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Back to Tachibana numbers of Riemannian manifolds

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Abstract. The Tachibana number $t_p(M)$ of a closed n-dimensional Riemannian manifold (M, g) is defined as the dimension of the vector space of conformally Killing p-forms, for $1 \leq p \leq n-1$, on (M, g). In this paper, we will prove vanishing and estimate propositions for Tachibana numbers $t_p(M)$ of closed n-dimensional Riemannian manifolds, serving as analogues to the vanishing and estimate theorems for their Betti numbers $b_p(M)$.

1. Introduction

The Tachibana number $t_p(M)$ of a closed n-dimensional Riemannian manifold (M, g) is defined as the dimension of the vector space over the field of real numbers $\mathbb{R}$ of conformally Killing p-forms $(1 \leq p \leq n-1)$ on (M, g), see [25, 28].

The idea is based on the following concept: the known Betti number $b_p(M)$, $0 < p < n$, is also equal to the dimension of the vector space over $\mathbb{R}$ of harmonic p-forms defined on (M, g). At the same time, Tachibana numbers possess certain duality property $t_p(M) = t_{n-p}(M)$, which is similar to the Poincare duality for Betti numbers $b_p(M) = b_{n-p}(M)$ of a closed n-dimensional Riemannian manifold (M, g).

We recall here that conformal Killing p-forms, or equivalently, conformal Killing-Yano tensors of degree p on an n-dimensional $(1 \leq p \leq n-1)$ Riemannian manifold (M, g) were defined by Tachibana and his student Kashiwada approximately sixty years ago (see [12, 31]) as a natural generalization of conformally Killing vector fields, or in other words, infinitesimal conformal transformations (see [34, pp. 47–48]).

Conformal Killing p-forms have been extensively studied by many geometers (see, for example, [1, 10, 11, 15, 27, 29], etc.). These studies were motivated by the various applications of these forms (see, for example, [4, 23, 26], [8, pp. 414, 426], etc.). In particular, Benn and Charlton demonstrated that conformal Killing-Yano tensors (of any degree p) give rise to Dirac symmetry operators on spin manifolds of arbitrary dimension and signature (see [4]). A significant focus of these studies was on the conditions for the vanishing of conformal Killing forms on closed Riemannian manifolds (see [12, 15, 31]).

In this paper, we will prove vanishing and estimate propositions for Tachibana numbers $t_p(M)$ of closed n-dimensional Riemannian manifolds as analogues to the vanishing and estimate theorems of Betti numbers $b_p(M)$ for any $1 \leq p \leq n-1$ (see, for example, [34, pp. 70–73], [19, p. 351], [32]).

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2. Conformal Killing forms and Tachibana numbers

2.1 Let (M, g) be an n-dimensional ($n \geq 2$) connected Riemannian manifold with the Levi-Civita connection ∇, and $\Lambda^p M$ be the bundle of differential p-forms over (M, g) for $p = 1, 2, \ldots, n-1$. Bourguignon in [6] proved the existence of a basis $\{D_1, D_2, D_3\}$ in the space of natural (with respect to isometric diffeomorphisms) first-order differential operators on the space $C^\infty(\Lambda^p M)$ with values in the space of homogeneous tensors on (M, g). The first operator, D_1, is the exterior derivative $d : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1} M)$, and the second operator, D_2, is the codifferential $\delta : C^\infty(\Lambda^{p+1} M) \to C^\infty(\Lambda^p M)$. In contrast, the third basis operator, D_3, was not given explicitly for $p > 1$. When $p = 1$, it was mentioned (see [6], Appendix) that the kernel of D_3 is formed by the infinitesimal conformal transformations of (M, g). In our papers [23] and [22], we utilized some classical theorems of Weyl about the representation theory of the orthogonal group $O(n, \mathbb{R})$ and showed that the third operator D_3 takes the form

$$D_3 = \nabla - \frac{1}{p+1}\, d - \frac{1}{n-p+1}\, g \wedge \delta \tag{1}$$

where the symbol $\wedge$ denotes the alternating procedure which given by the formula

$$(g \wedge \delta\omega)(X_0, X_1, \ldots, X_p) = \sum_{a=2}^{p} (-1)^a g(X_0, X_a)\delta\omega(X_1, \ldots, X_{a-1}, X_{a+1} \ldots, X_p)$$

for any vector fields $X_0, X_1, \ldots, X_p \in C^\infty(TM)$. We proved that the kernel of D_3 consists of conformal Killing p-forms (see, for example, [4, 14, 22, 23]). At the same time, the equation $D_3\,\omega = 0$ is called the *conformal Killing-Yano equation* (see, for example, [4]).

The formal adjoint operator D_3^* for D_3 has the form (for details, see [24])

$$D_3^* = \nabla^* - \frac{1}{p+1}\,\delta - \frac{1}{n-p+1}\, d \circ \mathrm{trace}_g,$$

then

$$D_3^* D_3 = \frac{p}{p+1}\left(\nabla^*\nabla - \frac{1}{p+1}\,\delta d - \frac{1}{n-p+1}\, d\,\delta\right). \tag{2}$$

We named $D_3^* D_3$ as the *Tachibana operator* (see, for example, [27]). The operator $D_3^* D_3$ is an elliptic and self-adjoint operator defined on differential p-forms $\omega \in C^\infty(\Lambda^p M)$ for $p = 1, \ldots, n-1$, and its kernel consists of conformal Killing p-forms if (M, g) is a closed Riemannian manifold (see [4, 13, 27]). Therefore, if (M, g) is a closed Riemannian manifold, then the dimension of the vector space of conformal Killing p-forms is $t_p(M) = \dim_{\mathbb{R}} \ker D_3^* D_3 < \infty$. We recall that $t_p(M)$ was referred to as the Tachibana number in [25, 28], by analogy with the Betti number $b_p(M) = \dim_{\mathbb{R}} (\ker D_1 \cap \ker D_2)$ of a closed manifold (M, g). In addition, we can also conclude that $t_p(M) = t_{n-p}(M)$, since the conformal Killing-Yano equation is invariant under Hodge duality (see [4, 15]). Moreover, any $t_p(M)$ is a conformal invariant of (M, g), since if $\omega \in C^\infty(\Lambda^p M)$ is a conformal Killing form, then $\bar{\omega} = e^{(p+1)f}\omega$ is a conformal Killing p-form with respect to the conformally equivalent metric $\bar{g} = e^{2f} g$ (see [4, 15]).

Remark 1 For $p = 1$ the operator $D_3^* D_3$ is the *Ahlfors Laplacian* (see [7, 20]) and if (M, g) is a closed manifold, then its kernel consists of infinitesimal conformal transformations.

2.2 We proved in [29] that the Tachibana number $t_p(\mathbb{S}^n) = \frac{(n+2)!}{(p+1)!(n-p+1)!}$ for a unit n-dimensional sphere $\mathbb{S}^n$ in Euclidean space $\mathbb{R}^{n+1}$, and at the same time, $t_p(\mathbb{R}^n/\Gamma) = \frac{n!}{p!(n-p)!}$ for $1 \leq p \leq n-1$. Moreover, it is known (see [2]) that any Hamiltonian form on a Kähler manifold defines a conformally Killing 2-form. Utilizing the global classification of Kähler manifolds that admit a Hamiltonian 2-form (see [3]), it is possible to compile a list of Kähler manifolds with $t_2(M) > 1$. In addition, if (M, g) is a four-dimensional closed Riemannian manifold with a zero Ricci tensor and a nonzero first Betti number $b_1(M)$, then $t_1(M) = t_3(M) = 4$ and $t_2(M) = 6$ (see our Corollary 4.3).

3. Curvature and Tachibana numbers

3.1 In any coordinate neighborhood U on (M, g) with local coordinate system $\{x^1, \ldots, x^n\}$ the metric g can be given by its components $g_{ij} = g(\partial_i, \partial_j)$, were $\partial_i = \partial/\partial x^i$ and $i, j, k, \dots = 1, \ldots, n$. We denote by Ric the Ricci tensor for the metric g of a manifold (M, g). We construct this tensor by formulas $R_{ij} = R_{iklj} g^{kl}$ for the components $R_{iklj} = Rm(\partial_i, \partial_k, \partial_l, \partial_j)$ of the Riemann curvature tensor Rm. We also denote by $s = g^{ij} R_{ij}$ the scalar curvature of the metric g for $(g^{ij}) = (g_{ij})^{-1}$.

Let us consider the quadratic form $\mathcal{F}_p : \Lambda^p M \times \Lambda^p M \to R$ defined by the equality (see [34, p. 33])

$$\mathcal{F}_p(\omega) = R_{ij} \omega^i_{i_2 \ldots i_p} \omega^{j i_2 \ldots i_p} - \frac{p-1}{2} R_{ijkl} \omega^{ij}{}_{i_3 \ldots i_p} \omega^{k l i_3 \ldots i_p}, \tag{3}$$

where $\omega^{i_1 \ldots i_p} =: g^{i_1 j_1} \ldots g^{i_p j_p} \omega_{j_1 \ldots j_p}$ for the local components $\omega_{i_1 i_2 \ldots i_p} = \omega(\partial_{i_1}, \ldots, \partial_{i_p})$ of $\omega \in \Lambda^p M$. In turn, Kora proved the identity (see [15])

$$\mathcal{F}_p(\omega) = \mathcal{F}_{n-p}(*\omega), \tag{4}$$

where $* : \Lambda^p M \to \Lambda^{n-p} M$ is the Hodge star operator (see [34, pp. 5-6], [19, p. 162]).

Paper [12] tells us that there are no nonparallel conformally Killing p-forms ($p \le n/2$) on any n-dimensional closed Riemannian manifold (M, g) if the quadratic form $\mathcal{F}_p : \Lambda^p M \times \Lambda^p M \to \mathbb{R}$ is nonpositive everywhere on (M, g). Furthermore, there are no nonzero conformally Killing p-forms ($p \le n/2$) on any n-dimensional closed Riemannian manifold (M, g) if the quadratic form $\mathcal{F}_p : \Lambda^p M \times \Lambda^p M \to \mathbb{R}$ is negative everywhere on (M, g). In turn, Kora, using (4), generalized this result as follows: if a conformal Killing p-form ω satisfies $\mathcal{F}_p(\omega) \le 0$ for any $1 \le p \le n-1$, then it is parallel. Especially, if $\mathcal{F}_p(\omega)$ is negative definite, then there exists no conformal Killing p-form other than the zero form. In particular, $t_p(\mathbb{H}^n) = 0$ for the real closed hyperbolic space $\mathbb{H}^n$.

3.2 The Riemann curvature tensor Rm of (M, g) naturally induces two self-adjoint curvature operators: $\hat{R}$ acts on the space of two-forms $\Lambda^2(T_x M)$ at an arbitrary point $x \in M$ via

$$\hat{R}(\omega)_{ij} = R_{ijkl} \omega^{kl}, \tag{5}$$

and $\mathring{R}$ acts on the space $S^2_0(T_x M)$ of trace-free symmetric two-tensors via

$$\mathring{R}(\varphi)_{ij} = R_{iklj} \varphi^{kl}. \tag{6}$$

Following the terminology of Kashiwada and Nishikawa (see [13] and [18]), we call the symmetric bilinear form $\mathring{R}$ in (6) the curvature operator of the second kind, to distinguish it from the curvature operator $\hat{R}$ defined in (5), which Nishikawa called the curvature operator of the first kind.

We can interpret $\hat{R}$ as the symmetric bilinear form $\hat{R} : \Lambda^2(T_x M) \times \Lambda^2(T_x M) \to \mathbb{R}$. Then at an arbitrary point $x \in M$ we choose orthogonal unit vectors $X, Y \in T_x M$, then by direct calculation we obtain $g(\hat{R}(X \wedge Y), X \wedge Y) = 4\, sec(X, Y)$ (see [13]). Then it is easy to see that if $\hat{R} \le 0$ (resp., $\hat{R} < 0$) as a symmetric bilinear form on $\Lambda^2(T_x M)$, then the sectional curvature of (M, g) is non-positive (resp., negative). In this case, the Ricci curvature and scalar curvature also are non-positive (resp., negative).

On the other hand, we can interpret $\mathring{R}$ as the symmetric bilinear form $\mathring{R} : S^2_0(T_x M) \times S^2_0(T_x M) \to \mathbb{R}$. In this case, by direct calculation we obtain $g(\mathring{R}(X \odot Y), X \odot Y) = 2\, sec(X, Y)$, where $\odot$ denotes the symmetric product (see [13]). Then it is easy to see that if $\mathring{R} \le 0$ (resp., $\mathring{R} < 0$) as a symmetric bilinear form on $S^2_0 M$, then the sectional curvature of (M, g) is non-positive (resp., negative). In this case, the Ricci curvature and scalar curvature also are non-positive (resp., negative).

The geometry and topology of manifolds with non-negative or positive operators of both the first and second kinds have long fascinated scholars in global Riemannian geometry (see [19, Chapters 7-9], [5, 13, 16, 18, 21], etc.). In turn, in [30], we provided the classification of connected, complete, locally irreducible Riemannian manifolds with a nonpositive curvature operator of the first kind that admit L^2-integrable conformal Killing forms. Furthermore, we proved that if the curvature operator of the first kind (respectively, second kind) is negative, then a closed manifold (M, g) admits no nonzero conformally Killing p-forms, and consequently, $t_p(M) = 0$ for $p = 1, 2, \dots, n-1$ (see [25, 26, 29]). These theorems serve as analogues to the two vanishing theorems for the Betti numbers of closed Riemannian manifolds (see [34, pp. 70-73], [19, p. 351], [17, 32]).

4. Tachibana numbers of a manifold with nonpositive curvature operator of the second kind

4.1 In this section we generalize our result and results of Kashiwada and Kora in the form of the following theorem.

Theorem 4.1. *Let (M, g) be an n-dimensional closed connected Riemannian manifold with nonpositive curvature operator of the second kind. Then, it is either a compact Euclidean space form or $t_p(M) = 0$ for $p = 1, 2, \dots, n-1$.*

Proof. For an arbitrary $\omega \in C^\infty(\Lambda^p M)$ the following integral formula holds (see formula (8) from [27]):

$$\frac{p+1}{p}\int_M g(D_3^* D_3\omega, \omega) dvol_g + p\int_M \mathcal{F}_p(\omega) dvol_g =$$

$$\int_M \left(\frac{p}{p+1}\,\|d\omega\|^2 + \frac{n-p}{n-p+1}\,\|\delta\omega\|^2\right) dvol_g \leq 0,$$

where $\|d\omega\|^2 = g(d\omega, d\omega)$ and $\|\delta\omega\|^2 = g(\delta\omega, \delta\omega)$.

Let ω be a conformal Killing p-form, then $D_3^* D_3\omega = 0$ and, therefore, it satisfies the integral equation

$$p\int_M \mathcal{F}_p(\omega) dvol_g = \int_M \left(\frac{p}{p+1}\,\|d\omega\|^2 + \frac{n-p}{n-p+1}\,\|\delta\omega\|^2\right) dvol_g \geq 0. \tag{7}$$

In turn, Tachibana and Ogiue rewrote the quadratic form $\mathcal{F}_p(\omega)$ as (see [32])

$$\mathcal{F}_p(\omega) = \frac{1}{6}\, g(\mathring{R}(\bar{\varphi}), \bar{\varphi}) + \frac{2p(n-2p)}{3n}\, R_{ij}\omega^{i}_{i_2\dots i_p}\omega^{ji_2\dots i_p} + \frac{2p^2}{3n^2}\, s\,\|\omega\|^2, \tag{8}$$

where $\bar{\varphi} := \varphi^{[i_1 i_2 \dots i_p]}$ is the symmetric trace less two-tensor with local components

$$\varphi^{[i_1 i_2 \dots i_p]}_{jk} = \sum_{a=1}^{p}\left(\omega_{i_1 i_2 \dots i_{a-1} j i_{a+1} \dots i_p}\, g_{k i_a} + \omega_{i_1 i_2 \dots i_{a-1} k i_{a+1} \dots i_p}\, g_{j i_a}\right) - \frac{2p}{n}\, g_{jk}\omega_{i_1 i_2 \dots i_p}$$

for each set of values of indices $[i_1 i_2 \dots i_p]$ such that $1 \leq i_1 < i_2 < \dots < i_p \leq n$. Next, if we assume that (M, g) has a non-positive curvature operator of the second kind, then $\mathcal{F}_p(\omega) \leq 0$ for $n \geq 2p$. Furthermore, using (4), we conclude that the inequality $\mathcal{F}_p(\omega) \leq 0$ holds for any $p = 1, 2, \dots, n-1$.

In this case, from (7) we obtain $\mathcal{F}_p(\omega) = 0, \nabla\omega = 0$ and hence $\|\omega\|^2 = \text{const}$ for any $p = 1, 2, \dots, n-1$. In this case, from (8) we derive the identity $s \cdot \|\omega\|^2 = 0$ for $\omega \in C^\infty(\Lambda^p M)$, since $\mathring{R}$, Ric and s are non-positive. If there is at least one point $x \in M$ at which $s(x) < 0$, then ω must be equal to zero at the point $x \in M$. In this case ω is equal to zero everywhere on (M, g), since $\|\omega\|^2 = \text{const}$. As a result, we can conclude that the p-th Tachibana number $t_p(M) = 0$ for any $p = 1, 2, \dots, n-1$. Consequently, (M, g) admits no non-zero conformal Killing p-forms for all $1 \leq p \leq n-1$. Otherwise, we have $s = \sum_{i \neq j} sec(e_i, e_j) = 0$, where $sec(e_i, e_j) \leq 0$ and $e_1, \dots, e_n$ is any orthonormal basis of $T_x M$ at an arbitrary point $x \in M$. Then (M, g) is a manifold with zero sectional curvature. This is equivalent to saying that the Riemannian curvature tensor vanishes. Therefore, (M, g) is a closed connected flat Riemannian manifold and, hence, is isometric to a quotient $\mathbb{R}^n/\Gamma$, where Γ is a group of isometries acting freely and properly discontinuously (see [33, pp. 68-69]). These manifolds are known as compact Euclidean space forms (see also [33, pp. 105; 125]). Our theorem has been proven. □

Remark 3 For the harmonic p-form $\omega \in C^\infty(\Lambda^p M)$, the integral formula, similar to formula (7), has the form (see [34, p. 70]):

$$p \int_M \mathcal{F}_p(\omega) dvol_g = -\int_M \|\nabla\omega\|^2 dvol_g \leq 0. \tag{9}$$

Therefore, by using equations (8) and (9), it is straightforward to prove the following proposition (cf. [17]): Let (M, g) be an n-dimensional closed connected Riemannian manifold with a nonnegative curvature operator of the second kind. Then, it is either a compact Euclidean space form or $b_p(M) = 0$ for all $p = 1, 2, \ldots, n-1$.

4.2 In addition, the following corollary is true.

Corollary 4.2. *Let (M, g) be an n-dimensional closed connected Riemannian manifold with nonpositive curvature operator of the second kind. If its Tachibana number $t_p(M) \neq 0$ for some $1 \leq p \leq n-1$, then it is the boundary of a compact Riemannian manifold.*

Proof. Let us consider an n-dimensional closed Riemannian manifold (M, g) with a non-negative curvature operator of the second kind and $t_p(M) \neq 0$ for some $p \neq 0, n$. In this case, from equation (7) and (8), we derive the identity $s \cdot \|\omega\|^2 = 0$ for a non-zero conformal Killing form $\omega \in C^\infty(\Lambda^p M)$. Then $s = 0$ everywhere on (M, g), and consequently, the sectional curvature of (M, g) also identically vanishes everywhere on (M, g). Thus, (M, g) is a flat manifold (see also [31]). At the same time, as proven in [9], every closed, connected and flat Riemannian manifold is the boundary of a compact manifold. As a result, our first corollary has been proven. □

Remark 4 By using equations (8) and (9), it is straightforward to prove the following proposition: Let (M, g) be an n-dimensional closed and connected Riemannian manifold with nonnegative curvature operator of the second kind. If its Betti number $b_p(M) \neq 0$ for some $1 \leq p \leq n-1$, then it is the boundary of a compact Riemannian manifold.

And finally, we formulate and prove our last proposition.

Corollary 4.3. *Let (M, g) be a four-dimensional closed Riemannian manifold with zero Ricci tensor and nonzero first Betti number $b_1(M)$, then $t_1(M) = t_3(M) = 4$ and $t_2(M) = 6$.*

Proof. It is well known (see [35]) that every closed four-dimensional Riemannian manifold with zero Ricci tensor and nonzero first Betti number is flat or in other words, is an n-dimensional Euclidean space form. Moreover, if (M, g) is an n-dimensional Euclidean space form, then $t_p(M) = \frac{n!}{p!(n-p)!}$ for $1 \leq p \leq n-1$ (see [28]). This proposition follows from the fact that we may choose a local coordinate system $x^1, \ldots, x^n$ in which $\nabla_k \omega_{i_1 \ldots i_p} := \partial_k \omega_{i_1 \ldots i_p} = 0$ (see [33, pp. 44-45]) , [30, p. 212]). In particular, a parallel form is a conformal Killing form. Then the Tachibana number estimate follows from the fact that a parallel form is completely determined by its value at a point. Then $t_1(M) = t_3(M) = 4$ and $t_2(M) = 6$ if (M, g) is a closed four-dimensional Euclidean space form. Our second corollary has been proven. □

Remark 5 Based on the property of parallelism, if a form is parallel, its value at any point on the manifold can be obtained from its value at a specific point through parallel transport. Thus, knowing the form at one point allows the entire form on the manifold to be completely reconstructed. This property has important implications in Riemannian geometry and global analysis, as it allows for the simplification of studying the geometric and analytical properties of manifolds by using local data of the form to gain information about its global properties.

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