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# Approximation properties of a general sequence of $\lambda-$Szász-Kantorovich-Schurer operators

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**Abstract.** In the present manuscript, we study the approximation properties of a new sequence of modified Szász Kantorovich Schurer operators which depends on parameters $\lambda \in [-1,1]$ and $\rho > 0$. Further, we prove a Korovkin-type approximation theorem to discuss uniform convergence of these sequences of operators and obtain the order of approximation of these operators in terms of classical modulus of continuity. Moreover, univariate and bivariate versions of these sequences of operators are introduced in their respective blocks. Rate of convergence, order of approximation, local approximation, global approximation in terms of weight function and A-statistical approximation result are investigated via first and second-order modulus of smoothness, Lipschitz classes, Peetre's K-functional in different spaces of functions.

## 1. Introduction

Bernstein (1912) [1] introduced an important sequence of polynomials which are known as Bernstein polynomials in order to demonstrate the proof of Weierstrass approximation theorem as:

$$\mathcal{B}_m(g;u) = \sum_{j=0}^{m} g\left(\frac{j}{m}\right) q_{m,j}(u), 0 \le j \le m, \tag{1}$$

where $g$ is a continuous function defined on $[0,1]$ and $q_{m,j}(u) = \binom{j}{m} u^j (1-u)^{m-j}$. The sequences of operators given in (1) are restricted in $C[0,1]$ only. To investigate the approximation properties on unbounded interval, i.e., $[0,\infty)$, Szász (1950) [2] presented a new generalization of operator (1) as:

$$S_m(g;u) = \sum_{j=0}^{\infty} g\left(\frac{j}{m}\right) Q_{m,j}(u), u \in [0,\infty), \tag{2}$$

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where $g \in C[0,\infty)$ and $Q_{m,j}(u) = e^{-mu}\frac{(mu)^j}{j!}$. The operators given in (2) are restricted for the space of continuous functions only. Many modifications are investigated for the operators (2) by the several mathematicians, viz. Mohiuddine et al. ([3]-[5]), Mursaleen et al. ([6]-[8]), Braha et al. ([9]-[11]), Alotaibi et al. ([12, 13]), Aslan et al. ([15]-[17]), Rao et al. ([18]-[23]), Ansari et al. [24]and Özger et al. [25]. In 2010, Ye et al. [27] constructed a new Bézier bases with shape parameter $\lambda$ by

$$\widetilde{c}_{m,0}(\lambda;u) = q_{m,0}(u) - \frac{\lambda}{m+1} q_{m+1,1}(u),$$

and

$$\widetilde{c}_{m,j}(\lambda;u) = q_{m,j}(u) + \lambda\left(\frac{m-2j+1}{m^2-1} q_{m+1,j}(u) - \frac{m-2j-1}{m^2-1} q_{m+1,j+1}(u)\right), \tag{3}$$

where $\lambda \in [-1,1]$ and $1 \leq j \leq m-1$. To achieve flexibility in approximation results, Cai et al. [28] introduced a new generalization of Bernstein operators (1) as follows:

$$\mathcal{B}_{m,\lambda}(g;u) = \sum_{j=0}^{m} \widetilde{c}_{m,j}(\lambda;u) g\left(\frac{j}{m}\right), \tag{4}$$

where $g \in C[0,1]$ and $\widetilde{c}_{m,j}(.;.)$ are defined in (3).

**Remark 1.1.** *The sequence of operators introduced in* (4) *are particular case of classical Bernstein operators given in* (1) *for* $\lambda = 0$.

Further, they studied several approximation properties in terms of Voronovskaja type theorem and the modulus of continuity. These operators given in (4) are restricted for the class of the continuous functions only. To approximate Lebesgue measurable functions, Acu et al. [29] introduced a Kantorovich variant of these operators.

In continuation, Kumar [30] presented a new kind of Kantorovich variant of the $\lambda$- Bernstein operators based on the two non-negative parameters $\alpha, \beta$ and $0 \leq \alpha \leq \beta$ to get better approximation results as follows:

$$\mathcal{K}^{\alpha,\beta}_{m,\lambda}(h;u) = \sum_{j=0}^{m} \widetilde{c}_{m,j}(\lambda;u) \int_0^1 h\left(\frac{j+t^\alpha}{m+\beta}\right) dt, \ for\ each\ u \in [0,1], \tag{5}$$

where $\widetilde{c}_{m,j}(\lambda;u)$ is defined in (3).

In addition of above litrature, Qi et al. [26] introduced Szász type operators based on shape parameter $\lambda$ as follows:

$$T_{s,\lambda}(h;u) = \sum_{j=0}^{\infty} \widetilde{Q}_{s,j}(\lambda;u) h\left(\frac{j}{s}\right), \tag{6}$$

where the basis function $\widetilde{Q}_{s,j}(.;.)$ as:

$$\widetilde{Q}_{s,0}(\lambda;u) = Q_{s,0}(u) - \frac{\lambda}{s+1} Q_{s+1,1}(u),$$

and

$$\widetilde{Q}_{s,j}(\lambda;u) = Q_{s,j}(u) + \lambda\left(\frac{s-2j+1}{s^2-1} Q_{s+1,j}(u) - \frac{s-2j-1}{s^2-1} Q_{s+1,j+1}(u)\right), \tag{7}$$

where $u \in [0,\infty)$ and $h \in C[0,\infty)$ which are termed as $\lambda$-Szász operators.

**Remark 1.2.** *$\lambda-$Szász operators defined in* (6) *is a particular case of Szász* (2) *operators. For $\lambda = 0$, the operators discussed in* (6) *turn into* (2)*. The sequences of these operators are restricted for continuous functions only.*

To approximate in the wider class, i.e., the space of Lebesgue measurable functions, Aslan [31] constructed a Kantorovich variant of $\lambda-$Szász operators as follows:

$$E_{s,\lambda}(h:u) = s\sum_{j=0}^{\infty} \widetilde{Q}_{s,j}(\lambda;u)\int_{\frac{j}{s}}^{\frac{j+1}{s}} h(t)dt, u \in [0,\infty), \tag{8}$$

where $\widetilde{Q}_{s,j}(.;.)$ are same as in (7). Motivated with the above development, we present a general sequence of Szász Kantrovich Schurer type operators to achieve flexibility in the approximation results in terms of parameter $\rho$ as follows:

$$F^{\rho}_{s+l,\lambda}(g;y) = \sum_{j=0}^{\infty} \widetilde{Q}_{s+l,j}(\lambda,y)\int_0^1 g\left(\frac{j+\nu^{\rho}}{s+l+1}\right)d\nu, \tag{9}$$

where $\rho, l > 0$ and $\widetilde{Q}_{s+l,j}(.;.)$ is found in (7) by replacing $s$ by $s+l$.
In subsequent sections some estimates are calculated in terms of central moments and test functions. Further, we prove a Korovkin-type approximation theorem and obtain the order of approximation of these operators. Moreover, univariate and bivariate version of these sequences of operators are introduced in their respective blocks. Rate of convergence, order of approximation, local approximation, global approximation in terms of weight function and A-statistical approximation result are investigated via first, second-order modulus of smoothness, Lipschitz classes, Peetre's K-functional in different spaces of functions.

## 2. Some Estimates

**Lemma 2.1.** *[26] We recall the following equalities:*

$$\begin{aligned}
T_{s,\lambda}(1;z) &= 1,\\
T_{s,\lambda}(t;z) &= z + \left[\frac{1-e^{-(s+1)z}-2z}{s(s-1)}\right]\lambda,\\
T_{s,\lambda}(t^2;z) &= z^2 + \frac{z}{s} + \left[\frac{2z+e^{-(s+1)z}-1-4(s+1)z^2}{s^2(s-1)}\right]\lambda.
\end{aligned}$$

**Lemma 2.2.** *Let $e_j(t) = t^j$ be the test function. Then, for the given operator* (9)*, $\rho > 0$, $s+l \in \mathbb{N}$, one has*

$$\begin{aligned}
F^{\rho}_{s+l,\lambda}(e_0,y) &= 1,\\
F^{\rho}_{s+l,\lambda}(e_1,y) &= \frac{(s+l)y}{s+l+1} + \frac{1}{(\rho+1)(s+l+1)} + \left[\frac{1-e^{-(s+l+1)y}-2y}{(s+l+1)(s+l-1)}\right]\lambda = W^{\rho}_{s+l,\lambda},\\
F^{\rho}_{s+l,\lambda}(e_2,y) &= \frac{y^2(s+l)^2+y(s+l)}{(s+l+1)^2} + \left[\frac{2y+e^{-(s+l+1)y}-4(s+l+1)y^2}{(s+l+1)^2(s+l-1)}\right]\lambda\\
&+ \frac{2y}{(s+l+1)^2(\rho+1)} + \left[\frac{2-2e^{-(s+l+1)y}-2y}{(s+l+1)^2(\rho+1)(s+l-1)}\right]\lambda + \frac{1}{(2\rho+1)(s+l+1)^2}.
\end{aligned}$$

*Proof.* *In terms of Lemma 2.1, Lemma 2.2 can easily be proved.*

**Lemma 2.3.** *Let $\mu_j(t) = (e_j(t) - y)^j = \xi_y^j(t)$, $j \in \mathbb{N}$ be the central moments of $F^{\rho}_{s+l,\lambda}(.;.)$ presented in* (9)*. Then, we have*

$$F^{\rho}_{s+l,\lambda}(e_1(t) - y); y) = \frac{y}{s+l+1} + \frac{1 + e^{-(s+l+1)y} + 2y}{(s+l+1)(s+l-1)} + \frac{1}{(\rho+1)(s+l+1)} = E^{\rho}_{s+l,\lambda},$$

$$\begin{aligned} F^{\rho}_{s+l,\lambda}((e_1(t) - y)^2; y)) &= \frac{y^2(s+l)^2 - 2(s+l)y^2(s+l+1)^2 + y(s+l)}{(s+l+1)^2} + \frac{2y - 2y(s+l+1)}{(s+l+1)(\rho+1)} \\ &+ \Big[\frac{2y + e^{-(s+l+1)y} - 4(s+l+1)y^2}{(s+l+1)^2(s+l-1)} \\ &+ \frac{2 - 2e^{-(s+l+1)y} - 4y}{(s+l+1)^2(s+l-1)(\rho+1)} - \frac{2y + 2ze^{-(s+l+1)y + 4y^2}}{(s+l+1)(s+l-1)}\Big]\lambda = H^{\rho}_{s+l,\lambda}. \end{aligned}$$

Lemma 2.3 can easily be obtained in terms of Lemma 2.2.

## 3. Rate of convergence of $F^{\rho}_{s+l,\lambda}(.;.)$

**Definition 3.1.** *Let $g \in C[0,\infty)$. Then, modulus of continuity for a uniformly continuous function $g$ is presented as:*

$$\omega(g;\gamma) = \sup_{|s_1 - s_2| \le \gamma} \{|g(s_1) - g(s_2)|, s_1, s_2 \in [0,\infty)\}.$$

For a uniformly continuous function $g$ in $C[0,\infty)$ and $\gamma > 0$, one has

$$|g(s_1) - g(s_2)| \le \left(1 + \frac{(s_1 - s_2)^2}{\gamma^2}\right)\omega(g;\gamma). \tag{10}$$

**Theorem 3.2.** *For the operators $F^{\rho}_{s+l,\lambda}(.;.)$ introduced by* (9) *and for each $g \in C[0,\infty) \cap E$, $F^{\rho}_{s+l,\lambda}(g;y) \longrightarrow g(y)$ on each compact subset of $[0,\infty)$, where $E = \left\{g : y \ge 0, \frac{g(y)}{1+y^2} \text{ is convergent as } y \longrightarrow \infty\right\}$.*

*Proof.* In the light of Korovkin-type theorem 4.1.4 property (*iv*) [32], it is enough to show that $F^{\rho}_{s+l,\lambda}(e_i;y) \to e_i(y)$, for $i = 0,1,2$. In terms of Lemma 2.2, it is obvious $F^{\rho}_{s+l,\lambda}(e_i;y) \to e_i(y)$ as $s + l \to \infty$ for $i = 0,1,2$. Which completes the proof of Theorem 3.2. □

**Theorem 3.3.** *[36] Let $\mathcal{L} : C[p,q] \to \mathcal{B}[c,d]$ be a positive linear operator and suppose $\beta_y$ be the function defined by*

$$\beta_y(w) = |w - y|, (w,y) \in [c,d] \times [p,q].$$

*If $g \in C_B([p,q])$, for any $y \in [p,q]$ and $\eta > 0$, the operator $L$ verifies:*

$$|(Lg)(y) - g(y)| \le |g(y)||(Le_0)(y) - 1|\{(Le_0)(y) + \eta^{-1}\sqrt{(Le_0)(y)(L\beta_y^2(y))}\}\omega_y(\eta).$$

**Theorem 3.4.** *Let $g \in C_B[0,\infty)$. Then, for the operator $F^{\rho}_{s+l,\lambda}(.;.)$ given by* (9)*, we obtain*

$$|F^{\rho}_{s+l,\lambda}(g,y) - g(y)| \le 2\omega(g;\eta),$$

*where $\eta = \sqrt{F^{\rho}_{s+l,\lambda}(H^{\rho}_{s+l,\lambda};y)}$ and $H^{\rho}_{s+l,\lambda}$ is defined in 2.3.*

*Proof.* In term of Lemma 2.1, 2.2 and Theorem 3.2, we have

$$\left|F^{\rho}_{s+l,\lambda}(g,y) - g(y)\right| \le \left\{1 + \eta^{-1}\sqrt{F^{\rho}_{s+l,\lambda}(H^{\rho}_{s+l,\lambda};y)}\right\}\omega(g;\eta),$$

which proves the Theorem 3.4 choosing $\eta = \sqrt{F^{\rho}_{s+l,\lambda}(H^{\rho}_{s+l,\lambda};y)}$. □

## 4. Local approximation

Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions equipped with norm, $||g|| = \sup_{0\le u<\infty} |g(u)|$. For any $g \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional is defined as:

$$K_2(g;\delta) = inf\left\{||g-h|| + \delta||g^{''}|| : g \in C_B^2[0,\infty)\right\},$$

where $C_B^2[0,\infty) = \left\{g \in C_B[0;\infty) : g^{'}, g^{''} \in C_B[0,\infty)\right\}$.
By DeVore and Lorentz ([35], $p$.177, Theorem (2.4), there is fixed real constant $C > 0$. As a result it exists

$$K_2(g,h) \le C\omega_2(g, \sqrt{\delta}), \tag{11}$$

where $\omega_2(.;.)$ is the modulus of smoothness of second order which is defined as:

$$\omega_2(g;\sqrt{\delta}) = \sup_{0<g\le\sqrt{\eta}} \sup_{u\in[0,\infty)} |g(u+2h) - 2g(u+g) + g(u)|.$$

Here, for $g \in C_{\mathcal{B}}[0,\infty), y \ge 0$ and $s+l > 1$, the auxiliary operator is taken into consideration $\widehat{F}^{\rho}_{s+l,\lambda}(.;.)$ as follows:

$$\widehat{F}^{\rho}_{s+l,\lambda}(g;y) = F^{\rho}_{s+l,\lambda}(g:y) + g(y) - g\left(W^{\rho}_{s+l,\lambda}\right). \tag{12}$$

**Lemma 4.1.** *Let $g \in C_B^2[0,\infty)$. Then, one obtain*

$$|\widehat{F}^{\rho}_{s+l,\lambda}(g:y) - g(y)| \le \xi_{s+l}(y)||g^{''}||,$$

*where*

$$\begin{aligned}\xi_{s+l}(y) &= \frac{y^2(s+l)^2 + y(s+l)}{(s+l+1)^2} + \left[\frac{2y + e^{-(s+l+1)y} - 4(s+l+1)y^2}{(s+l+1)^2(s+l-1)}\right]\lambda \\ &+ \frac{2y}{(s+l+1)^2(\rho+1)} + \left[\frac{2-2e^{-(s+l+1)y} - 2y}{(s+l+1)^2(\rho+1)(s+l-1)}\right]\lambda + \frac{1}{(2\rho+1)(s+l+1)^2}.\end{aligned}$$

*Proof.* For the auxiliary operators which are discussed in (12), we obtain

$$\widehat{F}^{*}_{s+l,\lambda}(1;y) = 1, \widehat{F}^{*}_{s+l,\lambda}(\phi_y;y) = 0 \; and \; |\widehat{F}^{*}_{s+l,\lambda}(g;y)| \le 3||g||. \tag{13}$$

Using Taylor series expension for $g \in C_B^2[0,\infty)$, we get

$$g(t) = g(y) + (t-y)g^{'}(y) + \int_y^t (t-w)f^{''}(w)dw. \tag{14}$$

Using operator (9) in (14) on both sides, one has

$$\widehat{F}^{*}_{s+l,\lambda}(g,y) - g(y) = g^{'}(y)\widehat{F}^{*}_{s+l,\lambda}(t-y;y) + \widehat{F}^{*}_{s+l,\lambda}\left(\int_y^t (t-w)g^{''}(w)dw\right). \tag{15}$$

From (12) and (15), we obtain

$$\widehat{F}^{*}_{s+l,\lambda}(g;y) - g(y) = \widehat{F}^{*}_{s+l,\lambda}\Big(\int_y^t (t-w)g^{''}dw;y\Big)$$

$$= F^{\rho}_{s+l,\lambda}\Big(\int_y^t (t-w)g^{''}(w)dw;y\Big) - \int_y^{W^{\rho}_{s+l,\lambda}} \Big(W^{\rho}_{s+l,\lambda} - y\Big)g^{''}(w)dw.$$

$$|\widehat{F}^{*}_{s+l,\lambda}(g:y)-g(y)|\leq\left|(t-w)\,g^{''}(w)dw;y\right|+\left|\int_{y}^{W^{\rho}_{s+l,\lambda}}\left(W^{\rho}_{s+l,\lambda}-w\right)g^{''}(w)dw\right|.$$

Since,

$$\left|\int_{y}^{1}(t-w)g^{''}(w)dw\right|\leq(t-y)^{2}\|g^{''}\|. \tag{16}$$

Then, we have

$$\left|\int_{y}^{W^{\rho}_{s+l,\lambda};y)}\left(W^{\rho}_{s+l,\lambda}-w\right)g^{''}(w)dw\right| \leq\left(F^{\rho}_{s+l,\lambda}(t-w;y)\right)^{2}\|g^{''}\|.$$

In view of of (13), (16) and (**??**), we have $|\widehat{F}^{*}_{s+l,\lambda}(g;y)-g(y)|\leq\xi^{y}_{n}\|g^{''}\|$.
Hence, completes the proof of the above lemma 4.1 □

**Theorem 4.2.** *Let $g\in C^{2}_{\mathcal{B}}[0,\infty)$. Then, there exist a constant $C>0$ such that*

$$|F^{\rho}_{s+l,\lambda}(g;y)-g(y)|\leq C\omega_{2}(g;\sqrt{\xi^{y}_{s+l}})+\omega(g;F^{\rho}_{s+l,\lambda}(\xi^{y}_{s+l};y)),$$

*where $\xi^{y}_{s+l}(y)$ is defined by the Lemma 4.1.*

*Proof.* For $h\in C^{2}_{\mathcal{B}}[0,\infty)$, $g\in C_{\mathcal{B}}[0,\infty)$ and by the definition of $\widehat{F}^{*}_{s+l,\lambda}(.;.)$, one has

$$\begin{aligned}|F^{\rho}_{s+l,\lambda}(g;y)-g(y)|\leq|\widehat{F}^{\rho}_{s+l,\lambda}(g-h;y)|+|(g-h)(y)|+|\widehat{F}^{\rho}_{s+l,\lambda}(g;y)-g(y)|\\+\left|g\left(F^{\rho}_{s+l,\lambda}\right)-g(u)\right|.\end{aligned}$$

With the aid of Lemma 4.1 and relation in (13), we get

$$\begin{aligned}\left|F^{\rho}_{s+l,\lambda}(g;y)-g(y)\right| &\leq 4\|g-h\|+|F^{\rho}_{s+l,\lambda}(g:y)-g(y)|+\left|g\left(F^{\rho}_{s+l,\lambda}(e_{1};y)\right)-g(u)\right|\\ &\leq 4\|g-h\|+\xi^{y}_{s+l}(y)\|g^{''}\|+\omega(g;F^{\rho}_{s+l,\lambda}(\xi_{s+l};y)).\end{aligned}$$

In terms of Peetre's K-functional definition, we obtain

$$\left|F^{\rho}_{s+l,\lambda}(g;y)-g(y)\right|\leq C\omega_{2}\left(g;\sqrt{\xi^{y}_{s+l}}\right)+\omega(g;F^{\rho}_{s+l,\lambda}(\xi_{s+l};y)),$$

which completes the proof of Theorem 4.2. □

Let $\eta_1>0$ and $\eta_2>0$, are two fixed real values. We recall Lipschitz-type space here as:
$Lip^{\eta_1\eta_2}_{M}(\gamma):=\left\{g\in C_{B}[0,\infty):|g(t)-g(y)|\leq M\dfrac{|t-y|^{\gamma}}{(t+\eta_1 y+\eta_2 y^2)^{\gamma/2}}:t,y\in(0,\infty)\right\}$, $M>0$ is a constant and $0<\gamma\leq1$.

**Theorem 4.3.** *Let $g\in Lip^{\eta_1,\eta_2}_{M}(\gamma)$. Then, by the operators (9), we get*

$$\left|F^{\rho}_{s+l,\lambda}(g;y)-g(y)\right|\leq M\left(\frac{\eta_{s,\lambda}(y)}{\eta_1 y+\eta_2 y^2}\right)^{\dfrac{\gamma}{2}}, \tag{17}$$

*where $\gamma\in[0,1]$ and $\eta_{s+l,\lambda}(y)=F^{\rho}_{s+l,\lambda}(\xi^{2}_{s+l};y)$.*

*Proof.* For $\gamma = 1$, one has

$$\left|F^{\rho}_{s+l,\lambda}(g;y) - g(y)\right| \leq F^{\rho}_{s+l,\lambda}(|g(t) - g(y); y) \leq MF^{\rho}_{s+l,\lambda}\left(\frac{|t-y|}{(t+\eta_1 y + \eta_2 y^2)^{1/2}}; y\right).$$

It is obvious, that

$$\frac{1}{t+\eta_1 y+\eta_2 y^2} < \frac{1}{(\eta_1 y + \eta_2 y^2)},$$

for all $y \in [0;\infty)$, we have

$$\begin{aligned}\left|F^{\rho}_{s+l,\lambda}(g;y) - h(y)\right| &\leq \frac{M}{(\eta_1 y+\eta_2 y^2)^{1/2}}\left(F^{\rho}_{s+l,\lambda}(s-y)^2; y\right)^{1/2} \\ &\leq M\left(\frac{\eta_{s,\lambda}(y)}{\eta_1 y+\eta_2 y^2}\right)^{1/2}.\end{aligned}$$

Using Hölder inequality, the Theorem 4.3 holds good for $\gamma = 1$, $\gamma \in [0,1)$ with $q_1 = 2/\gamma$ and $q_2 = 2/2-\gamma$, we have

$$\begin{aligned}\left|F^{\rho}_{s+l,\lambda}(g;y) - g(y)\right| &\leq \left(F^{\rho}_{s+l,\lambda}(|g(t)-g(y)|^{\gamma/2}; y)\right)^{\gamma/2} \\ &\leq M\left(F^{\rho}_{s+l,\lambda}\left(\frac{|t-y|^2}{t+\eta_1 y+\eta_2 y^2}; y\right)\right)^{\gamma/2}.\end{aligned}$$

Since $\dfrac{1}{t+\eta_1 y+\eta_2 y^2} < \dfrac{1}{\eta_1 y+\eta_2 y^2}$ for all $y \in (0,\infty)$, we have

$$|F^{\rho}_{s+l,\lambda}(g;y) - h(y)| \leq M\left(\frac{F^{\rho}_{s+l,\lambda}(|t-y|^2;y)}{\eta_1 y+\eta_2 y^2}\right)^{\gamma/2} \leq M\left(\frac{\eta_{s+l,\lambda}(y)}{\eta_1 y+\eta_2 y^2}\right)^{\dfrac{\gamma}{2}}.$$

Hence, we completes the proof of Theorem 4.3.
Now, we recall $r^{th}$ term order Lipschitz-type maximal function suggested by Lenze [34] as:

$$\widetilde{\omega}(g;y) = sup_{s\neq y, t\in(0,\infty)}\frac{|g(w)-f(y)|}{|w-y|^r}, y \in [0;\infty), \tag{18}$$

and $r \in (0,1]$. □

**Theorem 4.4.** *Let $g \in C_{\mathcal{B}}[0,\infty)$ and $t \in (0,1]$. Then, we get*

$$\left|F^{\rho}_{s+l,\lambda}(g;y) - g(y)\right| \leq \widetilde{\omega}_t(g;y)\,(\eta_r)^{r/2}.$$

*Proof.* We know that

$$\left|F^{\rho}_{s+l,\lambda}(g;y) - g(y)\right| \leq F^{\rho}_{s+l,\lambda}\left(|g(r)-g(y)|;y\right).$$

From equation (18), we yield

$$\left|R^{\rho}_{s,\lambda}(g;y) - g(y)\right| \leq \widetilde{\omega}_t\left((g;y)(R^{\rho}_{s,\lambda}|r-y|^t;y)\right).$$

By Hölder's inequality with $q_1 = 2/r$ and $q_2 = 2/2-r$, we have

$$\left|R^{\rho}_{s,\lambda}(g;y) - g(y)\right| \leq \widetilde{\omega}_r(g;y)\left(F^{\rho}_{s+l,\lambda}|s-y|^2;y\right)^{r/2},$$

which completes the proof of the theorem. □

## 5. Global Approximations

To establish the next result, we recall some notation from [? ]. Assume that $B_{1+y^2}[0,\infty)=\left\{g(y):|g(y)|\leq M_g(1+y^2)\right\}$, is weighted functional space, $M_g$ is a constant that is determined by $g$ and in $B_{1+y^2}[0,\infty)$, $C_{1+y^2}[0,\infty)$ is the space continuous functions with the norm

$$\|g\|_{1+y^2}=\sup_{y\in[0,\infty)}\frac{|g(y)|}{1+y^2},$$

and

$$C^k_{1+y^2}[0,\infty)=\left\{g\in C_{1+y^2}[0,\infty):\lim_{|y|\longrightarrow\infty}\frac{g(y)}{1+y^2}=M_g\right\},$$

where $M_g$ is a constant that depends on g.

**Theorem 5.1.** *Let $F^{\rho}_{s+l,\lambda}(.;.)$, be the operators given by* (9) *and*

$$F^{\rho}_{s+l,\lambda}(.;.):C^k_{1+y^2}[0;\infty)\to B_{1+y^2}[0;\infty).\ \textit{Then, we obtain}$$

$$\lim_{s\to\infty}\|F^{\rho}_{s+l,\lambda}(g;.)-g\|_{1+y^2}=0,\ \textit{where}\ g\in C^k_{1+y^2}[0,\infty).$$

*Proof.* To prove this result, it is required to check that

$$\lim_{s\to\infty}\|F^{\rho}_{s+l,\lambda}(e_i;.)-e_i\|_{1+y^2}=0, i=\{0,1,2\}.$$

Using Lemma 2.2, we obtain for $i=0$

$$\|F^{\rho}_{s+l,\lambda}(e_0;.)-e_0\|_{1+y^2}=\sup_{y\in[0,\infty)}\frac{|F^{\rho}_{s+l,\lambda}(e_0;.)-1|}{1+y^2}=0.$$

For $i=1$

$$\|F^{\rho}_{s+l,\lambda}(e_1;.)-e_1\|_{1+y^2}=\sup_{y\in[0;\infty)}\left[\frac{\frac{(s+l)y}{s+l+1}+\frac{1}{(\rho+1)(s+l+1)}+\left[\frac{1-e^{-(s+l+1)y}-2y}{(s+l+1)(s+l-1)}\right]\lambda-y}{1+y^2}\right]$$
$$=\left(\frac{s+l}{s+l+1}-1\right)\sup_{y\in[0,\infty)}\frac{y}{1+y^2}+\frac{1}{(\rho+1)(s+l+1)}+\left[\frac{1-e^{-(s+l+1)y}-2y}{(s+l+1)(s+l-1)}\right]\lambda\sup_{y\in[0,\infty)}\frac{1}{1+y^2}.$$

This implies that $\|F^{\rho}_{s+l,\lambda}(e_1;.)-e_1\|_{1+y^2}\to 0$ as $s+l\to\infty$.
Same as above, we can calculate for $i=2$, $\|F^{\rho}_{s+l,\lambda}(e_2;.)-e_2\|_{1+y^2}\to 0\ as\ s+l\to\infty$.
Hence, we arrived at our desired result. □

**Theorem 5.2.** *Let $g\in C^s_\mu[0,\infty)$, and $\mu$ is positive real number. Then,*
$$lim_{s+l\to\infty}\sup\nolimits_{y\in[0,\infty)}\frac{|F^{\rho}_{s+l,\lambda}(g;y)-g(y)|}{(1+y^2)^{1+\mu}}=0.$$

*Proof.* For any fixed real number $y_0 > 0$, we obtain

$$\begin{aligned}
\sup \quad_{y\in[0,\infty)} \frac{|F^{\rho}_{s+l,\lambda}(g;y) - g(y)|}{(1+y^2)^{1+\mu}} &\leq \sup_{y\leq y_0} \frac{|F^{\rho}_{s+l,\lambda}(g;y) - g(y)|}{(1+y^2)^{1+\mu}} \\
+ \quad \sup_{y\geq y_0} \frac{|F^{\rho}_{s+l,\lambda}(g;y) - g(y)|}{(1+y^2)^{1+\mu}} &\leq \|F^{\rho}_{s+l,\lambda}(g;.) - g(y)\|_{C[0,y_0]} \\
+ \quad \|g\|_{\rho} \sup_{y\geq y_0} \frac{|F^{\rho}_{s+l,\lambda}(g;y) - g(y)|}{(1+y^2)^{1+\mu}} &+ \sup_{y\geq y_0} \frac{|g(y)|}{(1+y^2)^{1+\mu}} \\
= S_1 + S_2 + S_3 \; say, \qquad (15)
\end{aligned}$$

we obtain

$$S_3 = \sup_{y\geq y_0} \frac{|g(y)|}{(1+y^2)^{1+\mu}} \leq \sup_{y\leq y_0} \frac{\|g\|_{\rho}(1+y^2)}{(1+y^2)^{1+\mu}} \leq \frac{\|g\|_{\rho}(1+y^2)}{(1+y_0^2)^{\mu}}.$$

In the light of Lemma 2.2. Therefore arbitrary $\epsilon > 0$, and corresponding $s_1 \in \mathbb{N}$ such that

$$\sup_{y\in[y_0,\infty)} \frac{F^{\rho}_{s+l,\lambda}(1+y^2,y)}{1+y^2} \leq \frac{(1+y_0^2)^{\mu}}{\|g\|_{\rho}} \frac{\epsilon}{3} + 1.$$

For all $s \geq s_1$

$$E_2 = \|g\|_{1+y^2} \sup_{y\in[y_0,\infty)} \frac{F^{\rho}_{s+l,\lambda}(1+y^2;y)}{1+y^2} \leq \frac{(1+y_0^2)^{\mu}}{\|g\|_{\rho}} + \frac{\epsilon}{3},$$

for all $s \geq s_1$.

$$E_2 = \|g\|_{1+y^2} \sup_{y\in[0;\infty)} \frac{F^{\rho}_{s+l,\lambda}(1+s^2;y)}{1+y^2} \leq \frac{\|g\|_{1+y^2}}{(1+y_0^2)^{\mu}} + \frac{\epsilon}{3} \; for \; all \; s \geq s_1.$$

Therefore

$$E_2 + E_3 < \frac{\|g\|_{1+y^2}}{(1+y^2)^{\mu}} + \frac{\epsilon}{3}. \qquad (11)$$

On choosing $y_0$ be a large number such that

$$\frac{\|g\|_{1+y^2}}{(1+y^2)^{\mu}} < \frac{\epsilon}{6} \; we \; obtain$$

$$E_2 + S_3 < \frac{2\epsilon}{3} \; for \; all \; s + l \geq s_1.$$

By Theorem 5.2 there corresponding $s_2 \geq s$ such that

$$E_1 = \|F^{\rho}_{s+l,\lambda}(g;.) - g\|_{C[0,y_0]} < \frac{\epsilon}{3} \; where \; s_2 \geq s + l.$$

Let $S_3 = max(s_1, s_2)$, we get

$$\sup_{y\in[0,\infty)} \frac{|F^{\rho}_{s+l,\lambda}(g;y) - l(y)|}{(1+y^2)^{1+\mu}} < \epsilon.$$

Hence, completes the proof of theorem 5.2. □

## 6. A-statistical Approximation

First, we include some basic definitions and notations for the concept of A-statistical convergence. Let $A = (a_{ml})$, *where* $m, l \in \mathbb{N}$), be a positive infinite summability matrix. For a given sequence $y := (y_l)$, the A-transform of $y$ denoted by $Ay : (Ay)_m$ defined as follows:

$$(Ay)_m = \sum_{l=0}^{\infty} a_{ml} Y_l.$$

Considering the series converges for every m. A is said to be regular if $lim_m (Ay)_m = L$. Whenever $lim_m y_m = L$. Then, $y = y_m$, is said to be a A-statistically convergent to L, i.e., $st_A - lim_m y_m = L$. If for every $\epsilon > 0$, $lim_m \sum_{l:|y_l - L| \geq \epsilon} a_{yl} = 0$.
Now, interchanging A by $C_1$, the Cesáro matrix of order one reduces to the statical convergence from the A-statistical convergence. Similarly, let $A = I$ the idenitity matrix. Then, the ordinary convergence and A-statistical convergence are simultaneous.

**Theorem 6.1.** *Let* $A = (a_{ml})$, *be a positive regular suitability matrix* $y \geq 0$. *Then, we obtain* $st_A - lim_s \|F^{\rho}_{s+l,\lambda}(g;.) - g\|_{1+y^2} = 0$, *for all* $g \in C^l_{1+y^2}[0,\infty)$.

*Proof.* [33]( p.191 Theorem.3), it is enough to show that $\delta_1 = 0$,

$$st_A - lim_s \|F^{\rho}_{s+l,\lambda}(e_i;.) - e_i\|_{1+y^2} = 0 \; for \; i = \{0,1,2\}. \tag{8}$$

By Lemma 2.2, we obtain

$$\begin{aligned}
\|F^{\rho}_{s+l,\lambda}(e_i;.) - e_i\|_{1+y^2} &= \sup_{y\in[0,\infty)} \frac{1}{1+y^2} \left| \frac{sy}{s+1} + \frac{1}{(\rho+1)(s+1)} + \left[\frac{1-e^{-(s+1)y} - 2y}{(s+1)(s-1)}\right]\lambda \right| \\
&\leq \left|\frac{s}{s+1}\right| sup_{y\in[0,\infty)} \frac{y}{1+y^2} + \left| \frac{1}{(\rho+1)((s+1))} + \left[\frac{1-e^{-(s+1)y} - 2y}{(s+1)(s-1)}\right]\lambda \right| \sup_{y\in[0,\infty)} \frac{1}{1+y^2}.
\end{aligned}$$

Now, for given $\epsilon > 0$, we define the following sets

$$\begin{aligned}
S_1 &= \left\{ s : \|F^{\rho}_{s+l,\lambda}(e_1;.) - e_1\| \geq \epsilon \right\}, \\
S_2 &= \left\{ s : \frac{s}{s+1} \geq \frac{\epsilon}{2} \right\}, \\
S_3 &= \left\{ s : \frac{1}{(\rho+1)((s+1))} + \left[\frac{1-e^{-(s+1)y} - 2y}{(s+1)(s-1)}\right]\lambda \geq \frac{\epsilon}{2} \right\}.
\end{aligned}$$

Then, we obtain $S_1 \subset S_2 \cup S_3$ which shows that

$$\sum_{l_1 \in M_1} a_{ml_1} \leq \sum_{l_1 \in M_2} a_{ml} + \sum_{l_1 \in M_3} a_{ml}.$$

Hence, form (8) we obtain.

$$st_A - lim_s \|F^{\rho}_{s+l,\lambda}(e_1;.) - e_1\|_{1+y^2} = 0. \tag{9}$$

Similarly, once can show that

$$st_A - lim_s \|F^{\rho}_{s+l,\lambda}(e_2;.) - e_2\|_{1+y^2} = 0.$$

Hence, we arrived our desired result. □

## 7. λ-Szász-Kantorovich-Schurer-Bivariate Operators

Take $\mathcal{F}^2 = \{(y_1, y_2) : 0 \leq y_1 < \infty, 0 \leq y_2 < \infty\}$ and $C(\mathcal{F}^2)$ is the class of all continuous function on $F^2$ equipped with norm $\|g\|_{C(F^2)} = \sup_{(y_1,y_2)\in\mathcal{F}^2} |g(y_1, y_2)|$. Then, for all $g \in C(\mathcal{F}^2)$ and $(s_1 + l, s_2 + l) \in \mathbb{N} \times \mathbb{N}$, we define a bivariate sequence as follows:

$$F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(g; y_1, y_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{S}_{s_1+l,s_2+l,j,k}(\lambda_1, \lambda_2, y_1, y_2) \int_0^1 \int_0^1 g\left(\frac{j + t_1^{\rho}}{s_1 + l + 1}, \frac{k + t_2^{\rho}}{s_2 + l + 1}\right) dt_1 dt_2. \tag{10}$$

where

$$\tilde{S}_{s_1+l,s_2+l,j,k}(\lambda_1, \lambda_2, y_1, y_2) = \tilde{S}_{s_1+l,j}(\lambda_1, y_1)\tilde{S}_{s_2+l,k}(\lambda_2, y_2)$$

and

$$\begin{aligned}
\tilde{S}_{s_1+l,j}(\lambda_1, y_1) &= t_{s_1+l,j}(y_1) + \lambda_1\left(\frac{s_1 + l - 2j + 1}{s_1 + l^2 - 1} t_{s_1+l+1,j}(y_1) - \frac{s_1 + l - 2j - 1}{(s_1 + l)^2 - 1} t_{s_1+l+1,j+1}(y_1)\right),\\
\tilde{S}_{s_2+l,k}(\lambda_2, y_2) &= t_{s_2+l,k}(y_2) + \lambda_2\left(\frac{s_2 + l - 2k + 1}{(s_2 + l)^2 - 1} t_{s_2+l+1,k}(y_2) - \frac{s_2 + l - 2k - 1}{(s_2 + l)^2 - 1} t_{s_2+l+1,k+1}(y_2)\right).
\end{aligned}$$

**Lemma 7.1.** *Let $e_{j,k} = y_1^j z_2^k$. Then, for the operators defined in* (10), *we get*

$$\begin{aligned}
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{0,0}; y_1, y_2) &= 1,\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{1,0}; y_1, y_2) &= \frac{(s_1 + l)y_1}{s_1 + l + 1} + \frac{1}{(\rho + 1)(s_1 + l + 1)} + \left[\frac{1 - e^{-(s_1+l+1)y_1} - 2y_1}{(s_1 + l + 1)(s_1 + l - 1)}\right]\lambda_1,\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{0,1}; y_1, y_2) &= \frac{(s_2 + l)y_2}{s_2 + l + 1} + \frac{1}{(\rho + 1)(s_2 + l + 1)} + \left[\frac{1 - e^{-(s_2+l+1)y_2} - 2y_2}{(s_1 + l + 1)(s_2 + l - 1)}\right]\lambda_2,\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{2,0}; y_1, y_2) &= \frac{y_1^2(s_1 + l)^2 + y_1 s_1 + l}{(s_1 + l + 1)^2} + \left[\frac{2y_1 + e^{-(s_1+l+1)y_1} - 4(s_1 + l + 1)y_1^2}{(s_1 + l + 1)^2(s_1 + l - 1)}\right]\lambda_1\\
&+ \frac{2y_1}{(s_1 + l + 1)^2(\rho + 1)} + \left[\frac{2 - 2e^{-(s_1+l+1)y_1} - 2y_1}{(s_1 + l + 1)^2(\rho + 1)(s_1 + l - 1)}\right]\lambda_1\\
&+ \frac{1}{(2\rho + 1)(s_1 + l + 1)^2},\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{0,2}; y_1, y_2) &= \frac{z_2^2(s_2 + l)^2 + z_2 s_2 + l}{(s_2 + l + 1)^2} + \left[\frac{2y_2 + e^{-(s_2+l+1)y_2} - 4(s_2 + l + 1)y_2^2}{(s_2 + l + 1)^2(s_2 + l - 1)}\right]\lambda_2\\
&+ \frac{2y_2}{(s_2 + l + 1)^2(\rho + 1)} + \left[\frac{2 - 2e^{-(s_2+l+1)y_2} - 2y_2}{(s_2 + l + 1)^2(\rho + 1)(s_2 + l - 1)}\right]\lambda_2\\
&+ \frac{1}{(2\rho + 1)(s_2 + l + 1)^2}.
\end{aligned}$$

*Proof.* From 2.2 and linearity property, we get

$$\begin{aligned}
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{0,0}; y_1, y_2) &= F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_0; y_1, y_2)F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_0; y_1, y_2),\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{1,0}; y_1, y_2) &= F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_1; y_1, y_2)F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_0; y_1, y_2),\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{0,1}; y_1, y_2) &= F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_0; y_1, y_2)F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_1; y_1, y_2),\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{2,0}; y_1, y_2) &= F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_2; y_1, y_2)F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_0; y_1, y_2),\\
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_{0,2}; y_1, y_2) &= F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_0; y_1, y_2)F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(e_2; y_1, y_2).
\end{aligned}$$

□

In the light of above equalities and Lemma 2.2, we prove Lemma 7.1.
For each $g \in C(\mathcal{F}^2)$ and $\eta > 0$, second order modulus of continuity is given by

$$\omega\left(g; \delta_{n_1}, \eta_{n_2}\right) = \sup\{\mid g(t, s+l) - g(y_1, y_2) \mid : (t, s+l), (z_1, z_2) \in \mathcal{F}^2\},$$

with $\mid t - z_1 \mid \leq \eta_{n_1}, \mid s + l - z_2 \mid \leq \eta_{n_2}$ given by modulus of $p-$continuity:

$$\omega_1\left(g; \eta\right) = \sup_{0 \leq z_2 \leq \infty} \sup_{|x_1 - x_2| \leq \eta} \{\mid g(x_1, y_2) - g(x_2, y_2) \mid\},$$

$$\omega_2\left(g; \eta\right) = \sup_{0 \leq y_1 \leq \infty} \sup_{|y_1 - y_2| \leq \eta} \{\mid g(y_1, y_1) - g(y_1, y_2) \mid\}.$$

**Theorem 7.2.** *Let $g \in C(\mathcal{F}^2)$. Then, we have*

$$\mid F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(g; y_1, y_2) - g(y_1, y_2) \mid \leq 2\Big(\omega_1(g; \delta_{y_1,n_1}) + \omega_2(g; \delta_{n_2,y_2})\Big).$$

*Proof.* Taking Cauchy-Schwartz, one has

$$\begin{aligned}
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(g; y_1, y_2) &- g(y_1, y_2) \mid \leq F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid g(t, s+l) - g(y_1, y_2) \mid; y_1, y_2\right) \\
&\leq F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid g(t, s+l) - g(y_1, s+l) \mid; y_1, y_2\right) \\
&+ F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid g(y_1, s+l) - g(y_1, y_2) \mid; y_1, y_2\right) \\
&\leq F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\omega_1(g; \mid t - y_1 \mid); y_1, y_2\right) \\
&+ F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\omega_2(g; \mid s + l - y_2 \mid); y_1, y_2\right) \\
&\leq \omega_1(g; \delta_{n_1})\left(1 + \delta_{n_1}^{-1} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(\mid t - y_1 \mid; y_1, y_2)\right) \\
&+ \omega_2(g; \delta_{n_2})\left(1 + \delta_{n_2}^{-1} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(\mid s + l - y_2 \mid; y_1, y_2)\right) \\
&\leq \omega_1(g; \delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}}\sqrt{F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}((t - y_1)^2; y_1, y_2)}\right) \\
&+ \omega_2(g; \delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}}\sqrt{F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}((s + l - y_2)^2; y_1, y_2)}\right).
\end{aligned}$$

If we choose $\delta^2_{n_1} = \delta^2_{n_1,y_1} = F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}((t - y_1)^2; y_1, y_2)$ and $\delta^2_{n_2} = \delta^2_{n_2,y_2} = F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}((s + l - y_2)^2; y_1, y_2)$. Then, we simply achieve our objectives. □

Here, using the Lipschitz class for bivariate functions, we analyse convergence. Taking $M > 0$ and $\nu \in [0, 1]$, maximal Lipschitz function space on $E \times E \subset \mathcal{F}^2$ given by

$$\begin{aligned}
\mathcal{L}_{\nu,\nu}(E \times E) &= \Big\{g : \sup(1 + t)^\nu (1 + s + l)^\nu \left(g_{\nu,\nu}(t, s + l) - g_{\nu,\nu}(y_1, y_2)\right) \\
&\leq M \frac{1}{(1 + y_1)^\nu} \frac{1}{(1 + y_2)^\nu}\Big\},
\end{aligned}$$

where $g$ is bounded and continuous on $\mathcal{F}^2$, and

$$g_{\nu,\nu}(t, s + l) - g_{\nu,\nu}(y_1, y_2) = \frac{\mid g(t, s + l) - g(y_1, y_2) \mid}{\mid t - y_1 \mid^\nu \mid s + l - y_2 \mid^\nu}; \quad (t, s + l), (y_1, y_2) \in \mathcal{F}^2.$$

**Theorem 7.3.** *Let $g \in \mathcal{L}_{\nu,\nu}(E \times E)$. Then, for each $\nu, \nu \in [0,1]$, and $M > 0$, such that*

$$\begin{aligned} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(g;y_1,y_2) - g(y_1,y_2) \mid &\leq M\Big\{\Big(\left(d(y_1,E)\right)^{\nu} + \left(\delta^2_{n_1,y_1}\right)^{\frac{\nu}{2}}\Big) \\ &\times \Big(\left(d(y_2,E)\right)^{\nu} + \left(\delta^2_{n_2,y_2}\right)^{\frac{\nu}{2}}\Big) \\ &+ \left(d(y_1,E)\right)^{\nu}\left(d(y_2,E)\right)^{\nu}\Big\}, \end{aligned}$$

*where $\delta_{n_1,y_1}$ and $\delta_{n_2,y_2}$ defined by Theorem 7.2.*

*Proof.* Take $\mid y_1 - x_0 \mid = d(y_1,E)$ and $\mid y_2 - y_0 \mid = d(y_2,E)$. For any $(y_1,y_2) \in \mathcal{F}^2$, and $(x_0,y_0) \in E \times E$. Let $d(y_1,E) = \inf\{\mid y_1 - y_2 \mid : y_2 \in E\}$. Then, we write

$$\mid g(t,s+l) - g(y_1,y_2) \mid \leq M \mid g(t,s+l) - g(x_0,y_0) \mid + \mid g(x_0,y_0) - g(y_1,y_2) \mid . \tag{11}$$

Apply $F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(.;.)$, we obtain

$$\begin{aligned} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(h;y_1,y_2) &- g(y_1,y_2) \mid \\ &\leq F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid g(y_1,y_2) - g(x_0,y_0) \mid + \mid g(x_0,y_0) - g(y_1,y_2) \mid\right) \\ &\leq MF^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid t - x_0 \mid^{\nu} \mid s+l-y_0 \mid^{\nu}; y_1,y_2\right) \\ &+ M \mid y_1 - x_0 \mid^{\nu} \mid y_2 - y_0 \mid^{\nu} . \end{aligned}$$

For all $C, D \geq 0$ and $\nu \in [0,1]$, the inequality $(C+D)^{\nu} \leq C^{\nu} + D^{\nu}$, thus

$$\mid t - x_0 \mid^{\nu} \leq \mid t - y_1 \mid^{\nu} + \mid y_1 - x_0 \mid^{\nu},$$

$$\mid s+l-y_0 \mid^{\nu} \leq \mid s+l-y_2 \mid^{\nu} + \mid y_2 - y_0 \mid^{\nu} .$$

Therefore

$$\begin{aligned} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(h;y_1,y_2) - g(y_1,y_2) \mid &\leq MF^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid t - y_1 \mid^{\nu} \mid s+l-y_2 \mid^{\nu}; y_1,y_2\right) \\ &+ M \mid y_1 - x_0 \mid^{\nu} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid s+l-y_2 \mid^{\nu}; y_1,y_2\right) \\ &+ M \mid y_2 - y_0 \mid^{\nu} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid t - y_1 \mid^{\nu}; y_1,y_2\right) \\ &+ 2M \mid y_1 - x_0 \mid^{\nu} \mid y_2 - y_0 \mid^{\nu} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mu_{0,0}; y_1,y_2\right). \end{aligned}$$

On apply Hölder inequality on $F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(.;.,.)$, we get

$$\begin{aligned} F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}\left(\mid t - y_1 \mid^{\nu} \mid s+l-y_2 \mid^{\nu}; y_1,y_2\right) &= \mathcal{S}_{\infty} + \mathcal{L}^{\lambda_1}_{n_1,k}\left(\mid t - y_1 \mid^{\nu}; y_1,y_2\right) \\ &\times \mathcal{S}_{\in} + \mathcal{L}^{\lambda_2}_{n_2,l}\left(\mid s+l-y_2 \mid^{\nu}; y_1,y_2\right) \\ &\leq \left(F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(\mid t - y_1 \mid^{2}; y_1,y_2)\right)^{\frac{\nu}{2}} \\ &\times \left(F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(\mu_{0,0}; y_1,y_2)\right)^{\frac{2-\nu}{2}} \\ &\times \left(F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(\mid s+l-z_2 \mid^{2}; y_1,y_2)\right)^{\frac{\nu}{2}} \\ &\times \left(F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(\mu_{0,0}; y_1,y_2)\right)^{\frac{2-\nu}{2}} . \end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
F^{\rho,\lambda_1,\lambda_2}_{s_1+l,s_2+l}(h;y_1,y_2)-g(y_1,y_2)\,| \quad &\leq \quad M\left(\delta^2_{n_1,y_1}\right)^{\frac{\nu}{2}}\left(\delta^2_{n_2,y_2}\right)^{\frac{\nu}{2}} \\
&+ \quad 2M\left(d(y_1,E)\right)^{\nu}\left(d(y_2,E)\right)^{\nu} \\
&+ \quad M\left(d(y_1,E)\right)^{\nu}\left(\delta^2_{n_2,y_2}\right)^{\frac{\nu}{2}}+L\left(d(y_2,E)\right)^{\nu}\left(\delta^2_{n_1,y_1}\right)^{\frac{\nu}{2}}.
\end{aligned}
$$

We have complete the proof. □

## 8. Conflict of interest

The authors declared that they have no conflict of interest.

## 9. Data availability statement

Data sharing not applicable

## References

[1] Bernšteın S.: Démonstration du théoreme de Weierstrass fondée sur le calcul des probabilities, Comm. Soc. Math. Kharkov. 1912, 13, 1-2.

[2] Szász O.: Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Standards., 1950 45(3), 239-45.

[3] Mohiuddine S.A., Özger F.: Approximation of functions by Stancu variant of Bernstein–Kantorovich operators based on shape parameter $\alpha$, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas (RACSAM), (2020), 114:70.

[4] Mohiuddine S.A., Ahmad N., Özger F., Alotaibi A., Hazarika B.: Approximation by the parametric generalization of Baskakov–Kantorovich operators linking with Stancu operators, Iran J Sci Technol Trans Sci 2021, 45, 593-605.

[5] Mohiuddine, S.A., Kajla, A., Mursaleen, M. et al.: Blending type approximation by $\tau$-Baskakov-Durrmeyer type hybrid operators. Adv Differ Equ 2020, 467 (2020).

[6] Mursaleen, M., Alotaibi, A.: Korovkin type approximation theorem for functions of two variables through statistical A-summability. Adv Differ Equ 2012, 65 (2012).

[7] Mursaleen M., Rahman S., Ansari K.J.: Approximation by Jakimovski-Leviatan-Stancu-Durrmeyer Type Operators, Filomat, 33 **(6)**, (2019), 1517-1530.

[8] Mursaleen M., Ahasan M. and Ansari, K.J.: Bivariate Bernstein–Schurer–Stancu type GBS operators in -analogue. Adv Differ Equ 2020, 76 (2020).

[9] Braha N.L., Mansour T. and Mursaleen M.: Some Properties of Kantorovich-Stancu-Type Generalization of Szász Operators including Brenke-Type Polynomials via Power Series Summability Method, Journal of Function Spaces, 2020 — Article ID 3480607.

[10] Braha N.L., Someweightedequi-statisticalconvergenceand Korovkin type-theorem, Results in Mathematics, 2016, 70 **(3)**,433–446.

[11] Braha N. L., "Some properties of newmodified Szász Mirakyanoperators inpolynomialweight spaces viapower summabilitymethod,"BulletinofMathematicalAnalysisand Applications, 2018, 10 **(3)**, 53-65.

[12] Alotaibi A.: On the Approximation by Bivariate Szász–Jakimovski–Leviatan-Type Operators of Unbounded Sequences of Positive Numbers, Mathematics. 2023, 11(4),1009.

[13] Alotaibi A.: Approximation of GBS type q-Jakimovski-Leviatan-Beta integral operators in Bögel space, Mathematics. 2022, 10(5), 675.

[14] Ye Z., Long X., Zeng X. M.: Adjustment algorithms for Bézier curve and surface, Intl. Conf. on Compu. Sci. Edu., (2010), 1712-1716.

[15] Aslan R., İzgi A.: Approximation by one and two variables of the Bernstein-Schurer-type operators and associated GBS operators on symmetrical mobile interval, J. Fun. Spaces. 2021, 3;2021:1-2.

[16] Aslan R., Mursaleen M.: Approximation by bivariate Chlodowsky type Szász–Durrmeyer operators and associated GBS operators on weighted spaces, J. Inequal. Appl., 2022, 23, 2022(1), 26.

[17] Aslan R., Mursaleen M.: Some approximation results on a class of new type $\lambda-$Bernstein polynomials, J. Math. Inequal. 2022, 16/ 445-62.

[18] Rao N., Heshamuddin M., and Shadab M.: Approximation properties of bivariate Szász Durrmeyer operators via Dunkl analogue, Filomat, 2021, 35, 4515-4532.

[19] Rao N., Malik P., and Rani M.: Blending type Approximations by Kantorovich variant of $\alpha$-Baskakov operators, Palestine Journal of Mathematics, 2022, 11 (3), 402-413.

[20] Rao N., Yadav A. K., Mursaleen M., Sinhs B. K., and Jha N. K.: Szász-Beta operators via Hermite Polynomial, Journal of King Saud University - Science, 2024, 36(4), 103120.

[21] Rao N., Malik P.: $\alpha$-Baskakov-Durrmeyer type operators and their approximation properties, Filomat, 2023, 37 (3), 935-948.

[22] Rao, N., Wafi, A., and Khatoon, S.: Better Rate of Convergence by Modified Integral Type Operators, Differential Geometry, Algebra, and Analysis. ICDGAA 2016. Springer Proceedings in Mathematics and Statistics, vol 327. Springer, Singapore.
[23] Rao N., and Wafi A.: Modified Szász Operators Involving Charlier Polynomials Based on Two Parameters, Thai. J. Math., 2021, 19 (1), 131-144.
[24] Ansari K. J., Özger F., Ödemiş Ö. Z.: Numerical and theoretical approximation results for Schurer–Stancu operators with shape parameter $\lambda$, Compu. Appl. Mathematics, 2022, 41(4), 181.
[25] Özger F., Aljimi E., Temizer E. M.: Rate of weighted statistical convergence for generalized blending-type Bernstein-Kantorovich operators, Mathematics., 2022, 10(12), 2027.
[26] Qi Q., Guo D., Yang G.: Approximation properties of $\lambda$-Szàsz-Mirakian operators, Int. J. Eng. Res., 2019, 12, 662-669.
[27] Ye Z., Long X., Zeng X. M.: Adjustment algorithms for Bézier curve and surface, Inter. Conf. Comp. Sci. Edu., 2010, 1712–1716.
[28] Cai, Q. B., Lian, B. Y., Zhou G.: Approximation properties of $\lambda$-Bernstein operators, J. Inequal. Appl., 2018, 61.
[29] Acu A. M., Manav N., Sofonea D. F.: Approximation properties of $\lambda-$Kantorovich operators. J. Inequ. Appl. 2018, 2018(1), 1-2.
[30] Kumar A.: Approximation properties of generalized $\lambda-$Bernstein–Kantorovich type operators, 2021, 70(1):505-20., Rend. Circ. Mat. Palermo, II. Series, 2021, 70(1), 505–520.
[31] Aslan, R.: Some approximation results on $\lambda$-Szasz-Mirakjan-Kantorovich operators. FUJMA, 4, 2021, 150-158.
[32] Altomare F., Campiti, M.: Korovkin-Type Approximation Theory and Its Applications., D. G. Stud. Math., 2011, 994, 17.
[33] Duman, O. and Orhan C.: Statistical approximation by positive linear operators, Studia Math., 2004, 16(2), 187-197.
[34] Lenze B., On Lipschitz type maximal functions and their smoothness spaces, Ned. Akad. Indag. Math., 1988, 50, 53-63.
[35] DeVore R. A., Lorentz G. G., Constructive approximation. Springer Science and Business Media; 1993.
[36] Shisha O., B. Mond.: The degree of convergence of linear positive operators, Proc. Nat. Acad. Science, 1968, 60 (4), 1196-1200.