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On lacunary strong invariant convergence with order γ

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Abstract. We known that the most vital part of summability theory is the sequence spaces. In this paper by using a modulus and the ρ- function we shall introduce the new sequence space of order γ and also discuss some important properties related to this space. Additional we establish some inclusion relations among the these sequence spaces in some detail. We should strongly indicate that our new sequence space is more general than that of corresponding sequence space defined by Waszak, [25].

1. Introduction and background

In this section we shall start by presenting the definition of modulus function. Ruckle [11] and Maddox [8] presented the following notion of modulus function:

Definition 1.1. A function $j : [0, \infty) \to [0, \infty)$ is called a *modulus function* provided that

1. $j(x) = 0$ if and only if $x = 0$,
2. $j(x + y) \leq j(x) + j(y)$ for all $x \geq 0$ and $y \geq 0$,
3. j is increasing, and
4. j is continuous from the right of 0.

Observe by (2) and (4) it follows instantly that j is continuous on $[0, \infty)$. Moreover, by (2) for all natural number n, $j(nx) \leq nj(x)$. Different extensions and applications of modulus function have been considered in [2, 4, 10, 13, 15].

Moreover if there is constant $K > 1$ such that $\rho(2t) \leq K\rho(t)$, we say that ρ-function ρ satisfy (Δ_2)-condition for all large t (see, [25].

We know that the most important part of summability theory is the convergence of the sequences, the new concepts related to the convergence such as almost, invariant and statistical etc. were studied. The idea of the statistical convergence order α, $0 < \alpha \leq 1$ was introduced by Gadjiev and Orhan [6]. It should be note that this concept was studied by the different authors in [3, 16–18, 21, 23]) in detail.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞, (which is the set of bounded sequences), is said to be an invariant mean or a σ-mean if and only if

1. $\phi(\alpha) \geq 0$ when the sequence $\alpha = (\alpha_n)$ has $\alpha_n \geq 0$ for all n;
2. $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and

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3. $\phi(\alpha_{\sigma(n)}) = \phi(\alpha_n)$ for all $\alpha \in l_\infty$.

When $\sigma(n) = n+1$, the $\sigma-$ means are the classical Banach limits on l_∞ and f is the set of almost convergent sequences(see, [1]).

The mappings σ are assumed one-to-one and such that $\sigma^k(n) \neq n$ for all positive integers n and k, whose $\sigma^k(n)$ denotes the kth iterate of the mapping σ at n.

Let $T\alpha = (T\alpha_k) = (\alpha_{\sigma^k(m)})$, we show that (see Schaefer [24])

$$V_\sigma = \left\{\alpha \in l_\infty : \lim_m t_{m,n}(\alpha) = L \text{ uniformly in } n,\ L = \sigma - \lim \alpha\right\}$$

where

$$t_{m,n}(\alpha) = \frac{\alpha_n + \alpha_{\sigma(n)} + \cdots + \alpha_{\sigma^m(n)}}{m+1},\ t_{-1,n}(\alpha) = 0.$$

Strongly $\sigma-$ convergent sequences is defined by Mursaleen replacing the Banach limits by invariant means, see for detail in [9].

Savas [12] presented the sequence space of lacunary invariant convergent in the following way.

$$V_\sigma^\vartheta = \left\{\alpha = (\alpha_k) : \lim_q \frac{1}{v_q} \sum_{k \in I_q} (\alpha_{\sigma^k(m)}) - l) = 0, \text{for some } l,\ \text{uniformly in } m\right\}.$$

If $\sigma(m) = m+1$, then we obtain AC_ϑ which is defined by Das and Mishra [5] as follows:

$$AC_\vartheta = \left\{\alpha = (\alpha_k) : \lim_q \frac{1}{v_q} \sum_{k \in I_q} (\alpha_{k+m} - L) = 0,\ \text{for some } l \text{ uniformly in } m\right\}.$$

2. Main results

We now start by formulating the following definition. In fact this space is more general than the space defined earlier by Savas [14]. In the whole article, we suppose that j and ρ are modulus and ρ functions, $p = (p_n)$ is a sequence of positive real numbers. Further, we suppose that $U = (u_{nk})(n, k = 1, 2, ...)$ is a real matrix, $\vartheta = (k_q)$ is a lacunary sequence and $0 < \gamma \leq 1$. We now write,

$$L_\vartheta^\gamma(U, \rho, j, \sigma, p) = \left\{\alpha = (\alpha_k) : \lim_q \frac{1}{v_q^\gamma} \sum_{n \in I_q} j\Big(\left|\sum_{k=1}^\infty u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} = 0,\ \text{uniformly in } m\right\},$$

where $\vartheta = (k_q)$ is an increasing integer sequence such that $k_0 = 0$ and $v_q = k_q - k_{q-1} \to \infty$, as $r \to \infty$. Let $I_q = (k_{q-1}, k_q]$,(see, [12]) and also ρ-function is a continuous non-decreasing function $\rho(t)$ such that $\rho(0) = 0, \rho(t) > 0$, for $t > 0$ and $\rho(t) \to \infty$ as $t \to \infty$ (see, [25]).

When $\alpha \in L_\vartheta^\gamma(U, \rho, j, \sigma, p)$, we say that the sequence α is lacunary strong (U, ρ, σ)- convergent of order γ to l with respect to a modulus j.

Later on generalizations of lacunary sequence are considered in many articles by various authors(see, [7, 19, 20, 22]).

Some well-known spaces are also obtained by specializing $\sigma(m)$ and γ.

If $p_k = p$, for all k, we have

$$L_\vartheta^\gamma(U, \rho, j, \sigma)_p = \left\{\alpha = (\alpha_k) : \lim_q \frac{1}{v_q^\gamma} \sum_{n \in I_q} j\Big(\left|\sum_{k=1}^\infty u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p} = 0,\ \text{uniformly in } m\right\}.$$

If $\sigma(m) = m + 1$, our definition will be become

$$\hat{L}_{\vartheta}^{\gamma}(U, \rho, j, p) = \left\{ \alpha = (\alpha_k) : \lim_q \frac{1}{v_q^{\gamma}} \sum_{n \in I_q} j\Big(\left| \sum_{k=1}^{\infty} u_{nk} \rho(|x_{k+m} - l|) \right| \Big)^{p_n} = 0, \text{ uniformly in } m \right\},$$

(see, [13]). When $\gamma = 1$, we get

$$L_{\vartheta}(U, \rho, j, \sigma, p) = \left\{ \alpha = (\alpha_k) : \lim_q \frac{1}{v_q} \sum_{n \in I_q} j\Big(\left| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \right| \Big)^{p_n} = 0, \text{ uniformly in } m \right\}$$

(see, [14]).

If $\sigma(m) = m + 1$ and $\gamma = 1$, our main sequence space reduces to

$$\hat{L}_{\vartheta}(U, \rho, j, \sigma, p) = \left\{ \alpha = (\alpha_k) : \lim_q \frac{1}{v_q^{\gamma}} \sum_{n \in I_q} j\Big(\left| \sum_{k=1}^{\infty} u_{nk} \rho(|x_{k+m} - l|) \right| \Big)^{p_n} = 0, \text{ uniformly in } m \right\}$$

(see, [15]).

Theorem 2.1. *$L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p)$ is a linear space over the complex field $\mathbb{C}$.*

Theorem 2.2. *Suppose ρ-function $\rho(t)$ satisfy the condition (Δ_2) and $p = (p_n)$. Then $N_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p)$ is a paranormed space with the paranorm defined by*

$$g(\alpha) = \sup \left(\frac{1}{v_q} \sum_{n \in I_q} j \left(\left| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)}|) \right| \right)^{p_n} \right)^{\frac{1}{M}}$$

where $M = \max\left(1, \sup_n p_n\right)$

The proofs of the above theorems are routine verification by using standard techniques and hence are omitted.

In the following we include our another important result.

Theorem 2.3. *Suppose $\vartheta = (k_q)$ and $\vartheta' = (s_q)$ are two lacunary sequences such that $I_q \subset J_q$ for all $q \in \mathbb{N}$ and let γ and β be defined by $0 < \gamma \le \beta \le 1$,*

If

$$\liminf_{q \to \infty} \frac{v_q^{\gamma}}{\ell_q^{\beta}} > 0, \tag{1}$$

then

$$L_{\vartheta}^{\beta}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta'}^{\gamma}(U, \rho, j, \sigma, p).$$

If

$$\lim_{q \to \infty} \frac{\ell_q}{v_q^{\beta}} = 1 \tag{2}$$

and j is bounded, then $L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p) \subset L_{\vartheta'}^{\beta}(U, \rho, j, \sigma, p)$.

Proof. Suppose that $I_q \subset J_q$ for all $q \in \mathbb{N}$ and let (1) be satisfied. Assume $\alpha = (\alpha_k) \in L_{\vartheta}^{\gamma}(U, \rho, j, \sigma), p$. Then given γ and β such that $< \gamma \leq \beta \leq 1$ and a positive real number p, we consider

$$\frac{1}{l_q^{\beta}} \sum_{n \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p \leq \frac{v_q^{\gamma}}{l_q^{\beta}} \frac{1}{v_q^{\gamma}} \sum_{n \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p$$

for all $q \in \mathbb{N}$, where $I_q = \left(k_{q-1}, k_q\right], J_q = \left(s_{q-1}, s_q\right], v_q = k_q - k_{q-1}$ and $\ell_q = s_q - s_{q-1}$. and we get that $L_{\vartheta}^{\beta}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta'}^{\gamma}(U, \rho, j, \sigma, p)$.
(ii) Let $\alpha = (\alpha_k) \in L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p)$ and assume that (2) holds. Since j is bounded, then there will exist some $T > 0$ such that $j\Big(\big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\big|\Big) \leq T$ for all k. Now, since $I_q \subset J_q$ for all $q \in \mathbb{N}$, we write

$$\begin{aligned}
\frac{1}{\ell_q^{\beta}} \sum_{n \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p &\leq \frac{1}{\ell_q^{\beta}} \sum_{k \in J_q - I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p \\
&+ \frac{1}{\ell_q^{\beta}} \sum_{k \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p \\
&\leq \left(\frac{\ell_q - v_q}{\ell_q^{\beta}}\right) T^p + \frac{1}{\ell_r^{\beta}} \sum_{k \in I_q} \mathrm{j}\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p \\
&\leq \left(\frac{\ell_q - v_q^{\beta}}{v_q^{\beta}}\right) T^p + \frac{1}{v_q^{\beta}} \sum_{k \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p \\
&\leq \left(\frac{\ell_q}{v_q^{\beta}} - 1\right) T^p + \frac{1}{v_q^{\gamma}} \sum_{k \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p
\end{aligned}$$

for every $q \in \mathbb{N}$. Finally $\alpha = (\alpha_k) \in N_{\vartheta'}^{\beta}(U, \rho, j, \sigma, p)$. □

From Theorem 2.3 we get the following.

Corollary 2.4. *Let $\vartheta = \left(k_q\right)$ and $\vartheta' = \left(s_q\right)$ be two lacunary sequences such that $I_q \subset J_q$ for all $q \in \mathbb{N}$.*
If (2.1) holds then,
(i) $L_{\vartheta'}^{\gamma}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p)$, for each $0 < \gamma \leq 1$,
(ii) $L_{\vartheta'}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p)$, for each $0 < \gamma \leq 1$,
(iii) $L_{\vartheta'}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta}(U, \rho, j, \sigma, p)$,
If (2.2) holds and j is bounded, then
(i) $L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta'}^{\gamma}(U, \rho, j, \sigma, p)$, for $0 < \gamma \leq 1$,
(ii) $L_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta'}(U, \rho, j, \sigma, p)$, for each $0 < \gamma \leq 1$,
(iii) $L_{\vartheta}(U, \rho, j, \sigma, p) \subseteq L_{\vartheta'}(U, \rho, j, \sigma, p)$.

Theorem 2.5. *Let $0 < \gamma < \varrho \leq 1$ and p be a positive real number, then $L_{\vartheta}^{\gamma}(U, \rho, j, \sigma)_p \subseteq L_{\vartheta}^{\varrho}(U, \rho, j, \sigma)_p$, and the inclusion is strict.*

Proof. Let $\alpha = (\alpha_k) \in L_{\vartheta}^{\gamma}(U, \rho, j, \sigma)_p$. Then, given γ and ϱ such that $0 < \gamma < \varrho \leq 1$ and a positive real number p, we may write

$$\frac{1}{v_q^{\varrho}} \sum_{n \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p \leq \frac{1}{v_q^{\gamma}} \sum_{n \in I_q} j\Big(\Big|\sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|)\Big|\Big)^p$$

and this gives us that $L^{\gamma}_{\vartheta}(U,\rho,j,\sigma)_p \subseteq L^{\varrho}_{\vartheta}(U,\rho,j,\sigma)_p$.

If we take $\alpha = (\alpha_k)$ such that

$$\alpha_k = \begin{cases} 1, & \text{if } k \text{ is square} \\ 0, & \text{otherwise.} \end{cases}$$

Then, it can be easily verified that the inclusion is strict . □

Now we can get the following interesting corollary.

Corollary 2.6. *Let* $0 < \gamma < \varrho \leq 1$ *and* p *be a positive real number. Then,*
(i) If $\gamma = \varrho$, *then* $L^{\gamma}_{\vartheta}(U,\rho,j,\sigma)_p = L^{\varrho}_{\vartheta}(U,\rho,j,\sigma)_p$,
(ii) $L^{\gamma}_{\vartheta}(U,\rho,j,\sigma)_p \subseteq L_{\vartheta}(U,\rho,j,\sigma)_p$ *for each* $\gamma \in (0,1]$ *and* $0 < p < \infty$.

The following is also a main result of us.

Theorem 2.7. *If*

$$w^{\gamma}(U,\rho,j,\sigma,p) = \left\{\alpha = (\alpha_k) : \lim_t \frac{1}{t^{\gamma}} \sum_{n=1}^{t} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} = 0, \text{ uniformly in } m\right\}.$$

then the following relations will be true:
(a) If $\liminf_q \tau_q > 1$ *then we write* $w^{\gamma}(U,\rho,j,\sigma,p) \subseteq L^{\gamma}_{\vartheta}(U,\rho,j,\sigma,p)$,
(b) If $\sup_q \tau_q < \infty$, *then we write* $L^{\gamma}_{\vartheta}(U,\rho,j,\sigma,p) \subseteq w^{\gamma}(U,\rho,j,\sigma,p)$.

Proof. (a) Suppose that $\alpha \in w^{\gamma}(U,\rho,j,\sigma,p)$. There exists $\varsigma > 0$ such that $\tau_q > 1+\varsigma$ for all $q \geq 1$ and we have $v^{\gamma}_q/k^{\gamma}_q \geq \varsigma^{\gamma}/(1+\varsigma)^{\gamma}$ for sufficiently large q. Then, for all m,

$$\begin{aligned} \frac{1}{k^{\gamma}_q}\sum_{n=1}^{k_q} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} &\geq \frac{1}{k^{\gamma}_q}\sum_{n\in I_q} j\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} \\ &= \frac{v^{\gamma}_q}{k^{\gamma}_q}\frac{1}{v^{\gamma}_q}\sum_{n\in I_q} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} \\ &\geq \frac{\varsigma^{\gamma}}{(1+\varsigma)^{\gamma}}\frac{1}{v^{\gamma}_q}\sum_{n\in I_q} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(||\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n}. \end{aligned}$$

Thus, $\alpha \in L^{\gamma}_{\vartheta}(U,\rho,j,\sigma,p)$.
(b) If $\limsup_q \tau_q < \infty$ then there will exist $T > 0$ which is $\tau_q < T$ for all $q \geq 1$. Let $\alpha \in L^{\gamma}_{\vartheta}(U,\rho,j,\sigma,p)$ and ε is an arbitrary positive number, then there will exist an index z_0 such that ,

$$K_z = \frac{1}{\nu^{\gamma}_z}\sum_{n\in I_q} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} < \varepsilon,$$

for every $z \geq z_0$ and all m.
Thus, we may also find $Q > 0$ such that $K_z \leq Q$ for all $z = 1,2,\ldots$. Let ν be any integer with $k_{q-1} \leq \nu \leq k_q$, then we consider, for all m

$$I = \frac{1}{\nu^{\gamma}}\sum_{n=1}^{\nu} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p} \leq \frac{1}{k^{\gamma}_{q-1}}\sum_{n=1}^{k_q} j\Big(\left|\sum_{k=1}^{\infty} u_{nk}\rho(|\alpha_{\sigma^k(m)} - l|)\right|\Big)^{p_n} = I_1 + I_2$$

where

$$I_1 = \frac{1}{k_{q-1}^{\gamma}} \sum_{z=1}^{z_0} \sum_{n \in I_s} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \Big| \Big)^{p_n}.$$

$$I_2 = \frac{1}{k_{q-1}^{\gamma}} \sum_{z=z_{0+1}}^{\nu} \sum_{n \in I_z} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \Big| \Big)^{p_n}.$$

It is obvious that,

$$\begin{aligned}
I_1 &= \frac{1}{k_{q-1}^{\gamma}} \sum_{z=1}^{z_0} \sum_{n \in I_z} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \Big| \Big)^{p_n} \\
&= \frac{1}{k_{q-1}^{\gamma}} \Big(\sum_{n \in I_1} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \Big| \Big)^{p_n} + \ldots + \sum_{n \in I_{z_0}} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)}|) \Big| \Big)^{p_n} \\
&\leq \frac{1}{k_{q-1}^{\gamma}} (v_1 K_1 + \ldots + v_{z_0} K_{z_0}), \\
&\leq \frac{1}{k_{q-1}^{\gamma}} z_0 k_{z_0}^{\gamma} sup_{1 \leq i \leq z_0} K_i, \\
&\leq \frac{z_0 k_{z_0}^{\gamma}}{k_{q-1}^{\gamma}} Q.
\end{aligned}$$

Moreover, for all m

$$\begin{aligned}
I_2 &= \frac{1}{k_{q-1}^{\gamma}} \sum_{z=z_0+1}^{\nu} \sum_{n \in I_z} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \Big| \Big)^{p_n} \\
&= \frac{1}{k_{q-1}^{\gamma}} \sum_{z=z_o+1}^{m} \Big(\frac{1}{\nu_z} \sum_{n \in I_z} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{\sigma^k(m)} - l|) \Big| \Big)^{p_n} h_s \\
&\leq \varepsilon \frac{1}{k_{q-1}^{\gamma}} \sum_{z=z_0+1}^{m} \nu_z, \\
&\leq \varepsilon \frac{k_q^{\gamma}}{k_{q-1}^{\gamma}}, \\
&= \varepsilon \tau_q^{\gamma} < \varepsilon . T.
\end{aligned}$$

Obviously $I \leq \frac{z_o k_{z_o}^{\gamma}}{k_{q-1}^{\gamma}} K + \varepsilon . T$. Hence, $x \in w^{\gamma}(U, \rho, j, \sigma, p)$. □

If we take $\sigma(m) = m + 1$, we can get the following corollary (see, [15]) is a consequence of the previous theorem.

Corollary 2.8. *If*

$$\hat{w}^{\gamma}(U, \rho, j, \sigma, p) = \left\{ \alpha = (\alpha_k) : \lim_t \frac{1}{t^{\gamma}} \sum_{n=1}^{t} j\Big(\Big| \sum_{k=1}^{\infty} u_{nk} \rho(|\alpha_{k+m} - l|) \Big| \Big)^{p_n} = 0, \ uniformly\ in\ m \right\}.$$

then the following relations will be true:
(a) If $\liminf_q \tau_q > 1$ *then we have* $\hat{w}^{\gamma}(U, \rho, j, \sigma, p) \subseteq \hat{L}_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p)$,
(b) If $\sup_q \tau_q < \infty$, *then we have* $\hat{L}_{\vartheta}^{\gamma}(U, \rho, j, \sigma, p) \subseteq \hat{w}^{\gamma}(U, \rho, j, \sigma, p)$.

3. Conclusion

In the paper, we have introduced a considerable space structure called lacunary strong (U, ρ, f) invariant space of order γ via infinite matrix . Also, some relations among these newly defined sequence spaces were established. These results unify and generalize the existing results. As a scope, we believe that this paper may attract the future researcher in this direction.

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