi_x(t) \to 0$ as $t \to x$. Applying $\mathcal{F}_{c,\Gamma}$ to the identity (18) and multiplying by c/q_c, we get

$$\begin{aligned}\frac{c}{q_c}\{\mathcal{F}_{c,\Gamma}(\eta,x) - \eta(x)\} &= \frac{c}{q_c}\frac{d\eta(x)}{dx}\mathcal{F}_{c,\Gamma}(t-x;x)\\ &\quad + \frac{c}{q_c}\frac{d^2\eta(x)}{2dx^2}\mathcal{F}_{c,\Gamma}((t-x)^2;x) + \frac{c}{q_c}\mathcal{F}_{c,\Gamma}((t-x)^2\xi_x(t);x).\end{aligned} \tag{19}$$

The following relations are satisfied by the results in (11), (12) and (13) :

$$\left[\text{stat}_{\mathcal{B}}, w_c\right] - \lim_{c\to\infty}\frac{c}{q_c}\mathcal{F}_{c,\Gamma}(t-x,x) = a\rho + \frac{1}{2} + \frac{\mathcal{A}'(1)}{\mathcal{A}(1)}; \tag{20}$$

$$\left[\text{stat}_{\mathcal{B}}, w_c\right] - \lim_{c\to\infty}\frac{c}{q_c}\mathcal{F}_{c,\Gamma}((t-x)^2,x) = x; \tag{21}$$

$$\left[\text{stat}_{\mathcal{B}}, w_c\right] - \lim_{c\to\infty}\frac{c^2}{q_c^2}\mathcal{F}_{c,\Gamma}((t-x)^4,x) = 3x^2. \tag{22}$$

The following inequality is attained if each parts of (19) is multiplied by c/q_c and if Cauchy-Schwarz inequality is used:

$$\frac{c}{q_c}\mathcal{F}_{c,\Gamma}((t-x)^2\xi_x(t);x) \le \sqrt{\frac{c^2}{q_c^2}\mathcal{F}_{c,\Gamma}((t-x)^4;x)}\sqrt{\mathcal{F}_{c,\Gamma}(\xi_x(t);x)}.$$

Therefore, this concludes the following result:

$$[\text{stat}_{\mathcal{B}}, w_c] - \lim\frac{c}{q_c}\mathcal{F}_{c,\Gamma}((t-x)^2\xi_x(t);x) = 0. \tag{23}$$

Substituting (20), (21), (22) and (23) into (19) concludes the proof. □

4. Conclusion with Experiments

In this section, we consider certain examples and discuss the numerical results.

Remark 4.1. *On consideration of $\mathcal{A}(t) = e^t$, the 2D Appell polynomials $\Gamma_d^{(\rho)}(x;a)$ reduce to the polynomials $\omega_d^{(\rho)}(x;a)$ with the generating function*

$$e^t e^{xt+at^r} = \sum_{d=0}^{\infty}\omega_d^{(\rho)}(x;a)\frac{t^d}{d!}$$

and the explicit representation:

$$\omega_d^{(\rho)}(x;a) = \sum_{e=0}^{[d/\rho]} \frac{d!\,(x+1)^{d-\rho s}\,a^e}{(d-\rho e)!\,e!}.$$

Let $\mathcal{F}_{c,\omega}(\eta;x)$ be the Kantorovich operators involving $\omega_d^{(\rho)}(x;a)$. Then,

$$\mathcal{F}_{c,\omega}(\eta;x) = \frac{c}{q_c}\frac{e^{-\frac{c}{q_c}x-a}}{e}\sum_{d=0}^{\infty}\frac{\omega_d^{(\rho)}(\frac{c}{q_c}x,a)}{d!}\int_{\frac{dq_c}{c}}^{\frac{(d+1)q_c}{c}}\eta(u)du. \tag{24}$$

For the operators $\mathcal{F}_{c,\omega}(\eta;x)$ given by (24) and $x\in[0,\Delta]$, we have:

$$\mathcal{F}_{c,\omega}(\eta;x)((e_1-x)^2;x) \quad = \quad \frac{q_c}{c}x + \frac{q_c^2}{c^2}\left(\frac{1}{3}+a\rho+a^2\rho^2+a\rho(\rho-1)+3+2a\rho\right). \tag{25}$$

Recall the following result of Gavrea and Raşa [30]:

Lemma 4.2 ([30]). *Let $\eta\in C^2[0,a]$ and $(M_c)_{c\geq 0}$ be a sequence of positive linear operators with the property $M_c(1;x)=1$. Then,*

$$|M_c(\eta;x)-\eta(x)| \leq \|\eta'\|\sqrt{M_c((e_1-x)^2;x)} + \frac{1}{2}\|\eta''\|\,M_c((e_1-x)^2;x).$$

Corollary 4.3. *Making use of the expression* (25) *in Lemma 4.2, we find that*

$$\begin{aligned}|\mathcal{F}_{c,\omega}(\eta;x)-\eta(x)| \quad \leq \quad & \|\eta'\|\sqrt{\frac{q_c}{c}x+\frac{q_c^2}{c^2}\left(\frac{1}{3}+a\rho+a^2\rho^2+a\rho(\rho-1)+3+2a\rho\right)} \\ & +\frac{1}{2}\|\eta''\|\left(\frac{q_c}{c}x+\frac{q_c^2}{c^2}\left(\frac{1}{3}+a\rho+a^2\rho^2+a\rho(\rho-1)+3+2a\rho\right)\right).\end{aligned}$$

Example 4.4. *We consider following functions in our experiments:*

(a) The function

$$\eta(x) = sin(x^2)$$

is considered in Table 1. The absolute errors of operators $\mathcal{F}_{c,\omega}$ and $L_{c,b}$ [2] are compared in Table 1. In Figure 3 and Figure 4, approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x)=sin(x^2)$ and related error of approximation are given. The parameters $\rho=2$ and $a=0.1$ are chosen to illustrate the figures. The absolute error of approximation decreases when the sequence q_c decreases.

(b) The function

$$\eta(x) = \frac{e^{\frac{x}{3}}}{1+x^2}$$

is considered in Figure 1 and Figure 2. In Figure 1 and Figure 2, approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x)=\frac{e^{\frac{x}{3}}}{1+x^2}$ and related error of approximation are given. The parameters $q_c=c^{1/100}$, $\rho=2$ and $a=0.001$ are chosen to illustrate the figures. The error of approximation decreases when the variable c increases.

(c) The function

$$\eta(x) = -2xe^{-3x}$$

is considered in Figure 5 and Figure 6. The approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x)=-2xe^{-3x}$ and related error of approximation are given. The parameters $q_c=c^{1/8}$, $\rho=2$ and $a=0.1$ are chosen to illustrate the figures. The error of approximation decreases when the variable c increases.

As a summary, different selections of the sequence q_c, and the parameters ρ and a provide an idea about construction of the proposed operators. All these numerical results show that the proposed operators provide less error of approximation and better convergence behavior with the selected parameters.

Table 1: Comparison of operators $\mathcal{F}_{c,\omega}$ (24) and $L_{c,b}$ [2] by absolute errors

Sequence q_c	c	Error bound of operators $\mathcal{F}_{c,\omega}$ at			Error bound of operators $L_{c,b}$ at		
		$x = 0.2$	$x = 0.5$	$x = 0.8$	$x = 0.2$	$x = 0.5$	$x = 0.8$
$q_c = c^{\frac{1}{8}}$	20	0.0855	0.1514	0.0947	0.0909	0.2505	0.3768
	30	0.0556	0.1076	0.0821	0.0655	0.1947	0.3024
	50	0.0331	0.0688	0.0604	0.0453	0.1454	0.2321
$q_c = c^{\frac{1}{4}}$	20	0.1302	0.2097	0.0880	0.1341	0.3351	0.4812
	30	0.0932	0.1615	0.0957	0.0975	0.2640	0.3941
	50	0.0585	0.1121	0.0840	0.0679	0.2004	0.3102
$q_c = c^{\frac{1}{2}}$	20	0.2877	0.2785	0.0500	0.3365	0.6640	0.8218
	30	0.2512	0.2822	0.0058	0.2573	0.5435	0.7063
	50	0.1851	0.2556	0.0532	0.1873	0.4293	0.5878

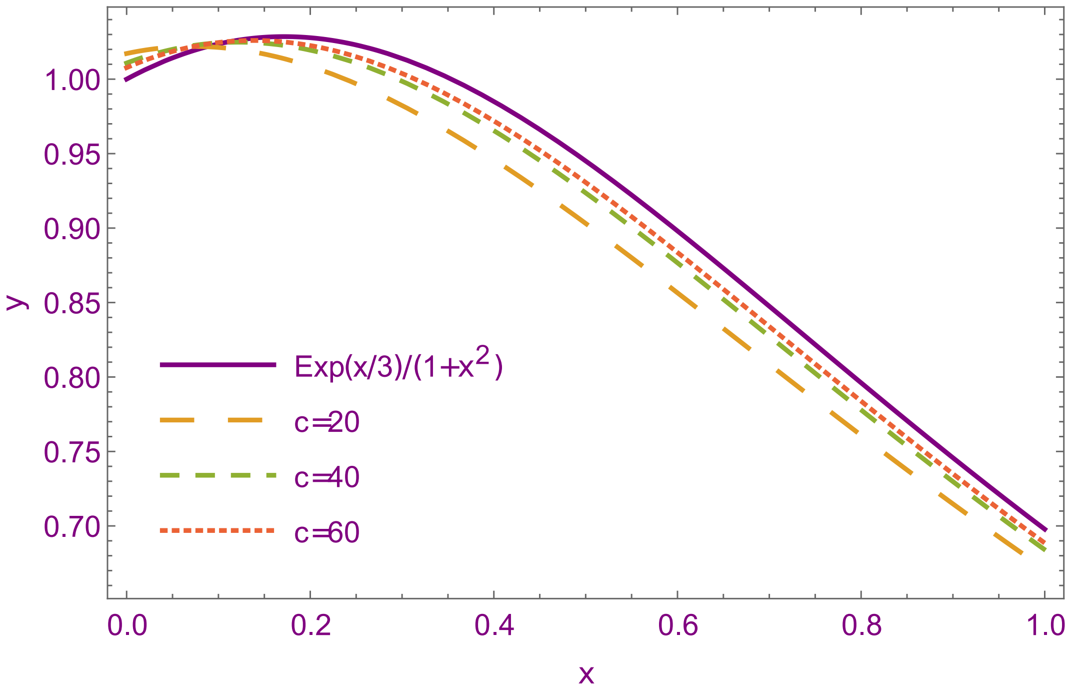

Figure 1: Approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x) = e^{x/3}/(1+x^2)$

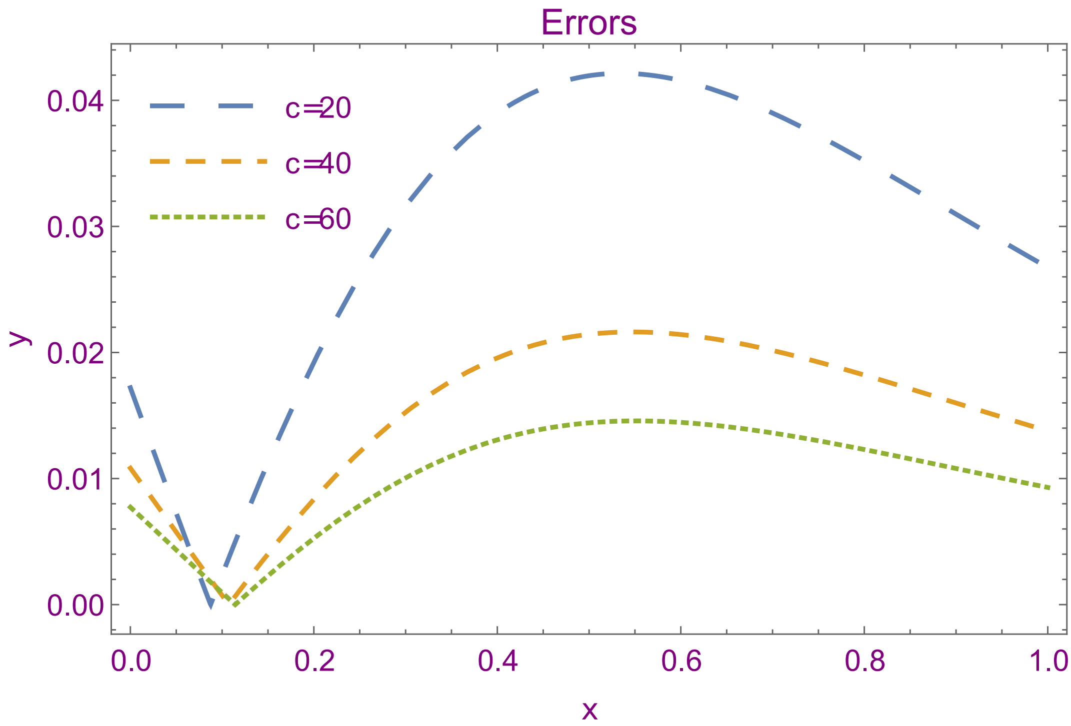

Figure 2: Error of approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x) = e^{-3x}$

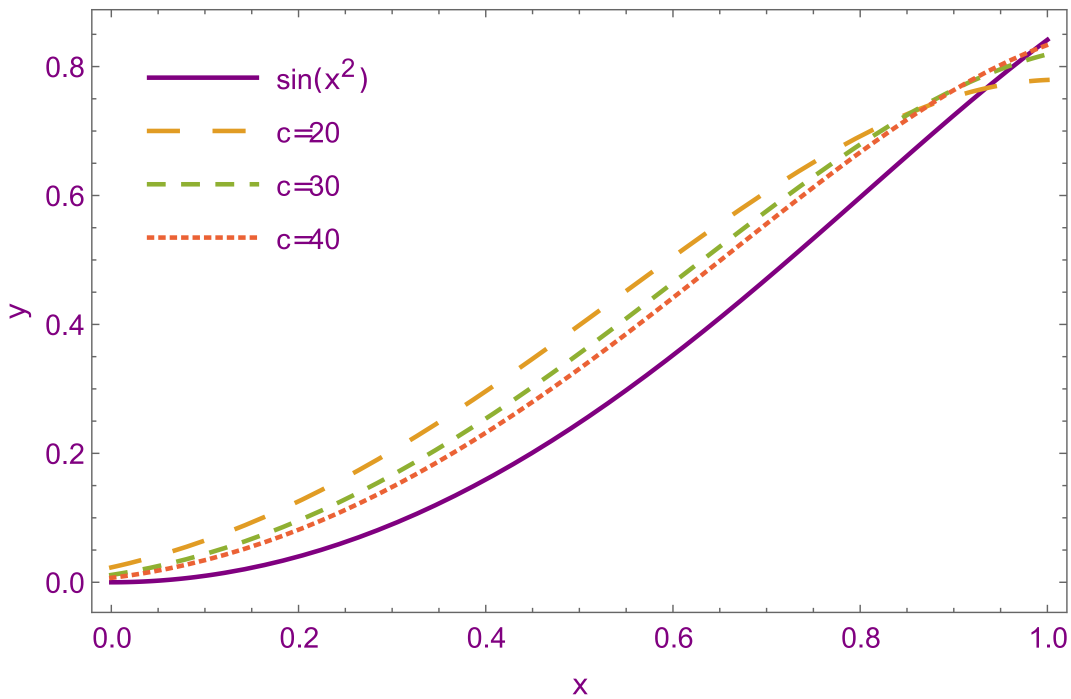

Figure 3: Approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x) = \sin(x^2)$

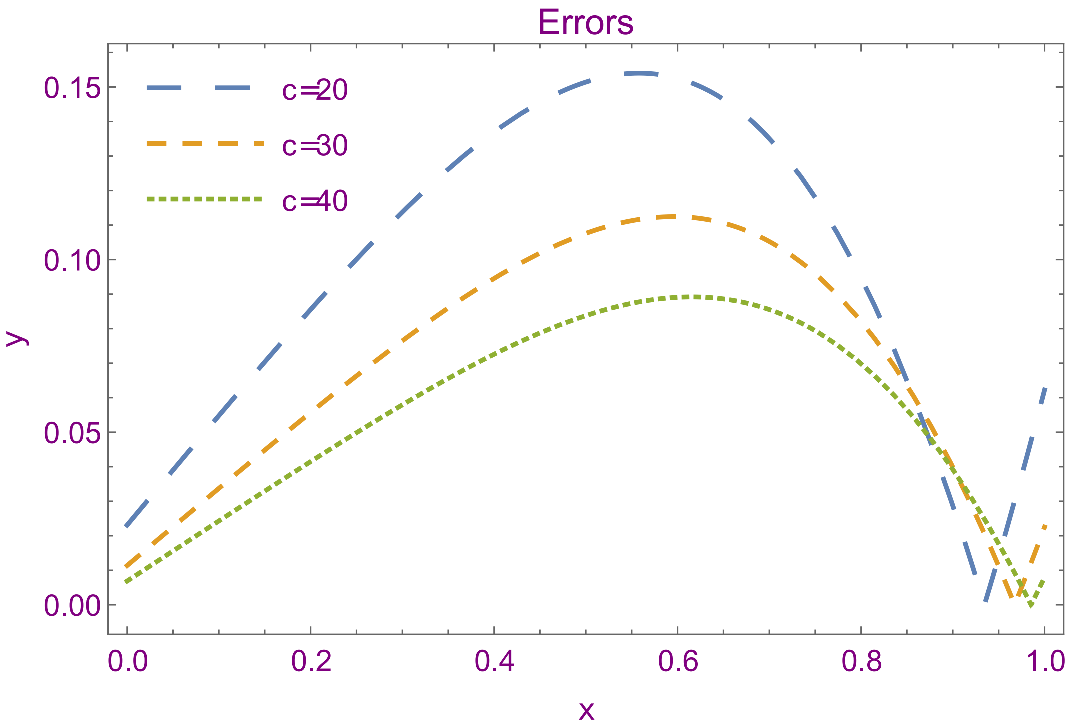

Figure 4: Error of approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x) = \sin(x^2)$

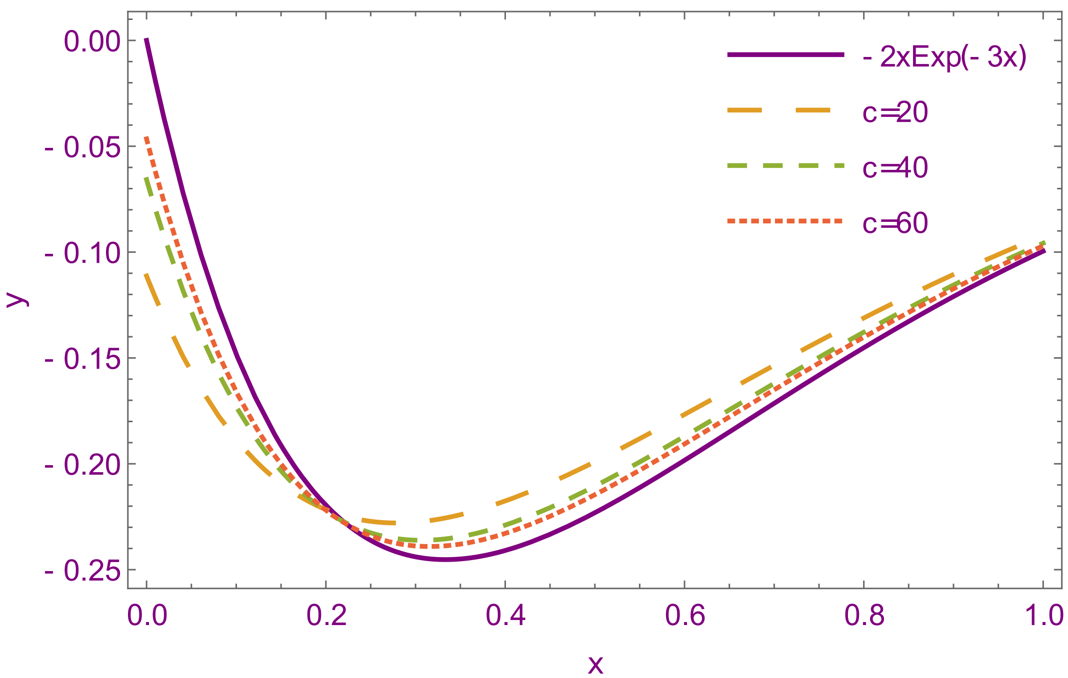

Figure 5: Approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x) = -2xe^{-3x}$

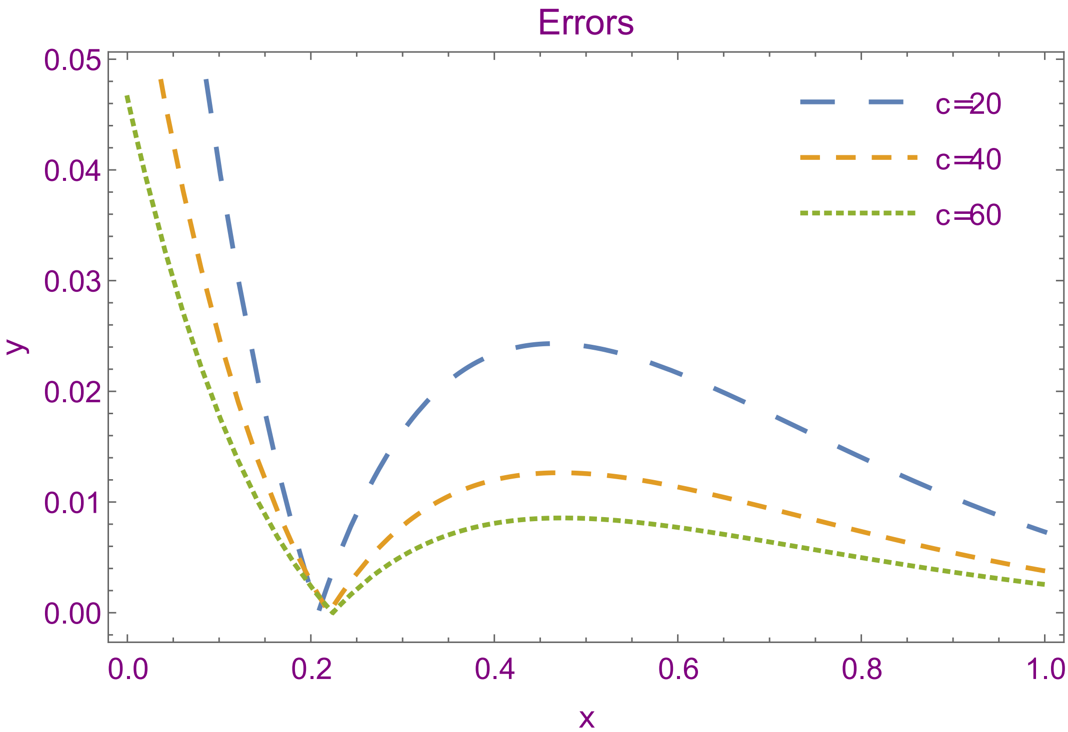

Figure 6: Error of approximation of operators $\mathcal{F}_{c,\omega}$ to $\eta(x) = -2xe^{-3x}$

5. Conclusion

This paper presented a novel construction of Kantorovich type Szász-Chlodowsky operators via 2D Appell polynomials. Some needed explicit estimates of moments and central moments of operators $\mathcal{F}_{c,\Gamma}$ were calculated. Next, the Korovkin-type approximation theorem is proved, and a connection between weighted $\mathcal{B}$-statistical convergence (WBSC) and the rate of WBSC concepts and its approximation results are provided. We also studied a Voronovskaja theorem related to WBSC. Finally, to show the approximation of the operators $\mathcal{F}_{c,\Gamma}$ to the certain functions, we considered some graphical and numerical examples and compared the convergence of the proposed operator with a linear positive operators known in the literature also showed that it provided better approximation results in terms of convergence behavior, calculation efficiency, and consistency.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and contributions.

Declarations

Author Contribution Statements The authors have made equal contributions in this work. All authors read and approved the final copy of this paper.

Funding This work received no funding.

Conflict of interest: The authors declare that there is no known competing financial interests or personal relationship that could have appeared to influence the work reported in this article.

Ethical standard: The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

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https://doi.org/10.3390/sym11030316