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Higher coderivations of formal triangular matrix coalgebras

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Abstract. In this paper, we introduce the notion of higher coderivations on coalgebras as an important generalization of coderivations, and show that there exists a one to one correspondence between the set of all higher coderivations $\{D_n\}$ with $D_0 = id_C$ and the set of all sequences of coderivations $\{f_n\}$ with $f_0 = 0$. Then we characterize the higher coderivations on the formal triangular matrix coalgebra $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$, and it is shown that the higher coderivations on Γ could be described by the higher coderivations on C and D, generalized higher coderivations on M, and two other related families of maps.

1. Introduction

Let A be an k-algebra. A linear map $d : A \to A$ is called a derivation on A if d satisfies the Leibniz rule $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. Derivation is a very important tool in algebra. It originates from analytical theory. In the mid-20th century, derivations began to be applied to algebras and introduced into algebraic systems to study the structure and properties of algebraic systems. In 1955, Singer and Wermer [20] applied derivations to normed algebras. In 1957, Posner [18] studied derivations on prime rings. Since then, the theory of derivation has been made great development, and many kinds of derivations such as Lie derivations, Jordan derivations and so on, have been introduced and developed [8, 9]. Derivations are useful to construct deformation formulas [2], homotopy Lie algebras [22] and they have applications in differential Galois theory [11]. Algebras with derivations are studied from an operadic point of view in [10]. In particularly, derivations of polynomial rings and their kernels have been the objects of great interest for a long time because of their connections with several conjunctures and popular problems in Commutative Algebra and Algebraic Geometry, such as the Jacobian conjecture and the Zariski cancellation problem [1, 4, 12].

There are many interesting generalizations of derivations, one of them being higher derivations. Hasse and Schmidt [5] considered a sequence $D = \{D_n\}_{n=0}^{\infty}$ of k-linear map on an algebra A satisfying the following conditions: $D_0 = id_A$ and $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$ for all $a, b \in A$ and any $n \geq 0$. Such a sequence

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$D = \{D_n\}_{n=0}^{\infty}$ is called a higher derivation of A, and is also known as a Hasse-Schmidt derivation of A. A basic example of higher derivations is the sequence of divided power differential operators $\{\frac{d^n}{n!}\}_{n=0}^{\infty}$, where d is a derivation of an algebra A over a field of characteristic zero. A higher derivation $D = \{D_n\}_{n=0}^{\infty}$ on A is said to be locally finite (resp., iterative) if D satisfies for any $a \in A$, there exists an integer $n \geq 0$ such that $D_m(a) = 0$ for any integer $m \geq n$ (resp., for any integers $i, j \geq 0$, $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$). If $D = \{D_n\}_{n=0}^{\infty}$ is a higher k-derivation on A, then D_1 is an k-derivation on A. The concept of higher derivations was originally introduced in order to remove some of the anomalies in the calculus of derivations on fields of characteristic $p \neq 0$. Nevertheless, higher derivations have important applications in characteristic zero situations. Heerema [6] showed a review of the theory of higher derivations on fields which form a background for the study of the uses of higher derivations in automorphism theory of complete local rings. Mirzavaziri [13] and Saymeh [19] proved that each higher derivation on an algebra A is a combination of compositions of derivations, hence, characterize all higher derivations in terms of derivations on A. Triangular matrix rings represent a very important construction and tool in classical ring theory and non-commutative algebra. Higher derivations on triangular algebras are considered in [26]. Higher derivations are useful in the theory of automorphisms of complete local rings. In fact, higher derivations can be applied to Galois theory of fields, to universal higher derivations and to separability criteria [27].

Higher derivations can be interpreted as coalgebra measurings. For each $1 \leq N \leq \infty$, we define a coalgebra $C^N = \oplus_{n=0}^{N} k x_n$ with structure maps Δ and ε given by $\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}$ and $\varepsilon(x_n) = \delta_{n0}$. Then a higher derivation $\{D_n\}_{n=0}^{N}$ on A defines a measuring $C^N \otimes A \to A$ by $x_n \cdot a = D_n(a)$ for all $1 \leq n \leq N$ and $a \in A$ [21, P.139-140]. We call a higher derivation $\{D_n\}_{n=0}^{N}$ inner if the associated measuring is inner, i.e. implemented by an invertible $\kappa \in \mathrm{Hom}_k(C^N, A)$ such that $c \cdot a = \sum_{(c)} \kappa(c_1) a \kappa^{-1}(c_2)$ for all $c \in C^N$ and $a \in A$, note that we can assume $\kappa(x_0) = \kappa^{-1}(x_0) = 1$. Writing $\kappa(x_n) = a_n$ and $\kappa^{-1}(x_n) = b_n$, we have that the higher derivation is inner if and only if $D_n(a) = \sum_{i=0}^{n} a_i a b_{n-i}$ for all n and $a \in A$, where $\{a_n\}_{n=0}^{N}$ and $\{b_n\}_{n=0}^{N}$ are sequences in A such that $a_0 = b_0 = 1$ and $\sum_{i=0}^{n} a_i b_{n-i} = \delta_{n0} = \sum_{i=0}^{n} b_i a_{n-i}$. This is equivalent with that given by Nowicki [17]. See [16, 26] for more results about higher derivations.

As a dual concept of derivation, in 1981, Doi [3] introduced the notion of coderivation in the study of cohomology of coalgebra, which paved the way for further research on coderivation. In 2012, Nakajima [15] introduced the concept of generalized coderivations and inner coderivations, and in 2015, as a generalization, Nakajima [7] introduced the concept of triple coderivations. In 2016, the first author with coauthors in [25] introduced cohomological invariants derived from the coalgebraic structure of Hopf algebras and to use them to study (mostly) infinite dimensional Hopf algebras. formal triangular matrix coalgebra also represent a very important construction and tool in coalgebra theory. In 2021, the coderivation on formal triangular matrix coalgebra was described in [24].

Motivated by these ideas and as a continuation of the work in [24], in this paper, we firstly introduce the notion of higher coderivations on coalgebras as an essential generalization of coderivations. Then we show that there exists a one to one correspondence between the set of all higher coderivations $\{D_n\}$ with $D_0 = id_C$ and the set of all sequences $\{f_n\}$ with $f_0 = 0$. Finally we characterize the higher coderivations on the formal triangular matrix coalgebra $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$.

This paper is organized as follows. In section 1, we will recall the definitions and results on formal triangular matrix coalgebra and coderivations. In section 2, we will introduce the notion of higher coderivations $\{D_n\}$ on coalgebra C, and give some examples of higher coderivations. We also prove that $\{D_n\}$ with $D_0 = id_C$ could be determined by a sequence of coderivations $\{f_n\}$. Thus we establish the one to one correspondence between the set of all higher coderivations $\{D_n\}$ with $D_0 = id_C$ and the set of all sequences of coderivations $\{f_n\}$ with $f_0 = 0$. In section 3, we characterize the higher coderivations on formal triangular matrix coalgebra $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$, where C and D are both coalgebras and M is a C-D-bicomodule. It is shown that the higher coderivations on Γ could be described by the higher coderivations on C and D, generalized higher coderivations on M, and two other related families of maps.

Throughout this paper, let k be a fixed field. Unless otherwise specified, all vector space, tensor product and homomorphisms are all over k. We put $\otimes$ shorthand for $\otimes_k$. For a coalgebra C, the structure maps of

C are denoted by $\Delta = \Delta_C : C \to C \otimes_k C$ and $\epsilon = \epsilon_C : C \to k$, and for $c \in C$, we use Sweedler's notation $\Delta(c) = \sum\limits_{(c)} c_1 \otimes c_2$. For convenience, we usually omit the summation and write $\Delta(c) = c_1 \otimes c_2$. We refer to the books of Sweedler [21] and Montgomery [14] for more details on coalgebras and comodules.

2. Preliminaries

In this section, we will recall the definitions of coderivations on coalgebras and formal triangular matrix coalgebras.

Let A be a k-algebra, and (C, Δ, ε) be a coalgebra. The space $\mathrm{Hom}_k(C, A)$ ia an associative algebra with respect to the convolution product $*$ defined by $(f * g)(c) = \sum_{(c)} f(c_1)g(c_2)$, and the identity element is $\eta\varepsilon : c \to \varepsilon(c)1_A$. In particularly, when $A = k$ the algebra structure $(\mathrm{Hom}_k(C, k), *, \varepsilon)$ is the same as the dual algebra C^* of C.

Recall from [15], a coderivation of a coalgebra (C, Δ, ε) is a linear map $\delta : C \to C$ satisfying

$$\Delta\delta = (\delta \otimes 1 + 1 \otimes \delta)\Delta.$$

Let (C, Δ, ε) be a k-coalgebra, recall that M is a left C-comodule via left comodule structure map $\rho^l : M \to C \otimes M$ means that the following conditions hold:

$$(id \otimes \rho^l)\rho^l = (\Delta \otimes id)\rho^l, \quad (\varepsilon \otimes id)\rho_M = id.$$

If M and N are two left C-comodules, a k-linear map $f : M \to N$ is called a left C-comodule map if

$$(id \otimes f)\rho_M = \rho_N f.$$

Similarly, one can define a right C-comodule M via right comodule structure map $\rho^r : M \to M \otimes C$.

Let C and D be coalgebras, and recall that a C-D-bicomodule M means that M is a left C-comodule and a right D-comodule via left and right comodule structure maps $\rho^l : M \to C \otimes M$ and $\rho^r : M \to M \otimes D$ respectively, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho^r} & M \otimes D \\ \downarrow{\rho^l} & & \downarrow{\rho^l \otimes 1} \\ C \otimes M & \xrightarrow{1 \otimes \rho^r} & C \otimes M \otimes D \end{array}$$

Thus the relations of ρ^l, ρ^r and Δ are as follows:

$$(\Delta \otimes 1)\rho^l = (1 \otimes \rho^l)\rho^l,\ (1 \otimes \Delta)\rho^r = (\rho^r \otimes 1)\rho^r,\ (1 \otimes \rho^r)\rho^l = (\rho^l \otimes 1)\rho^r.$$

Similar to the Sweedler notation, if M is a C-D-bicomodule, for all $m \in M$, for convenience, we usually omit the summation and write

$$\rho^l(m) = m_{<-1>} \otimes m_{<0>}, \quad \rho^r(m) = m_{(0)} \otimes m_{(1)}.$$

Let C and D be coalgebras, M be a C-D-bicomodule, and

$$\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix} = \left\{ \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} \mid \ \forall c \in C, d \in D, m \in M \right\},$$

then, recall from [23], Γ makes a coalgebra with the following comultiplication and counit:

$$\Delta\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix}, \ \Delta\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix},$$

$$\Delta\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & m_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix},$$

$$\varepsilon\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \varepsilon_C(c) + \varepsilon_D(d),$$

for all $c \in C, d \in D, m \in M$. We call Γ a formal triangular matrix coalgebra.

Lemma 2.1. *[24] Let* $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$ *be a formal triangular matrix coalgebra. For all* $c \in C, d \in D, m \in M$, *define linear maps* $\varepsilon_1, \varepsilon_2 : T \to k$ *as follows:*

$$\varepsilon_1\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \varepsilon_C(c), \quad \varepsilon_2\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \varepsilon_D(d).$$

Then we have

$$(id \otimes \varepsilon_1)\Delta\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.1}$$

$$(id \otimes \varepsilon_2)\Delta\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & d \end{pmatrix}, \tag{1.2}$$

$$(\varepsilon_1 \otimes id)\Delta\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} c & m \\ 0 & 0 \end{pmatrix}, \tag{1.3}$$

$$(\varepsilon_2 \otimes id)\Delta\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}. \tag{1.4}$$

First of all, we need to recall the following result on the coderivation on Γ.

Proposition 2.2. *[24] Let* $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$ *be a formal triangular matrix coalgebra, and* δ *a coderivation on* Γ. *For all* $\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} \in \Gamma$, *denote*

$$\delta\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} \varphi(c,m,d) & \gamma(c,m,d) \\ 0 & \psi(c,m,d) \end{pmatrix}.$$

Then there exist coderivations δ_C *on* C *and* δ_D *on* D, *and linear maps*

$$\xi : M \to C, \ \tau : M \to D, \ f : M \to M,$$

satisfying

(1) $\varphi(c,m,d) = \delta_C(c) + \xi(m), \gamma(c,m,d) = f(m), \psi(c,m,d) = \delta_D(d) + \tau(m)$.

(2) ξ *is a left* C*-comodule map, and* τ *is a right* D*-comodule map.*

(3) $\rho^l \circ f = (\delta_C \otimes id_M + id_C \otimes f) \circ \rho^l$, $\rho^r \circ f = (id_M \otimes \delta_D + f \otimes id_D) \circ \rho^r$.

3. Higher Coderivations

In this section, we will introduce the concept of higher coderivations on coalgebras, and give some recurrence relations on higher coderivations.

Definition 3.1. *Let C be a coalgebra and $\{D_n : C \to C \mid n \in \mathbb{N}\}$ a family of linear maps with $D_0 = id_C$, then $D = \left\{D_n\right\}_{n=0}^{\infty}$ is called a higher coderivation on C if*

$$\Delta D_n = \sum_{i=0}^{n}(D_i \otimes D_{n-i})\Delta. \tag{2.1}$$

Note that in the above relation, $\Delta D_1 = (id_C \otimes D_1 + D_1 \otimes id_C)\Delta$, that is, D_1 is a coderivations on C.

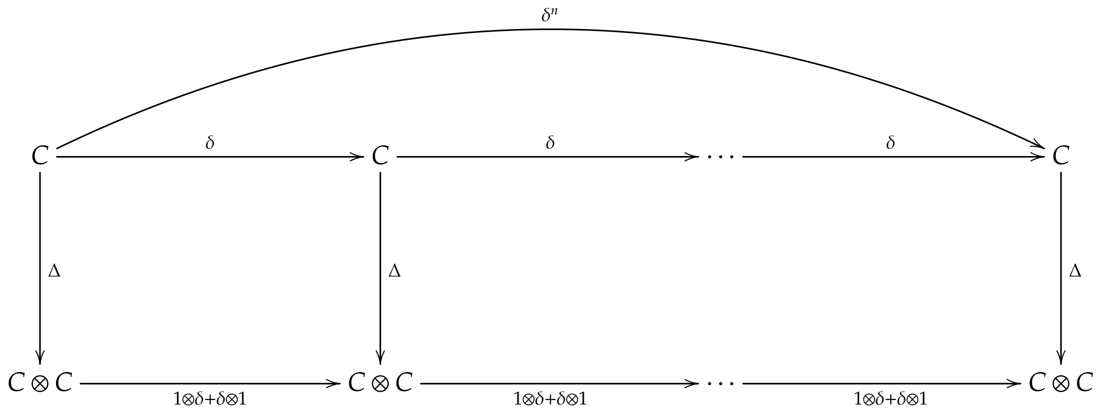
We denote by HCD(C) the set of all higher coderivations on the coalgebra (C, Δ, ε). Let us define an operation on HCD(C) as follows. For any $D = \left\{d_n\right\}_{n=0}^{\infty}$, $D' = \left\{d'_n\right\}_{n=0}^{\infty} \in$ HCD(C), the product $D{*}D' = \{d_n{*}d'_n\}_{n=0}^{\infty}$ is defined by

$$d_n * d'_n \triangleq \sum_{i+j=n} d_i \circ d'_j$$

for all non-negative integer n. In particular, $d_1 * d'_1 = d_1 + d'_1$. Notice that HCD(C) forms a monoid with respect to $*$.

Example 3.2. *Let $\delta : C \to C$ be a coderivation, then $D = \left\{\frac{\delta^n}{n!}\right\}_{n=0}^{\infty}$ is a higher coderivation. This kind of higher coderivation is called an ordinary higher coderivation.*

Indeed we have the following commutative diagram:



From the above diagram, we have

$$\Delta\delta^n = (1 \otimes \delta + \delta \otimes 1)^n \Delta = \sum_{i=0}^{n} \binom{n}{i} (\delta^i \otimes \delta^{n-i})\Delta = \sum_{i=0}^{n} \frac{n!}{i! \cdot (n-i)!}(\delta^i \otimes \delta^{n-i})\Delta.$$

Therefore

$$\Delta D_n = \Delta\frac{\delta^n}{n!} = \sum_{i=0}^{n} \frac{1}{i! \cdot (n-i)!}(\delta^i \otimes \delta^{n-i})\Delta = \sum_{i=0}^{n}(\frac{\delta^i}{i!} \otimes \frac{\delta^{n-i}}{(n-i)!})\Delta.$$

Thus D is a higher coderivation.

Proposition 3.3. *Let $D = \{D_n : C \to C\}_{n=0}^{\infty}$ be a family of linear maps. Then D is higher coderivation on C if and only if $\{D^*_n\}_{n=0}^{\infty}$ is a higher derivation on C^*, where C^* is the linear dual of C, and $D^*_n : C^* \to C^*$ is given by*

$$D^*_n(f) = fD_n,$$

for all $f \in C^*$.

Proof. For all $f, g \in C^*$,

$$D_n^*(f * g) = (f * g)D_n = (f \otimes g)\Delta D_n.$$

and

$$\begin{aligned}\sum_{i=0}^{n} D_i^*(f) * D_{n-i}^*(g) &= \sum_{i=0}^{n}(fD_i) * (gD_{n-i}) \\ &= \sum_{i=0}^{n}(fD_i \otimes gD_{n-i})\Delta \\ &= \sum_{i=0}^{n}(f \otimes g)(D_i \otimes D_{n-i})\Delta \\ &= (f \otimes g)\sum_{i=0}^{n}(D_i \otimes D_{n-i})\Delta,\end{aligned}$$

as claimed. □

We will show that not all higher coderivations arise from coderivations in the above way.

Lemma 3.4. *Let C be a coalgebra and $\alpha \in C^*$ be any fixed element, $\delta_0 = id_C$ and for $n \geq 1$, we define $\delta_n : C \to C$ as follow: for any $c \in C$,*

$$\delta_n(c) = \alpha^n(c_1)c_2 - \alpha^{n-1}(c_1)c_2\alpha(c_3) = \alpha^{n-1}(c_1)(\alpha(c_2)c_3 - c_2\alpha(c_3)).$$

Then $\delta = \{\delta_n\}_{n=0}^{\infty}$ is a higher coderivation on C, where $\alpha^0 = \varepsilon_C$.

Proof. Indeed, we show it by induction on n. For all $c \in C$,

$$\begin{aligned}\Delta\delta_1(c) &= \alpha(c_1)c_2 \otimes c_3 - c_1 \otimes c_2\alpha(c_3) \\ &= \alpha(c_1)c_2 \otimes c_3 - c_1\alpha(c_2) \otimes c_3 + c_1\alpha(c_2) \otimes c_3 - c_1 \otimes c_2\alpha(c_3) \\ &= \delta_1(c_1) \otimes c_2 + c_1 \otimes \delta(c_2),\end{aligned}$$

that is, $\Delta\delta_1 = (\delta_1 \otimes id)\Delta + (id \otimes \delta_1)\Delta$.

Obviously $\delta_n(c) = \alpha(c_1)\delta_{n-1}(c_2)$ for all $n > 0$. Now assume that the sequence $\{\delta_k\}_{k=0}^{n-1}$ satisfies

$$\Delta\delta_k = \sum_{i=0}^{k}(\delta_i \otimes \delta_{k-i})\Delta.$$

For δ_n, we have

$$\begin{aligned}\Delta\delta_n(c) &= \alpha(c_1)\Delta\delta_{n-1}(c_2) \\ &= \sum_{i=0}^{n-1} \alpha(c_1)(\delta_i(c_2) \otimes \delta_{n-1-i}(c_3)) \\ &= \sum_{i=1}^{n-1} \delta_{i+1}(c_1) \otimes \delta_{n-(i+1)}(c_2) + \alpha(c_1)c_2 \otimes \delta_{n-1}(c_3) \\ &= \sum_{i=2}^{n} \delta_i(c_1) \otimes \delta_{n-i}(c_2) + \alpha(c_1)c_2 \otimes \delta_{n-1}(c_3) \\ &= \sum_{i=2}^{n} \delta_i(c_1) \otimes \delta_{n-i}(c_2) + (\delta_1(c_1) \otimes \delta_{n-1}(c_2) + c_1 \otimes \alpha(c_2)\delta_{n-1}(c_3))\end{aligned}$$

$$= \sum_{i=2}^{n} \delta_i(c_1) \otimes \delta_{n-i}(c_2) + (\delta_1(c_1) \otimes \delta_{n-1}(c_2) + c_1 \otimes \delta_n(c_2))$$
$$= \sum_{i=0}^{n} \delta_i(c_1) \otimes \delta_{n-i}(c_2)$$
$$= \sum_{i=0}^{n} (\delta_i \otimes \delta_{n-i})\Delta(c).$$

The proof is completed. □

Lemma 3.5. *Let C be a coalgebra, $D = \{D_n\}_{n=0}^{\infty}$ be a higher coderivation on C. For any $m \geq 1$, we define a new family of mappings $\delta = \{\delta_n\}_{n=0}^{\infty}$ from C to C by*

$$\delta_n = \begin{cases} 0, & \text{if } m \nmid n, \\ D_r, & \text{if } n = rm. \end{cases}$$

Then $\delta = \{\delta_n\}_{n=0}^{\infty}$ is a higher coderivation on C.

Proof. (1) If $m \nmid n$, for all $0 \leq i \leq n$, either $\delta_i = 0$ or $\delta_{n-i} = 0$, hence $\delta_i \otimes \delta_{n-i} = 0$. Obviously $\Delta\delta_n = \sum_{i=0}^{n}(\delta_i \otimes \delta_{n-i}) = 0$.

(2) If $m \mid n$ and $n = rm$, then

$$\Delta\delta_n = \Delta D_r = \sum_{i=0}^{r}(D_i \otimes D_{n-i}),$$

and for all $0 \leq i \leq n$, either $\delta_i = 0, \delta_{n-i} = 0$ or $\delta_i \neq 0, \delta_{n-i} \neq 0$. Hence

$$\sum_{i=0}^{n}(\delta_i \otimes \delta_{n-i}) = \sum_{i=0}^{r}(\delta_{im} \otimes \delta_{(r-i)m}) = \sum_{i=0}^{r}(D_i \otimes D_{n-i}).$$

The proof is completed. □

Lemma 3.6. *Let C be a coalgebra and $D = \{D_n\}_{n=0}^{\infty}$ be a sequence of linear mappings of C. If there exist two sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ of elements in C^* satisfying the conditions: $\alpha_0 = \beta_0 = \varepsilon = 1_{C^*}$ and*

$$\sum_{i=0}^{n} \alpha_i\beta_{n-i} = \delta_{n,0}\varepsilon = \sum_{i=0}^{n} \beta_i\alpha_{n-i}$$

such that

$$D_n(c) = \sum_{j=0}^{n} \alpha_j(c_1)c_2\beta_{n-j}(c_3),$$

for all $c \in C$ and each non-negative integer n, where $\delta_{n,0}$ is the Kronecker sign. Then $D = \{D_n\}_{n=0}^{\infty}$ is a higher coderivation of C. This kind of higher coderivation is called an inner higher coderivation.

Proof. By the definition of D, from [17] we could see that D^* is a higher derivation on C^*, where

$$D_n^*(f)(c) = \sum_{i=0}^{n} \alpha_i(c_1)f(c_2)\beta_{n-i}(c_3),$$

for all $f \in C^*, c \in C$. Therefore by Proposition 3.3, D is a higher coderivation on C. □

Lemma 3.7. *Let C be a coalgebra and $\delta = \{\delta_n\}_{n=0}^{\infty}$ a higher coderivation on C. Then $\varepsilon_C\delta_n = 0$ for all positive integer n.*

Proof. We prove it by an induction on n. First of all, since δ_1 is a coderivation on C, by [24, Lemma 2.1], $\varepsilon_C\delta_1 = 0$.

Now we assume that $\varepsilon_C\delta_k = 0$ for all $k < n$. Then for all $c \in C$,

$$\begin{aligned}\varepsilon_C\delta_n(c) &= \varepsilon_C(\delta_n(c)_1)\varepsilon_C(\delta_n(c)_2)\\ &= \sum_{i=0}^{n} \varepsilon_C(\delta_i(c_1))\varepsilon_C(\delta_{n-i}(c_2))\\ &= 2\varepsilon_C(c_1)\varepsilon_C(\delta_n(c_2))\\ &= 2\varepsilon_C\delta_n(c),\end{aligned}$$

where the third identity holds by the inductive hypothesis. Thus $\varepsilon_C\delta_n = 0$. □

For the relation between higher coderivation and coderivation, we have the following result.

Proposition 3.8. *Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher coderivation on C with $D_0 = id_C$. Then there exists a sequence $\{f_n\}$ of coderivations on C such that for each nonnegative integer n,*

$$(n+1)D_{n+1} = \sum_{k=0}^{n} D_{n-k}f_{k+1}.$$

Proof. We give the proof by using induction on n. For $n = 0$, we have $D_0 = id_C$ and $D_0f_1 = f_1$. Thus if $f_1 = D_1$, then f_1 is a coderivation on C.

Now suppose that f_k is defined and is a coderivation for all $k \le n$. Putting

$$f_{n+1} = (n+1)D_{n+1} - \sum_{k=0}^{n-1} D_{n-k}f_{k+1},$$

Now we will show that the well-defined mapping f_k is a coderivation on C.

Since $\{D_n\}$ is a higher coderivation on C, we have that

$$\begin{aligned}\Delta f_{n+1} &= (n+1)\Delta D_{n+1} - \sum_{k=0}^{n-1} \Delta D_{n-k}f_{k+1}\\ &= (n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}(D_l \otimes D_{n-k-l})\Delta f_{k+1}.\end{aligned}$$

Since $f_1, \cdots, f_n$ are coderivations, we have that

$$\begin{aligned}\Delta f_{n+1} &= (n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}(D_l \otimes D_{n-k-l})(id_C \otimes f_{k+1} + f_{k+1} \otimes id_C)\Delta\\ &= (n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}(D_l \otimes D_{n-k-l}f_{k+1} + D_lf_{k+1} \otimes D_{n-k-l})\Delta.\end{aligned}$$

In the summation $\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}$, we have $0 \le k + l \le n$ and $k \ne n$. Thus if we put $r = k + l$, then we can write it as the form $\sum_{r=0}^{n}\sum_{k+l=r,k\ne n}$. Putting $l = r - k$, we indeed have

$$\Delta f_{n+1} = (n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta$$

$$
\begin{aligned}
&-\sum_{r=0}^{n}\sum_{0\le k+l\le n,k\neq n}(D_{r-k}\otimes D_{n-r}f_{k+1}+D_{r-k}f_{k+1}\otimes D_{n-r})\Delta\\
=&(n+1)\sum_{k=0}^{n+1}(D_k\otimes D_{n+1-k})\Delta-\sum_{r=0}^{n-1}\sum_{k=0}^{r}(D_{r-k}\otimes D_{n-r}f_{k+1}+D_{r-k}f_{k+1}\otimes D_{n-r})\Delta\\
&-\sum_{k=0}^{n-1}(D_{n-k}\otimes f_{k+1}+D_{n-k}f_{k+1}\otimes id_C)\Delta.
\end{aligned}
$$

By our assumption

$$(n+1)\Delta D_{n+1}=\sum_{k=0}^{n}\Delta D_{n-k}f_{k+1}.$$

Therefore

$$
\begin{aligned}
(n+1)\sum_{k=0}^{n+1}(D_k\otimes D_{n+1-k})\Delta &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_l\otimes D_{n-k-l})\Delta f_{k+1}\\
&=\sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_l\otimes D_{n-k-l})(id_C\otimes f_{k+1}+f_{k+1}\otimes id_C)\Delta\\
&=\sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_l\otimes D_{n-k-l}f_{k+1}+D_lf_{k+1}\otimes D_{n-k-l})\Delta.
\end{aligned}
$$

By a similar argument, we can write $\sum_{k=0}^{n}\sum_{l=0}^{n-k}$ as the form $\sum_{r=0}^{n}\sum_{k}^{r}$. Putting $l=r-k$, we have

$$
\begin{aligned}
(n+1)\sum_{k=0}^{n+1}(D_k\otimes D_{n+1-k})\Delta &= \sum_{r=0}^{n}\sum_{k=0}^{r}(D_{n-r}\otimes D_{n-r}f_{k+1}+D_{n-r}f_{k+1}\otimes D_{n-r})\Delta\\
&=\sum_{r=0}^{n-1}\sum_{k=0}^{r}(D_{r-k}\otimes D_{n-r}f_{k+1}+D_{r-k}f_{k+1}\otimes D_{n-r})\Delta\\
&\quad+\sum_{k=0}^{n}(D_{n-k}\otimes f_{k+1}+D_{n-k}f_{k+1}\otimes id_C)\Delta\\
&=\sum_{r=0}^{n-1}\sum_{k=0}^{r}(D_{r-k}\otimes D_{n-r}f_{k+1}+D_{r-k}f_{k+1}\otimes D_{n-r})\Delta\\
&\quad+\sum_{k=0}^{n-1}(D_{n-k}\otimes f_{k+1}+D_{n-k}f_{k+1}\otimes id_C)\Delta+(id_C\otimes f_{k+1}+f_{k+1}\otimes id_C)\Delta.
\end{aligned}
$$

Thus

$$\Delta f_{n+1}=(id_C\otimes f_{k+1}+f_{k+1}\otimes id_C)\Delta,$$

that is, f_{n+1} is a coderivation. The proof is completed. □

Theorem 3.9. *Let $\{D_n\}_{n=0}^{\infty}$ be a higher coderivation on an coalgebra C with $D_0=id_C$. Then there is a sequence $\{f_n\}$ of coderivations on C such that*

$$D_n=\sum_{i=1}^{n}\left(\sum_{\Sigma_{j=1}^{i}r_j=n}(\prod_{j=1}^{i}\frac{1}{r_j+\cdots+r_i})f_{r_i}\cdots f_{r_1}\right),$$

where the inner summation is taken over all positive integers r_j *with* $\sum_{j=1}^{i} r_j = n$.

Proof. We first show that if D_n is of the above form, then it satisfies the recursive relation of Proposition 3.8. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation, we put

$$a_{r_1,\cdots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i}.$$

Note that if $r_1 + \cdots + r_i = n + 1$, then

$$(n+1)a_{r_1,\cdots,r_i} = a_{r_2,\cdots,r_i}.$$

Moreover $a_{n+1} = \frac{1}{n+1}$. Therefore

$$\begin{aligned}
(n+1)D_{n+1} &= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j=n+1}(n+1)a_{r_1,\ldots,r_i} f_i \ldots f_1\right) + f_{n+1}\\
&= \sum_{i=2}^{n+1}\sum_{r_1=1}^{n+2-i}\left(\sum_{\sum_{j=2}^{i} r_j=n+1-r_1} a_{r_2,\cdots,r_i} f_{r_i}\cdots f_{r_2}\right) f_{r_1} + f_{n+1}\\
&= \sum_{r_1=1}^{n}\left(\sum_{i=2}^{n-r_1+1}\left(\sum_{\sum_{j=2}^{i} r_j=n-r_1+1} a_{r_2,\cdots,r_i} f_{r_i}\cdots f_{r_2}\right)\right) f_{r_1} + f_{n+1}\\
&= \sum_{r_1=1}^{n} D_{n-r_1+1} f_{r_1} + f_{n+1} = \sum_{k=0}^{n} D_{n-k} f_{k+1}.
\end{aligned}$$

The proof is completed. □

Theorem 3.10. *Let* C *be a coalgebra,* Φ *the set of all higher coderivations* $D = \{D_n\}_{n=0}^{\infty}$ *on* C *with* $D_0 = id_C$ *and* Ψ *be the set of all sequences* $\{f_n\}$ *of coderivations on* C *with* $f_0 = 0$. *Then there is a one to one correspondence between* Φ *and* Ψ.

Proof. Let $\{f_n\} \in \Psi$. Define $D_n : C \to C$ by $D_0 = id_C$ and

$$D_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j=n}\left(\prod_{j=1}^{i}\frac{1}{r_j + \cdots + r_i}\right) f_{r_i}\cdots f_{r_1}\right).$$

We show that $\{D_n\} \in \Phi$. By Theorem 3.9, $\{D_n\}$ satisfies the recursive relation

$$(n+1)\Delta D_{n+1} = \sum_{k=0}^{n} \Delta D_{n-k} f_{k+1}.$$

To show that $\{D_n\}$ is a higher coderivation, we use induction on n. For $n = 0$, we have $D_0 = id_C$. Let us assume that

$$\Delta D_k = \sum_{i=0}^{k} (D_i \otimes D_{k-i})\Delta$$

for $k \leq n$. Thus we have

$$(n+1)\Delta D_{n+1} = \sum_{k=0}^{n} \Delta D_{n-k} f_{k+1}$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} (D_l \otimes D_{n-k-l}) \Delta f_{k+1}$$
$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} (D_l \otimes D_{n-k-l})(I \otimes f_{k+1} + f_{k+1} \otimes I)\Delta$$
$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} (D_l \otimes D_{n-k-l} f_{k+1} + D_l f_{k+1} \otimes D_{n-k-l})\Delta,$$

and

$$\begin{aligned}(n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta &= \sum_{k=0}^{n+1}(k + n + 1 - k)(D_k \otimes D_{n+1-k})\Delta \\ &= \sum_{k=0}^{n+1}(kD_k \otimes D_{n+1-k})\Delta + \sum_{k=0}^{n+1}(D_k \otimes (n+1-k)D_{n+1-k})\Delta.\end{aligned}$$

Let $S = \sum_{k=0}^{n+1}(kD_k \otimes D_{n+1-k})\Delta$ and $T = \sum_{k=0}^{n+1}(D_k \otimes (n+1-k)D_{n+1-k})\Delta$, then

$$(n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta = S + T.$$

For S, we have

$$\begin{aligned}S &= \sum_{k=0}^{n+1}(kD_k \otimes D_{n+1-k})\Delta \\ &= \sum_{k=0}^{n+1}((n+1-k)D_{n+1-k} \otimes D_k)\Delta \\ &= \sum_{k=0}^{n}((n+1-k)D_{n+1-k} \otimes D_k)\Delta \\ &= \sum_{k=0}^{n}((\sum_{l=0}^{n-k} D_{n-l-k} f_{l+1}) \otimes D_k)\Delta \\ &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_{n-l-k} f_{l+1} \otimes D_k)\Delta \\ &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_l f_{k+1} \otimes D_{n-l-k})\Delta.\end{aligned}$$

For T, we have

$$\begin{aligned}T &= \sum_{k=0}^{n+1}(D_k \otimes (n+1-k)D_{n+1-k})\Delta \\ &= \sum_{k=0}^{n}(D_k \otimes (n+1-k)D_{n+1-k})\Delta\end{aligned}$$

$$= \sum_{k=0}^{n}(D_k \otimes (\sum_{l=0}^{n-k} D_{n-k-l}f_{l+1}))\Delta$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_k \otimes D_{n-k-l}f_{l+1})\Delta$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{n-k}(D_l \otimes D_{n-k-l}f_{k+1})\Delta.$$

Therefore

$$(n+1)\Delta D_{n+1} = S + T = (n+1)\sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta,$$

that is, $\Delta D_{n+1} = \sum_{k=0}^{n+1}(D_k \otimes D_{n+1-k})\Delta$. Thus $\{D_n\} \in \Phi$.

Conversely, suppose that $\{D_n\} \in \Phi$. Define $f_n : C \to C$ by $f_0 = 0$ and

$$f_n = nD_n - \sum_{k=0}^{n-2} D_{n-k-1}f_{k+1}.$$

Then Proposition 3.8 ensures us that $\{f_n\} \in \Psi$.

Now define $\varphi : \Psi \to \Phi$ by $\varphi(\{f_n\}) = \{D_n\}$, where

$$D_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}(\prod_{j=1}^{i}\frac{1}{r_j + \cdots + r_i})f_{r_i}\ldots f_{r_1}\right),$$

Clearly φ is a one to one correspondence. The proof is completed. □

4. Higher coderivations on formal triangualr matrix coalgebras

In this section, we will describe the higher coderivations on formal triangualr matrix coalgebras.

Proposition 4.1. *Let* $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$ *be a formal triangular matrix coalgebra and* $\{D_n\}_{n=0}^{\infty}$ *a higher coderivation on* Γ. *Then for all* $c \in C, d \in D, m \in M$,

$$D_n\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} \delta_n(c) + \xi_n(m) & f_n(m) \\ 0 & \delta'_n(d) + \tau_n(m) \end{pmatrix},$$

where $\delta_n : C \to C, \delta'_n : D \to D$, $\xi_n : M \to C, \tau_n : M \to D, f_n : M \to M$ *are all linear maps satisfying*

(1) $\{\delta_n\}_{n=0}^{\infty}$*(resp.,* $\{\delta'_n\}_{n=0}^{\infty}$*) is a higher coderivation on* C*(resp.,* D*);*

(2) $\Delta\xi_n = \sum_{i=0}^{n-1}(\delta_i \otimes \xi_{n-i})\rho^l$, $\Delta\tau_n = \sum_{i=0}^{n-1}(\tau_{n-i} \otimes \delta'_i)\rho^r$*;*

(3) Set $f_0 = id_M$*, then*

$$\rho^l f_n = \sum_{i=0}^{n}(\delta_i \otimes f_{n-i})\rho^l, \quad (3.1)$$
$$\rho^r f_n = \sum_{i=0}^{n}(f_{n-i} \otimes \delta'_i)\rho^r. \quad (3.2)$$

Proof. We prove this theorem by induction on n. First of all, by Proposition 3.5, the conclusion holds when $n = 1$.

Now assume that the conclusion holds for D_i, $i = 1, 2, \cdots, n-1$. For all $c \in C$, define

$$D_n\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_n(c) & \gamma_n(c) \\ 0 & \beta_n(c) \end{pmatrix},$$

where $\delta_n : C \to C, \beta_n : C \to D, \gamma_n : C \to M$ are maps. Then on one hand,

$$\begin{aligned}
\Delta D_n\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} &= \Delta\begin{pmatrix} \delta_n(c) & \gamma_n(c) \\ 0 & \beta_n(c) \end{pmatrix} \\
&= \begin{pmatrix} \delta_n(c)_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \delta_n(c)_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \gamma_n(c)_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \gamma_n(c)_{<0>} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \gamma_n(c)_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \gamma_n(c)_{(1)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \beta_n(c)_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \beta_n(c)_2 \end{pmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{i=0}^{n} (D_i \otimes D_{n-i})\Delta\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} &= \sum_{i=0}^{n} D_i\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes D_{n-i}\begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes D_n\begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} + D_n\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \sum_{i=1}^{n-1} D_i\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes D_{n-i}\begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \delta_n(c_2) & \gamma_n(c_2) \\ 0 & \beta_n(c_2) \end{pmatrix} + \begin{pmatrix} \delta_n(c_1) & \gamma_n(c_1) \\ 0 & \beta_n(c_1) \end{pmatrix} \otimes \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \sum_{i=1}^{n-1} \begin{pmatrix} \delta_i(c_1) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \delta_{n-i}(c_2) & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Since $\{D_n\}$ is a higher coderivation, we have

$$\begin{aligned}
&\delta_n(c)_1 \otimes \delta_n(c)_2 = \sum_{i=0}^{n} (\delta_i \otimes \delta_{n-i})\Delta(c), \\
&\gamma_n(c)_{<-1>} \otimes \gamma_n(c)_{<0>} = c_1 \otimes \gamma_n(c_2), \\
&\gamma_n(c)_{(0)} \otimes \gamma_n(c)_{(1)} = 0, \\
&\beta_n(c)_1 \otimes \beta_n(c)_2 = 0.
\end{aligned}$$

Therefore $\beta_n = 0, \gamma_n = 0$, and $\{\delta_n\}$ is a higher coderivation on C. Moreover

$$D_n\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_n(c) & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly we could obtain

$$D_n\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta'_n(d) \end{pmatrix},$$

for all $d \in D$, where $\{\delta'_n\}$ is a higher coderivation on D.

For all $m \in M$, define

$$D_n\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \xi_n(m) & f_n(m) \\ 0 & \tau_n(m) \end{pmatrix},$$

where $\xi_n : M \to C, f_n : M \to M, \tau_n : M \to D$ are linear maps. Then on one hand,

$$\Delta D_n \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \Delta \begin{pmatrix} \xi_n(m) & f_n(m) \\ 0 & \tau_n(m) \end{pmatrix}$$
$$= \begin{pmatrix} \xi_n(m)_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \xi_n(m)_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f_n(m)_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & f_n(m)_{<0>} \\ 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & f_n(m)_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & f_n(m)_{(1)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tau_n(m)_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \tau_n(m)_2 \end{pmatrix}.$$

On the other hand,

$$\sum_{i=0}^{n} (D_i \otimes D_{n-i}) \Delta \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \sum_{i=0}^{n} D_i \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix}$$
$$+ \sum_{i=0}^{n} D_i \begin{pmatrix} 0 & m_{(0)} \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix}.$$

For the first term,

$$\sum_{i=0}^{n} D_i \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix}$$
$$= \sum_{i=1}^{n-1} D_i \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes D_n \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix} + D_n \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix}$$
$$= \sum_{i=1}^{n-1} \begin{pmatrix} \delta_i(m_{<-1>}) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \xi_{n-i}(m_{<0>}) & f_{n-i}(m_{<0>}) \\ 0 & \tau_{n-i}(m_{<0>}) \end{pmatrix}$$
$$+ \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \xi_n(m_{<0>}) & f_n(m_{<0>}) \\ 0 & \tau_n(m_{<0>}) \end{pmatrix} + \begin{pmatrix} \delta_n(m_{<-1>}) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix}.$$

For the second term,

$$\sum_{i=0}^{n} D_i \begin{pmatrix} 0 & m_{(0)} \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix}$$
$$= \sum_{i=1}^{n-1} \begin{pmatrix} \xi_i(m_{(0)}) & f_i(m_{(0)}) \\ 0 & \tau_i(m_{(0)}) \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \delta'_{n-i}(m_{(1)}) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & m_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \delta'_n(m_{(1)}) \end{pmatrix} + \begin{pmatrix} \xi_n(m_{(0)}) & f_n(m_{(0)}) \\ 0 & \tau_n(m_{(0)}) \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix}.$$

Hence we get

$$\Delta \xi_n = \left(\sum_{i=0}^{n-1} \delta_i \otimes \xi_{n-i} \right) \rho^l, \ \Delta \tau_n = \left(\sum_{i=0}^{n-1} \tau_{n-i} \otimes \delta'_i \right) \rho^r,$$
$$\rho^l f_n = \left(\sum_{i=0}^{n-1} \delta_i \otimes f_{n-i} \right) \rho^l + (\delta_n \otimes id) \rho^l,$$

$$\rho^r f_n = \left(\sum_{i=0}^{n-1} f_{n-i} \otimes \delta'_i\right)\rho^r + (id \otimes \delta'_n)\rho^r.$$

Set $f_0 = id_M$, and we could finish the proof. □

Definition 4.2. *Let C and D be coalgebras, M be a C-D-bicomodule via ρ^l and ρ^r respectively, $\{\delta_n : C \to C\}$ and $\{\delta'_n : D \to D\}$ be higher coderivations. The family of linear maps $\{f_n : M \to M\}$ is called a generalized higher coderivation with respect to $(\{\delta_n\}, \{\delta'_n\})$ if the relations (3.1) and (3.2) hold.*

Lemma 4.3. *Let $\{D_n\}_{n=0}^{\infty}$ be defined in Proposition 4.1. Then for all $m \in M$,*

$$\xi_n(m) + \sum_{i=1}^{n} \varepsilon_D(\tau_i(m_{<0>}))\delta_{n-i}(m_{<-1>}) = 0, \tag{3.3}$$

$$\tau_n(m) + \sum_{i=1}^{n} \varepsilon_C(\xi_i(m_{(0)}))\delta'_{n-i}(m_{(1)}) = 0. \tag{3.4}$$

Proof. We only prove the identity (3.3), and (3.4) could be proved similarly. For all $m \in M$, on one hand,

$$(id \otimes \varepsilon_2)\Delta D_n \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_n(m) \\ 0 & \tau_n(m) \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}
&(id \otimes \varepsilon_2) \sum_{i=0}^{n} (D_i \otimes D_{n-i})\Delta \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\
&= (id \otimes \varepsilon_2) \sum_{i=0}^{n} D_i \begin{pmatrix} 0 & m_{(0)} \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix} \\
&\quad + (id \otimes \varepsilon_2) \sum_{i=0}^{n} D_i \begin{pmatrix} m_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes D_{n-i} \begin{pmatrix} 0 & m_{<0>} \\ 0 & 0 \end{pmatrix} \\
&\overset{\text{Lemma 4.4}}{=} \begin{pmatrix} \xi_n(m) & f_n(m) \\ 0 & \tau_n(m) \end{pmatrix} + \sum_{i=0}^{n-1} \varepsilon_D(\tau_{n-i}(m_{<0>})) \begin{pmatrix} \delta_i(m_{<-1>}) & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore

$$\xi_n(m) + \sum_{i=0}^{n-1} \varepsilon_D(\tau_{n-i}(m_{<0>}))\delta_i(m_{<-1>}) = 0,$$

or

$$\xi_n(m) + \sum_{i=1}^{n} \varepsilon_D(\tau_i(m_{<0>}))\delta_{n-i}(m_{<-1>}) = 0.$$

The proof is completed. □

Proposition 4.4. *Let $\Gamma = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$ be a formal triangular matrix coalgebra, and $\{D_n : \Gamma \to \Gamma\}_{n=0}^{\infty}$ a family of maps. Then the following assertions are equivalent:*

(1) $\{D_n\}_{n=0}^{\infty}$ given by

$$D_n \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} \delta_n(c) & f_n(m) \\ 0 & \delta'_n(d) \end{pmatrix}$$

is a higher coderivation on Γ, where $\{\delta_n\}$ (resp. $\{\delta'_n\}$) is a higher coderivation on C (resp. D), and $\{f_n : M \to M\}$ is a generalized higher coderivation with respect to $(\{\delta_n\}, \{\delta'_n\})$.

(2) $\{D_n\}_{n=0}^{\infty}$ *is a higher coderivation on* Γ *with* $\varepsilon_1 D_n = 0$ *for all positive integer* n.

(3) $\{D_n\}_{n=0}^{\infty}$ *is a higher coderivation on* Γ *with* $\varepsilon_2 D_n = 0$ *for all positive integer* n.

Proof. (1) $\Rightarrow$ (2). Clearly for all $n \in \mathbb{N}$, D_n is linear. For all $c \in C, d \in D, m \in M$,

$$\begin{aligned}
\Delta D_n \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} &= \Delta \begin{pmatrix} \delta_n(c) & f_n(m) \\ 0 & \delta'_n(d) \end{pmatrix} \\
&= \begin{pmatrix} \delta_n(c)_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \delta_n(c)_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f_n(m)_{<-1>} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & f_n(m)_{<0>} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & f_n(m)_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & f_n(m)_{(1)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \delta'_n(d)_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \delta'_n(d)_2 \end{pmatrix} \\
&= \sum_{i=0}^{n} \begin{pmatrix} \delta_i(c_1) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \delta_{n-i}(c_2) & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=0}^{n} \begin{pmatrix} \delta_i(m_{<-1>}) & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & f_{n-i}(m_{<0>}) \\ 0 & 0 \end{pmatrix} \\
&\quad + \sum_{i=0}^{n} \begin{pmatrix} 0 & f_{n-i}(m_{(0)}) \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \delta'_i(m_{(1)}) \end{pmatrix} + \sum_{i=0}^{n} \begin{pmatrix} 0 & 0 \\ 0 & \delta'_i(d_1) \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \delta'_{n-i}(d_2) \end{pmatrix} \\
&= \sum_{i=0}^{n} (D_i \otimes D_{n-i}) \Delta \begin{pmatrix} c & m \\ 0 & d \end{pmatrix}.
\end{aligned}$$

That is, $\{D_n\}$ is a higher coderivation. It is easy to check that $\varepsilon_1 D_n = 0$.

(2) $\Rightarrow$ (1). Assume that $\{D_n\}$ be a higher coderivation on Γ, then by Proposition 4.1,

$$D_n \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} \delta_n(c) + \xi_n(m) & f_n(m) \\ 0 & \delta'_n(d) + \tau_n(m) \end{pmatrix},$$

with $\delta_n, \delta'_n, \xi_n, f_n, \tau_n$ satisfying the conditions (1)-(3) of Proposition 4.1.

Since $\varepsilon_1 D_n = 0 (n > 0)$, we have

$$\varepsilon_C(\xi_n(m)) = 0.$$

The relation (3.4) implies

$$\tau_n(m) = 0.$$

Again by relation (3.3), we have

$$\xi_n(m) = 0.$$

Therefore

$$D_n \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} = \begin{pmatrix} \delta_n(c) & f_n(m) \\ 0 & \delta'_n(d) \end{pmatrix}.$$

In a similar way, one could also acquire the equivalence between (1) and (3). The proof is completed. □

Conflict of interest

The authors declare there is no conflicts of interest.

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