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The Chen type of conoid surfaces family in Minkowski 3-space $\mathbb{L}^3$

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Abstract. In this paper, the classification of the conoid surfaces family in Minkowski 3-space $\mathbb{L}^3$ is conducted under the condition $\Delta^J r_i = \lambda_i r_i$, $J = I, II, III$, where $\lambda_i \in \mathbb{R}$ and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II, and III.

1. Introduction

Euclidean submanifolds of finite type were introduced by Chen [6]. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space.

Meng and Liu [18] considered factorable surfaces along two lightlike directions and spacelike-lightlike directions in Minkowski 3-space and they also gave some classifcation theorems. Beneki et al. [5] indicated the Minkowski 3-space versions of helicoidal surfaces.

For an 2-dimensional submanifold of a Euclidean space $\mathbb{E}^3$, the mean curvature vector H satisfies Beltrami's formula:

$$\Delta r = -2H. \tag{1}$$

Formula (1) implies that the immersion is minimal if and only if it is harmonic. The family of submanifolds of finite type is huge, which contains many important families of submanifolds. Finite type submanifolds are the natural extension of minimal submanifolds. In [8], Chen has provided some open problems and conjectures on submanifolds of finite type.

A well known result due to Takahashi [28] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$. In [7], studied the submanifolds of Euclidean spaces satisfying $\Delta H = AH$. In [13], authors studied birotational hypersurface and the second Laplace–Beltrami operator in the four dimensional Euclidean space.

Here in this kind of research, many results can be found in [1, 4, 9–17, 20, 27].

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In this work, some fundamental notions of Minkowski 3-space $\mathbb{L}^3$ are introduced in Section 2. In Section 3, the family of minimal conoid surfaces is presented. The family of conoid surfaces satisfying $\Delta r_i = \lambda_i r_i$ (as well as $\Delta^{II} r_i = \lambda_i r_i$ and $\Delta^{III} r_i = \lambda_i r_i$) in $\mathbb{L}^3$ is determined in Section 4 (and respectively, in Section 5 and Section 6). Finally, in the last section, a conclusion is provided.

2. Preliminaries

Let $\mathbb{L}^3$ be a 3-dimensional Minkowski space with the scalar product of index 1 given by $g_{\mathbb{L}^3} = ds^2 = dx^2 + dy^2 - dz^2$, where (x, y, z) is a rectangular coordinate system of $\mathbb{L}^3$.

A vector V of $\mathbb{L}^3$ is said to be timelike if $g_{\mathbb{L}^3}(V, V) < 0$, spacelike if $g_{\mathbb{L}^3}(V, V) > 0$ or $V = 0$ and lightlike or null if $g_{\mathbb{L}^3}(V, V) = 0$ and $V \neq 0$. A surface in $\mathbb{L}^3$ is spacelike, timelike or lightlike if the tangent plane at any point is spacelike, timelike or lightlike, respectively.

The Lorentz scalar product of the vectors V and W is defined by $g_{\mathbb{L}^3}(V, W) = v_1 w_1 + v_2 w_2 - v_3 w_3$, where $V = (v_1, v_2, v_3)$, $W = (w_1, w_2, w_3) \in \mathbb{L}^3$.

We denote a surfaces M^2 in $\mathbb{L}^3$ by

$$r(u, v) = \big(r_1(u, v),\ r_2(u, v),\ r_3(u, v)\big).$$

The inmersion (M^2, r) is said to be of finite Chen-type k if the position vector r admits the following spectral decomposition

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_0 is a fixed vector and r_i are $\mathbb{L}^3$-valued eigenfunctions of the Laplacian of (M^2, r) [6]:

$$\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}, i = 1, 2, ..., k.$$

If λ_i are different, then M^2 is said to be of k-type. In particular, if one of λ_i, $(i = 1, 2, ..., k)$ is zero, then the immersion is said to be of *null k-type*.

The Laplace–Beltrami operator of a smooth function

$$\varphi : M^2 \to \mathbb{R}, (u, v) \mapsto \varphi(u, v)$$

with respect to the first fundamental form of the surface M^2 is the operator Δ, determined by [19]:

$$\Delta\varphi = \frac{-1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\varphi_u - F\varphi_v}{\sqrt{|EG - F^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{F\varphi_u - E\varphi_v}{\sqrt{|EG - F^2|}} \right) \right]. \tag{2}$$

The second differential parameter of Beltrami of a function $\varphi : M^2 \to \mathbb{R}, (u, v) \longmapsto \varphi(u, v)$ with respect to the second fundamental form of M^2 is the operator Δ^{II} which is defined by [19]:

$$\Delta^{II}\varphi = \frac{-1}{\sqrt{|LN - M^2|}} \left[\frac{\partial}{\partial u} \left(\frac{N\varphi_u - M\varphi_v}{\sqrt{|LN - M^2|}} \right) + \frac{\partial}{\partial v} \left(\frac{L\varphi_v - M\varphi_u}{\sqrt{|LN - M^2|}} \right) \right], \tag{3}$$

where $LN - M^2 \neq 0$ since the surface has no parabolic points.

If the third fundamental form III is non-degenerate, then the Laplacian Δ^{III} with respect to III can be defined formally on the pseudo-Riemannian manifold $(M^2,\ III)$.

The second Beltrami differential operator with respect to the third fundamental form III is defined by

$$\Delta^{III} = \frac{-1}{\sqrt{|e|}}\Big(\frac{\partial}{\partial x^i}\big(\sqrt{|e|}e^{ij}\frac{\partial}{\partial x^j}\big)\Big), \tag{4}$$

where $e = \det(e_{ij})$ and e^{ij} denote the components of the inverse tensor of e_{ij}.

If $r = (r_1, r_2, r_3)$ is a function of class C^2 then we set

$$\Delta^{III} r = (\Delta^{III} r_1, \Delta^{III} r_2, \Delta^{III} r_3).$$

The mean curvature and the Gauss curvature are

$$H = \frac{EN + GL - 2FM}{2\left|EG - F^2\right|},\ K_G = g_{\mathbb{L}^3}(\mathbf{N},\mathbf{N})\frac{LN - M^2}{EG - F^2},$$

where E, G, F are the coefficients of the first fundamental form, L, M, N are the coefficients of the second fundamental form.

In this paper, we classify conoid surfaces family in Minkowski 3-space $\mathbb{L}^3$ satisfying some algebraic equations in terms of the coordinate functions and the Laplacian operators with respect to the first, the second and the third fundamental form of the surface.

3. Minimal conoid surfaces family in $\mathbb{L}^3$

In this section we define and study conoid surfaces family in $\mathbb{L}^3$.

Definition 3.1. *A surface M^2 in $\mathbb{L}^3$ is a conoid surfaces family (CSF) if it can be parameterized by [11]:*

$$r(u,\ v) = (h(v),\ f(u)\cosh g(v),\ f(u)\sinh g(v)), \tag{5}$$

where $f = f(u)$, $g = g(v)$ and $h = h(v)$ denote the differentiable functions.

The coefficients of the first fundamental form of M^2 are given by

$$g_{11} = E = f'^2,\ \ g_{22} = G = h'^2 - f^2 g'^2,\ \ g_{12} = g_{21} = F = 0. \tag{6}$$

Let $\mathbf{N}$ denotes a unit normal vector field and put $g_{\mathbb{L}^3}(\mathbf{N},\mathbf{N}) = \varepsilon = \pm 1$, so that $\varepsilon = -1$ or $\varepsilon = 1$ according to M^2 is endowed with a Lorentzian or Riemannian metric, respectively.

The unit normal vector is given by

$$\boldsymbol{N} = \frac{1}{\sqrt{Q}}(fg',\ h'\sinh(g),\ h'\cosh(g)),$$

where $Q = g_{\mathbb{L}^3}(\boldsymbol{N},\boldsymbol{N})(f^2 g'^2 - h'^2)$ and

$$g_{\mathbb{L}^3}(\boldsymbol{N},\boldsymbol{N}) = \varepsilon,\quad \varepsilon = \begin{cases} 1, & M^2 \text{ is spacelike } f^2 g'^2 - h'^2 > 0, \\ -1, & M^2 \text{ is timelike } f^2 g'^2 - h'^2 < 0. \end{cases}$$

The constant ε is called the sign of M^2. The coefficients of the second fundamental form are given by

$$L = 0,\ \ M = -\frac{f'g'h'}{\sqrt{Q}},\ \ N = \frac{f(g'h'' - h'g'')}{\sqrt{Q}}. \tag{7}$$

Now, by using the natural frame $\{\boldsymbol{N}_u,\ \boldsymbol{N}_v\}$ of M^2 defined by

$$\boldsymbol{N}_u = \frac{1}{Q^{\frac{3}{2}}}(\boldsymbol{N}_u^1,\ \boldsymbol{N}_u^2,\ \boldsymbol{N}_u^3) \tag{8}$$

and

$$\boldsymbol{N}_v = \frac{1}{Q^{\frac{3}{2}}}(\boldsymbol{N}_v^1,\ \boldsymbol{N}_v^2,\ \boldsymbol{N}_v^3), \tag{9}$$

where

$$\begin{aligned}
\boldsymbol{N}_u^1 &= -g'f'h'^2,\ \ \boldsymbol{N}_u^2 = -ff'h'g'^2\sinh(g),\ \ \boldsymbol{N}_u^3 = -ff'h'g'^2\cosh(g),\\
\boldsymbol{N}_v^1 &= fh'(g'h''-h'g''),\\
\boldsymbol{N}_v^2 &= f^2g'(g'h''-h'g'')\sinh(g)+h'g'(g'^2f^2-h'^2)\cosh(g),\\
\boldsymbol{N}_v^3 &= f^2g'(g'h''-h'g'')\cosh(g)+h'g'(g'^2f^2-h'^2)\sinh(g),
\end{aligned}$$

the components $e_{ij};\ \ i,j=1,2$ of the third fundamental form in (local) coordinates are determined by

$$\begin{aligned}
e_{11} &= g_{\mathbb{L}^3}(\boldsymbol{N}_u,\boldsymbol{N}_u) = -\frac{g'^2f'^2h'^2}{Q^2},\\
e_{12} &= g_{\mathbb{L}^3}(\boldsymbol{N}_u,\boldsymbol{N}_v) = \frac{fg'f'h'(g'h''-h'g'')}{Q^2},\\
e_{22} &= g_{\mathbb{L}^3}(\boldsymbol{N}_v,\boldsymbol{N}_v) = \frac{-f^2(g'h''-h'g'')^2+g'^2h'^2Q}{Q^2}.
\end{aligned}$$

The expression of H is

$$H = \frac{-f(g'h''-h'g'')}{2Q^{\frac{3}{2}}}. \tag{10}$$

Then, M^2 is a minimal surface if and only if

$$g'h''-h'g''=0. \tag{11}$$

Case 1. Assume $g'=0$, then $g(v)=\alpha\in\mathbb{R}$. In such case, (11) is satisfied for any function h.
Case 2. Assume $h'=0$, then $h(v)=\beta\in\mathbb{R}$. In such case, (11) is satisfied for any function g.
Case 3. Assume $g'h'\neq 0$. Equation (11) writes as

$$\left(\frac{h'}{g'}\right)'=0.$$

A direct integration implies that there exist $\gamma,\ \delta\in\mathbb{R}$ such that

$$h(v)=\gamma g(v)+\delta.$$

Therefore, we have the following:

Theorem 3.2. *Let M^2 be a conoid surfaces family in $\mathbb{L}^3$. Let its mean curvature equal zero, then this surface will be one of the following:*
i) $g(v)=\alpha\in\mathbb{R}$ and h is any arbitrary function,
ii) $h(v)=\beta\in\mathbb{R}$ and g is any arbitrary function,
iii) $h(v)=\gamma g(v)+\delta,\ \gamma,\ \delta\in\mathbb{R}$.

4. Conoid surfaces family satisfying $\Delta r_i = \lambda_i r_i$

In this section, we explore the classification of the conoid surfaces family satisfying the relation

$$\Delta r_i = \lambda_i r_i, \tag{12}$$

where $\lambda_i \in \mathbb{R},\ i = 1, 2, 3$ and

$$(r_1(u, v),\ r_2(u, v),\ r_3(u, v)) = (h(v),\ f(u)\cosh g(v),\ f(u)\sinh g(v)).$$

If we use (2), the Laplacian Δ of M^2 is given by

$$\Delta \ = \ \frac{\varepsilon}{f'^2}\frac{\partial^2}{\partial u^2} - \frac{\varepsilon}{Q}\frac{\partial^2}{\partial v^2} + \frac{\varepsilon}{2f'^3 Q}(f' Q_u - 2f'' Q)\frac{\partial}{\partial u} + \frac{\varepsilon}{2Q^2}Q_v\frac{\partial}{\partial v}, \tag{13}$$

where $Q_u = \frac{\partial Q}{\partial u}$ and $Q_v = \frac{\partial Q}{\partial v}$.

By (5), (12) and (13), we get the following system of differential equations:

$$\frac{2Hg' f}{\sqrt{Q}} = \lambda_1 h, \tag{14}$$

$$\frac{2Hh' \sinh g}{\sqrt{Q}} = \lambda_2 f \cosh g, \tag{15}$$

$$\frac{2Hh' \cosh g}{\sqrt{Q}} = \lambda_3 f \sinh g. \tag{16}$$

Therefore, the problem of classifying the conoid surfaces family M^2, satisfying (12) and (5) is reduced to the integration of this system of ordinary differential equations.

Next we study this system concerning the values of the constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1. Let $\lambda_1 = 0$.

Then, the equation (14) gives rise to $H = 0$, which means that the surface is minimal. We get also, by the equations (15) and (16), $\lambda_2 = 0 = \lambda_3$.

Case 2. Let $\lambda_1 \neq 0$.

a) If $\lambda_2 = 0$ and $\lambda_3 = 0$ equations (15) and (16) imply that $h' = 0$.

Thus (10) implies $H = 0$. Considering it into (14) yields a contradiction.

b) If $\lambda_2 = 0$ and $\lambda_3 \neq 0$, we have $g = 0$. From (10) we have $H = 0$ a contradiction.

c) If $\lambda_2 \neq 0$ and $\lambda_3 = 0$. Then, from (16) we have $H = 0$ a contradiction.

d) If $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$. From (15) and (16) we derive

$$\lambda_2 + (\lambda_2 - \lambda_3)\sinh^2 g = 0. \tag{17}$$

We have two cases:

d.1) If $\lambda_2 = \lambda_3$. (17) yields a contradiction due to $\lambda_2 \neq 0$.

d.2) If $\lambda_2 \neq \lambda_3$. This implies $g' = 0$ which contradicts with $H \neq 0$.

Hence, we present the following.

Theorem 4.1. *Let $r : M^2 \to \mathbb{L}^3$ be an isometric immersion given by (5). Then, $\Delta r_i = \lambda_i r_i,\ \ \lambda_i \in \mathbb{R},\ i = 1, 2, 3$ if and only if M^2 has zero mean curvature.*

5. Conoid surfaces family satisfying $\Delta^{II} r_i = \lambda_i r_i$

Let r be a conoid surfaces family with non-degenerate second fundamental form in $\mathbb{L}^3$ satisfying the condition

$$\Delta^{II} r_i = \lambda_i r_i; \;\; \lambda_i \in \mathbb{R}, \;\; i = 1, 2, 3, \tag{18}$$

and

$$\Delta^{II} r = (\Delta^{II} r_1, \; \Delta^{II} r_2, \; \Delta^{II} r_3).$$

The Gaussian curvature K_G is given by

$$K_G = \frac{g'^2 h'^2}{Q^2}.$$

Since the surface has non-degenerate second fundamental form everywhere, we have

$$f' g' h' \neq 0, \quad \forall u, v \in \Omega \subset \mathbb{R}. \tag{19}$$

The Laplacian operator Δ^{II} of M^2 can be expressed by

$$\Delta^{II} = -\frac{1}{M}\left(\frac{N}{M}\right)_u \frac{\partial}{\partial u} - \frac{N}{M^2}\frac{\partial^2}{\partial u^2} - 2\frac{\partial^2}{\partial u \partial v}. \tag{20}$$

Using (5) and (20), we get

$$\Delta^{II} r = \begin{pmatrix} 0 \\ (-\frac{1}{M}\left(\frac{N}{M}\right)_u f' - \frac{N}{M^2} f'') \cosh g(v) + \frac{2f'g'}{M} \sinh g(v) \\ (-\frac{1}{M}\left(\frac{N}{M}\right)_u f' - \frac{N}{M^2} f'') \sinh g(v) + \frac{2f'g'}{M} \cosh g(v) \end{pmatrix} \tag{21}$$

Since M^2 satisfies (18), equation (21) gives rise to the following differential equations

$$0 = \lambda_1 h, \tag{22}$$

$$\left(\frac{1}{M}\left(\frac{N}{M}\right)_u f' + \frac{N}{M^2} f''\right) \cosh g(v) - \frac{2f'g'}{M} \sinh g(v) = -\lambda_2 f \; \cosh g(v), \tag{23}$$

$$\left(\frac{1}{M}\left(\frac{N}{M}\right)_u f' + \frac{N}{M^2} f''\right) \sinh g(v) - \frac{2f'g'}{M} \cosh g(v) = -\lambda_3 f \; \sinh g(v). \tag{24}$$

From (23) and (24), we find

$$\frac{2f'g'}{M} = -f \; \cosh g(v) \sinh g(v) (\lambda_2 - \lambda_3). \tag{25}$$

1) Let $\lambda_2 = \lambda_3$, from (25), we get $f'g' = 0$. So regardless of (19), it gives rise to a contradiction to the property of being non-degenerate.

2) Let $\lambda_2 \neq \lambda_3$, from (25), we have

$$f^2 (16 g'^2 - h'^2 \sinh^2(2g(v))(\lambda_2 - \lambda_3)^2) = 16 h'^2. \tag{26}$$

The partial derivative of (26) with respect to u yields $f' f = 0$. This leads to a contradiction due to (19). Therefore, we give the following.

Theorem 5.1. *Let $r : M^2 \to \mathbb{L}^3$ be an isometric immersion given by (5). There are no conoid surfaces family in $\mathbb{L}^3$ without parabolic points, satisfying the condition $\Delta^{II} r_i = \lambda_i r_i; \quad \lambda_i \in \mathbb{R}, \quad i = 1, 2, 3.$*

6. Conoid surfaces family satisfying $\Delta^{III} r_i = \lambda_i r_i$

In this section, we consider conoid surfaces family in $\mathbb{L}^3$ with non vanishing Gauss curvature. We introduce the finite Chen type conoid surfaces family with respect to the third fundamental form of the surface.

Let r be a conoid surfaces family with non vanishing Gauss curvature in $\mathbb{L}^3$ satisfying the condition

$$\Delta^{III} r_i = \lambda_i r_i; \quad \lambda_i \in \mathbb{R}, \quad i = 1,2,3. \tag{27}$$

The Laplacian operator Δ^{III} of M^2 can be expressed as

$$\begin{aligned}
\Delta^{III} &= \frac{Q^{\frac{3}{2}}}{f' g'^2 h'^2} \Bigg[\left(\frac{1}{h'^2} \left(\frac{h'}{g'}\right)' \left(\frac{f^2}{f'\sqrt{Q}}\right)_u - \left(\frac{\sqrt{Q}}{f'}\right)_u + f \left(\frac{g'}{h'\sqrt{Q}} \left(\frac{h'}{g'}\right)' \right)_v \right) \frac{\partial}{\partial u} \\
&+ \left(f' \left(\frac{1}{\sqrt{Q}}\right)_v + \frac{g'}{h'} \left(\frac{h'}{g'}\right)' \left(\frac{f}{\sqrt{Q}}\right)_u \right) \frac{\partial}{\partial v} + \left(\frac{1}{h'^2} \left(\frac{h'}{g'}\right)' \left(\frac{f^2}{f'\sqrt{Q}}\right) - \frac{\sqrt{Q}}{f'} \right) \frac{\partial^2}{\partial u^2} \\
&+ \frac{f'}{\sqrt{Q}} \frac{\partial^2}{\partial v^2} + \frac{2fg'}{h'\sqrt{Q}} \left(\frac{h'}{g'}\right)' \frac{\partial^2}{\partial u \partial v} \Bigg].
\end{aligned} \tag{28}$$

Using (5) and (28), we get

$$\Delta^{III} r = \begin{pmatrix} \mathcal{P}_1(u,v) \\ \mathcal{R}_1(u,v)\cosh g(v) + \mathcal{R}_2(u,v)\sinh g(v) \\ \mathcal{R}_1(u,v)\sinh g(v) + \mathcal{R}_2(u,v)\cosh g(v) \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned}
\mathcal{P}_1(u,v) &= \frac{1}{g'^2 h'} \left(h'' h' - f^2 g'' g' - g' h' \left(\frac{h'}{g'}\right)' \right) = -\frac{\varepsilon g''}{h' g'^3} Q, \\
\mathcal{R}_1(u,v) &= -\frac{f}{g'^2 h'^4} \Bigg[\left(\frac{h'}{g'}\right)' (2h'^2 - f^2 g'^2 + f^2 g'^2 h' g'' - g' h'' h'^2) \\
&- Q g' \left(h' \left(\frac{h'}{g'}\right)'' - g' \left(\left(\frac{h'}{g'}\right)' \right)^2 \right) \Bigg], \\
\mathcal{R}_2(u,v) &= -\frac{f}{g' h'^3} \left(\frac{h'}{g'}\right)' \left[h'^2 (1 + 3g') - 2 f^2 g'^3 \right].
\end{aligned}$$

Since M^2 satisfies (27), equation (29) gives rise to the following differential equations

$$\mathcal{P}_1(u,v) = \lambda_1 h, \tag{30}$$

$$\mathcal{R}_1(u,v)\cosh g(v) + \mathcal{R}_2(u,v)\sinh g(v) = \lambda_2 f \cosh g(v), \tag{31}$$

$$\mathcal{R}_1(u,v)\sinh g(v) + \mathcal{R}_2(u,v)\cosh g(v) = \lambda_3 f \sinh g(v). \tag{32}$$

From (31) and (32), we have

$$\mathcal{R}_1(u,v) = (\lambda_2 \cosh^2 g - \lambda_3 \sinh^2 g) f, \tag{33}$$

$$\mathcal{R}_2(u,v) = (\lambda_3 - \lambda_2) f \sinh g \cosh g. \tag{34}$$

Next we study this system concerning the values of the constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1. Let $\lambda_1 = 0$, from (30), we obtain $g'' = 0$, namely $g'(v) = \alpha$, $\alpha \in \mathbb{R} - \{0\}$.

Hence, we can rewrite $\mathcal{R}_2(u,v)$ as

$$\mathcal{R}_2(u,v) = -\frac{fh''}{\alpha^2 h'^3}\left[h'^2(1+3\alpha) - 2f^2\alpha^3\right]. \tag{35}$$

a) If $\lambda_2 \neq 0$ and $\lambda_3 = 0$. From (34) and (35), we have

$$\frac{h''}{\alpha^2 h'^3}\left[h'^2(1+3\alpha) - 2f^2\alpha^3\right] = \lambda_2 \sinh g \cosh g.$$

Taking the derivative with respect to u, one gets $h'' = 0$. From (35), we have $\mathcal{R}_2(u,v) = 0$ gives, $\lambda_2 = 0$ a contradiction.

b) If $\lambda_2 = 0$ and $\lambda_3 \neq 0$. With respect to (34) and (35), we can obtain:

$$\frac{h''}{\alpha^2 h'^3}\left[h'^2(1+3\alpha) - 2f^2\alpha^3\right] = -\lambda_3 \sinh g \cosh g.$$

Taking the derivative with respect to u, one gets $h'' = 0$. From (35), we have $\mathcal{R}_2(u,v) = 0$ gives, $\lambda_3 = 0$ a contradiction.

c) If $\lambda_2 = 0$ and $\lambda_3 = 0$. From (33) and (34), we have $h'' = 0$. Then, the equation (10) gives rise to $H = 0$, which means that the surface is minimal.

d) If $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$. From (33), we have

$$\frac{h''}{\alpha^3 h'^4}\left[2h'^2 - \alpha^2 f^2 - \alpha h'^2 h''\right] = -(\lambda_2 \cosh^2 g - \lambda_3 \sinh^2 g). \tag{36}$$

Taking the partial derivative of (36) with respect to u gives $ff'h'' = 0$.

We discuss by cases:

d.1) Case $ff' = 0$. In this case, $\det(g_{ij}) = 0$, a contradiction.

d.2) Case $h'' = 0$. Then (31) implies $\lambda_2 = 0$, a contradiction.

Case 2. Let $\lambda_1 \neq 0$. We discuss by cases:

2.1) Case $g'' = 0$. In the particular case $\mathcal{P}_1(u,v) = 0$, we obtain the following contradiction $\lambda_1 = 0$.

2.2) Case $g'' \neq 0$. Taking the partial derivative of (30) with respect to u gives $ff' = 0$. Then, $\det(g_{ij}) = 0$, a contradiction.

Then, we have the following.

Theorem 6.1. *Let $r : M^2 \to \mathbb{L}^3$ be an isometric immersion given by (5). Then, $\Delta^{III} r_i = \lambda_i r_i$, where $\lambda_i \in \mathbb{R}$, $i = 1,2,3$, if and only if M^2 has zero mean curvature.*

7. Conclusion

In this paper, we conduct a comprehensive classification of the family of conoid surfaces within Minkowski 3-space $\mathbb{L}^3$. This classification is based on the condition $\Delta^J r_i = \lambda_i r_i$, $J = I, II, III$, where $\lambda_i \in \mathbb{R}$ and Δ^J indicates the Laplace operator with respect to the fundamental forms I, II, and III. Initially, we introduce and discuss key notions and properties related to $\mathbb{L}^3$. Following this, we present the family of minimal conoid surfaces. Finally, we determine and analyze the conoid surfaces family that satisfies the conditions for each of the fundamental forms, providing a thorough understanding of their geometric properties within the Minkowski 3-space framework.

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