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A study of bi-slant lightlike submanifolds of PNsR-manifolds

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Abstract. The aim of this paper is to define new class of slant submanifolds of poly-Norden semi-Riemannian manifolds (*PNsR*-manifolds) includes slant, semi-slant and *CR*-lightlike submanifolds as its subcases which called bi-slant submanifolds. We present necessary and sufficient conditions for the distributions included in the definition of such lightlike submanifolds to be integrable. Also, we obtain some results with non-trivial examples of such submanifolds.

1. Introduction

Because of degeneracy of the induced metric, lightlike submanifolds differs noticable from their non-degenerate counterparts. Such differences results from the fact that tangent and normal bundle have a non-trivial intersection. This theory is developed by K.L. Duggal and A. Bejancu [13] (see also [14]). Then the geometry of lightlike submanifolds have been extensively investigated ([3, 4, 15]).

As a generalization of totally real submanifolds and complex submanifolds slant submanifolds of almost Hermitian manifolds introduced by B.Y. Chen [7]. Then this theory was extended different structure. Semi-slant submanifolds in almost Hermitian manifolds were introduced by N. Papagiuc [17]. Semi-slant submanifolds in Sasakian manifolds were studied by J.L. Cabrerizo [12]. Recently, bi-slant lightlike submanifolds were examined in [16] (see also [5, 18]).

By use of generalization of golden mean, V.W. Spinadel introduced metallic structure [25]. Let ρ_1 and ρ_2 be positive integers. Thus, members of the metallic means family are positive solution

$$x^2 - \rho_1 x - \rho_2 = 0,$$

and this number, which are known $(\rho_1, \rho_2)-$metallic numbers denoted by [9]

$$\sigma_{\rho_1,\rho_2} = \frac{\rho_1 + \sqrt{\rho_1^2 + 4\rho_2}}{2}.$$

A metallic manifold has a tensor field $\tilde{J}$ such that the equality $\tilde{J}^2 = \rho_1\tilde{J} + \rho_2 I$ is satisfied, where the eigenvalues of automorphism $\tilde{J}$ of the tangent bundle are σ_{ρ_1,ρ_2} and $\rho_1 - \sigma_{\rho_1,\rho_2}$ [9]. Metallic structure on the ambient manifold provides geometrical results on the submanifolds, since it is an impotant tool while examining of submanifolds (for more details [1, 2, 8, 11, 20]).

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Also, in [19] unlike the bronze mean given in [24], a new bronze mean have been studied. A new bronze mean given in [19] can not be expressed with σ_{ρ_1,ρ_2}. Recently, a new type of manifold which is called almost poly-Norden manifold has been examined in [6]. Then different types of submanifolds of poly-Norden (semi)-Riemannian manifolds have been published ([10, 21, 22]).

In this article, we studied the term of bi-slant lightlike submanifolds of *PNsR*-manifolds which includes *slant*, *semi-slant* and *CR-lightlike* submanifolds as its subcases.

2. Preliminaries

The positive solution of $x^2 - \omega x + 1 = 0$, is called bronze mean [19], which is given by

$$\rho_\omega = \frac{\omega + \sqrt{\omega^2 - 4}}{2}. \tag{1}$$

In [6], bu use of (1), B. Şahin defined a new type of manifold equipped with the bronze structure. A differentiable manifold $\tilde{G}$, with a $(1,1)-$tensor field $\tilde{\Phi}$ and semi-Riemannian metric $\tilde{g}$ satisfying

$$\tilde{\Phi}^2 = \omega\tilde{\Phi} - Id, \tag{2}$$

$$\tilde{g}(\tilde{\Phi}\partial_1, \tilde{\Phi}\partial_2) = \omega\tilde{g}(\tilde{\Phi}\partial_1, \partial_2) - \tilde{g}(\partial_1, \partial_2), \tag{3}$$

then $\tilde{\Phi}$ is called an almost *PNsR*-manifold. Using (3), we arrive at

$$\tilde{g}(\tilde{\Phi}\partial_1, \partial_2) = \tilde{g}(\partial_1, \tilde{\Phi}\partial_2),$$

for all $\partial_1, \partial_2 \in \Gamma(T\tilde{G})$. Throught the paper, we will suppose that ω different from zero (for more details [23]).

Definition 2.1. *[6] Let $(\tilde{G}, \tilde{g})$ be a semi-Riemannian manifold endowed with a poly-Norden structure $\tilde{\Phi}$. If the almost poly-Norden structure $\tilde{\Phi}$ is parallel with respect to the Levi-Civita connection $\tilde{\sharp}$, i.e.,*

$$\tilde{\sharp}\tilde{\Phi} = 0, \tag{4}$$

then $(\tilde{G}, \tilde{\Phi}, \tilde{g})$ is called a PNsR-manifold.

Example 2.2. *[6] Consider the 4-tuples real space $\mathbb{R}^4$ and define a map by*

$$\begin{array}{rcl} \tilde{\Phi} & : & \mathbb{R}^4 \to \mathbb{R}^4 \\ (u_1, u_2, u_3, u_4) & \to & (\rho_\omega u_1, \rho_\omega u_2, \bar{\rho}_\omega u_3, \bar{\rho}_\omega u_4), \end{array}$$

where $\rho_\omega = \frac{\omega + \sqrt{\omega^2-4}}{2}$ and $\bar{\rho}_\omega = \frac{\omega - \sqrt{\omega^2-4}}{2}$. Thus $(\mathbb{R}^4, \tilde{\Phi})$ is an example of almost poly-Norden manifold.

A submanifold (G^m, g) immersed in a semi-Riemannian manifold $(\tilde{G}^{m+n}, \tilde{g})$ is called a lightlike submanifold [13], if the metric g induced from $\tilde{g}$ is degenerate and the radical distribution $RadTG$ is of rank r, $1 \leq r \leq m$. Assume that $S(TG)$ is a screen distribution which is a semi-Riemannian complementary distribution of $RadTG$, that is,

$$TG = S(TG) \perp RadTG. \tag{5}$$

Considering a screen transversal vector bundle $S(TG^\perp)$, which is a semi-Riemannian complementary vector bundle of $RadTG$ in $TG^\perp$. For every local basis $\{\zeta_i\}$ of $RadTG$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TG^\perp)$ in $(S(TG^\perp))^\perp$ such that

$$\tilde{g}(N_i, \zeta_i) = \delta_{ij} \quad \text{and} \quad \tilde{g}(N_i, N_j) = 0,$$

it follows that there exists a lightlike transversal vector bundle $ltr(TG)$ locally spanned by $\{N_i\}$ [13].

Assume that $tr(TG)$ is a complementary (but not orthogonal) vector bundle to TG in $T\tilde{G}|_G$. Then, we get

$$tr(TG) = ltr(TG) \perp S(TG^{\perp}), \tag{6}$$

$$T\tilde{G}|_G = TG \oplus tr(TG), \tag{7}$$

which gives

$$T\tilde{G} = S(TG) \perp \{RadTG \oplus ltr(TG)\} \perp S(TG^{\perp}). \tag{8}$$

The Gauss and Weingarten formulae are given as

$$\tilde{\sharp}_{\partial_1}\partial_2 = \sharp_{\partial_1}\partial_2 + h(\partial_1, \partial_2), \tag{9}$$

$$\tilde{\sharp}_{\partial_1}N = -A_N\partial_1 + \sharp^t_{\partial_1}N, \tag{10}$$

for all $\partial_1, \partial_2 \in \Gamma(TG)$ and $N \in \Gamma(ltr(TG))$. $\tilde{\sharp}$, $\sharp$ and $\sharp^t$ are linear connections on $T\tilde{G}$, TG and $tr(TG)$, respectively.
In view of (9) and (10), for all $\partial_1, \partial_2 \in \Gamma(TG)$ and $N \in \Gamma(ltr(TG))$ and $W \in \Gamma(S(TG^{\perp}))$, we get

$$\tilde{\sharp}_{\partial_1}\partial_2 = \sharp_{\partial_1}\partial_2 + h^l(\partial_1, \partial_2) + h^s(\partial_1, \partial_2), \tag{11}$$

$$\tilde{\sharp}_{\partial_1}N = -A_N\partial_1 + \sharp^l_{\partial_1}N + D^s(\partial_1, N), \tag{12}$$

$$\tilde{\sharp}_{\partial_1}W = -A_W\partial_1 + \nabla^s_{\partial_1}W + D^l(\partial_1, W). \tag{13}$$

Then, by use of (9), (11)~(13) and metric connection $\tilde{\sharp}$, we arrive at

$$\tilde{g}(h^s(\partial_1, \partial_2), W) + \tilde{g}(\partial_2, D^l(\partial_1, W)) = \tilde{g}(A_W\partial_1, \partial_2), \tag{14}$$

$$\tilde{g}(D^s(\partial_1, N), W) = \tilde{g}(N, A_W\partial_1). \tag{15}$$

Denote the projection of TG on $S(TG)$ by $\check{P}$. For any $\partial_1, \partial_2 \in \Gamma(TG)$ and $\zeta \in \Gamma(RadTG)$, we get

$$\sharp_{\partial_1}\check{P}\partial_2 = \sharp^*_{\partial_1}\check{P}\partial_2 + h^*(\partial_1, \check{P}\partial_2), \tag{16}$$

$$\sharp_{\partial_1}\zeta = -A^*_{\zeta}\partial_1 + \sharp^{*t}_{\partial_1}\zeta. \tag{17}$$

From above equations, we have

$$\tilde{g}(h^l(\partial_1, \check{P}\partial_2), \zeta) = \tilde{g}(A^*_E\partial_1, \check{P}\partial_2), \tag{18}$$

$$\tilde{g}(h^*(\partial_1, \check{P}\partial_2), N) = \tilde{g}(A_N\partial_1, \check{P}\partial_2), \tag{19}$$

$$\tilde{g}(h^l(\partial_1, \zeta), \zeta) = 0, \quad A^*_{\zeta}\zeta = 0. \tag{20}$$

In general, $\sharp$ is not metric connection and we have

$$(\sharp_{\partial_1}\tilde{g})(\partial_2, \partial_3) = \tilde{g}(h^l(\partial_1, \partial_2), \partial_3) + \tilde{g}(h^l(\partial_1, \partial_3), \partial_2). \tag{21}$$

3. BI-SLANT LIGHTLIKE SUBMANIFOLDS OF *PNsR*-MANIFOLDS

Definition 3.1. *Let G be a lightlike submanifold of a PNsR-manifold $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then we say that G is a bi-slant lightlike submanifold if the following conditions are satisfied:*

i) $\tilde{\Phi}RadTG$ is a distribution such that $RadTG \cap \tilde{\Phi}RadTG = \{0\}$,

ii) There exists non-degenerate orthogonal distributions γ, $\mathring{\gamma}$ and $\hat{\gamma}$ on G such that

$$S(TG) = \{\tilde{\Phi}RadTG \oplus \tilde{\Phi}ltr(TG)\} \perp \gamma \perp \mathring{\gamma} \perp \hat{\gamma},$$

iii) The distribution γ is invariant, $\tilde{\Phi}\gamma = \gamma$,

iv) The distribution $\mathring{\gamma}$ is slant with angle $\theta_1 (\neq 0)$ i.e., for each $x \in G$ and non-zero vector $X \in (\mathring{\gamma})_x$, the angle θ_1 between $\tilde{\Phi}X$ and the vector space $(\mathring{\gamma})_x$ is non-zero constant, which is independent of the choice of $x \in G$ and $X \in (\mathring{\gamma})_x$.

v) The distribution $\hat{\gamma}$ is slant with angle $\theta_2 (\neq 0)$ i.e., for each $x \in G$ and non-zero vector $X \in (\hat{\gamma})_x$, the angle θ_2 between $\tilde{\Phi}X$ and the vector space $(\hat{\gamma})_x$ is non-zero constant, which is independent of the choice of $x \in G$ and $X \in (\hat{\gamma})_x$.

A bi-slant lightlike submanifold is said to be proper if $\mathring{\gamma} \neq \{0\}$, $\hat{\gamma} \neq \{0\}$ and $\theta_1 \neq \frac{\pi}{2}$, $\theta_2 \neq \frac{\pi}{2}$.

In view of above definition, we have

$$TG = RadTG \perp \{\tilde{\Phi}RadTG \oplus \tilde{\Phi}ltr(TG)\} \perp \gamma \perp \mathring{\gamma} \perp \hat{\gamma}. \tag{22}$$

Also,

if $\gamma = 0$ and one of $\mathring{\gamma}$ and $\hat{\gamma}$ is zero, then G is a *slant lightlike submanifold*,

if $\gamma \neq 0$ and one of $\mathring{\gamma}$ and $\hat{\gamma}$ is zero, then G is a *semi-slant lightlike submanifold*,

if $\gamma \neq 0$ and $\theta_1 = \frac{\pi}{2} = \theta_2$, then G is a *CR-lightlike submanifold*.

As mentioned in the abstract, bi-slant lightlike submanifold of a *PNsR*-manifold containes *slant*, *semi-slant* and *CR-lightlike* submanifolds. This makes the study of bi-slant submanifolds extremely difficult and interesting.

Example 3.2. *Let $(\mathbb{R}^{16}_4, \tilde{g})$ be a semi-Riemannian manifold with signature $(-,-,+,...,+,-,-,+,...,+)$ and $(u_1, u_2, ..., u_{16})$ be standard coordinate system of $\mathbb{R}^{16}_4$.*

Taking

$$\tilde{\Phi}(u_1, ..., u_{16}) = \begin{pmatrix} \rho_\omega u_1, \bar{\rho}_\omega u_2, \rho_\omega u_3, \bar{\rho}_\omega u_4, \bar{\rho}_\omega u_5, \bar{\rho}_\omega u_6, \rho_\omega u_7, \rho_\omega u_8, \\ \rho_\omega u_9, \bar{\rho}_\omega u_{10}, \bar{\rho}_\omega u_{11}, \rho_\omega u_{12}, \bar{\rho}_\omega u_{13}, \rho_\omega u_{14}, \rho_\omega u_{15}, \bar{\rho}_\omega u_{16} \end{pmatrix}$$

where $\rho_\omega = \frac{\omega + \sqrt{\omega^2 - 4}}{2}$ and $\bar{\rho}_\omega = \frac{\omega - \sqrt{\omega^2 - 4}}{2}$. Then $\tilde{\Phi}$ is a poly-Norden structure on $\mathbb{R}^{16}_4$.

Assume that G is a submanifold of $\mathbb{R}^{16}_4$ given by

$$u_1 = \bar{\rho}_\omega x_1 - x_2 + x_3, \quad u_2 = x_1 + \bar{\rho}_\omega x_2 - \bar{\rho}_\omega x_3,$$

$$u_3 = -\bar{\rho}_\omega x_1 + x_2 + x_3, \quad u_4 = x_1 + \bar{\rho}_\omega x_2 + \bar{\rho}_\omega x_3,$$

$$u_5 = u_6 = \rho_\omega x_4, \quad u_7 = u_8 = \bar{\rho}_\omega x_5,$$

$$u_9 = u_{11} = x_6, \quad u_{10} = u_{12} = x_7, \quad u_{13} = \bar{\rho}_\omega x_8,$$

$$u_{14} = \rho_\omega x_9, \quad u_{15} = \rho_\omega x_8, \quad u_{16} = \bar{\rho}_\omega x_9.$$

Then $TG = Sp\{\Psi_1, ..., \Psi_9\}$, where

$$\Psi_1 = \bar{\rho}_\omega \partial x_1 + \partial x_2 - \bar{\rho}_\omega \partial x_3 + \partial x_4,$$

$$\Psi_2 = -\partial x_1 + \bar{\rho}_\omega \partial x_2 + \partial x_3 + \bar{\rho}_\omega \partial x_4,$$

$$\Psi_3 = \partial x_1 - \bar{\rho}_\omega \partial x_2 + \partial x_3 + \bar{\rho}_\omega \partial x_4,$$

$$\Psi_4 = \rho_\omega \partial x_5 + \rho_\omega \partial x_6, \quad \Psi_5 = \rho_\omega \partial x_7 + \rho_\omega \partial x_8,$$

$$\Psi_6 = \partial x_9 + \partial x_{11}, \quad \Psi_7 = \partial x_{10} + \partial x_{12},$$
$$\Psi_8 = \bar{\rho}_\omega \partial x_{13} + \rho_\omega \partial x_{15}, \quad \Psi_9 = \rho_\omega \partial x_{14} + \bar{\rho}_\omega \partial x_{16}.$$

Thus, $RadTG = Sp\{\Psi_1\}$ and $S(TG) = Sp\{\Psi_2, ..., \Psi_9\}$ and $ltr(TG)$ is spanned by

$$N = \frac{1}{2(1+\bar{\rho}_\omega^2)}\left(-\bar{\rho}_\omega \partial x_1 - \partial x_2 - \bar{\rho}_\omega \partial x_3 + \partial x_4\right).$$

and $S(TG^\perp)$ is spanned by

$$W_1 = \bar{\rho}_\omega \partial x_5 - \bar{\rho}_\omega \partial x_6, \quad W_2 = \partial x_7 - \partial x_8,$$
$$W_3 = \partial x_9 - \partial x_{11}, \quad W_2 = \partial x_{10} - \partial x_{12},$$
$$W_5 = \rho_\omega \partial x_{13} - \bar{\rho}_\omega \partial x_{15}, \quad W_6 = \bar{\rho}_\omega \partial x_{14} - \rho_\omega \partial x_{16}.$$

It follows that $\tilde{\Phi}\Psi_1 = \Psi_2$, $\tilde{\Phi}N = \Psi_3$, $\tilde{\Phi}\Psi_4 = -\Psi_4$, $\tilde{\Phi}\Psi_5 = (\rho_\omega)^2\Psi_5$ which gives that γ is invariant, i.e., $\tilde{\Phi}\gamma = \gamma$ and $\gamma = Sp\{\Psi_4, \Psi_5\}$ and $\mathring{\gamma} = Sp\{\Psi_6, \Psi_7\}$, $\hat{\gamma} = Sp\{\Psi_8, \Psi_9\}$ are slant distributions. Therefore G is a bi-slant lightlike submanifold of $\mathbb{R}^{16}_4$.

For any vector field $\partial_1 \in \Gamma(TG)$, we take

$$\tilde{\Phi}\partial_1 = t\partial_1 + n\partial_1, \tag{23}$$

where $t\partial_1$ and $n\partial_1$ are the tangential and the transversal part of $\tilde{\Phi}\partial_1$, respectively. We show the projections on $RadTG$, $\tilde{\Phi}(RadTG)$, $\tilde{\Phi}(ltr(TG))$, γ, $\mathring{\gamma}$ and $\hat{\gamma}$ in TG by R_1, R_2, R_3,R_4, $\mathring{R}_5$ and $\hat{R}_6$ and respectively. Similarly, we show that the projections of $tr(TG)$ on $\tilde{\Phi}(ltr(TG))$ and $S(TG^\perp)$ by Q_1 and Q_2, respectively. Then, we get

$$\partial_1 = R_1\partial_1 + R_2\partial_1 + R_3\partial_1 + R_4\,\partial_1 + \mathring{R}_5\partial_1 + \hat{R}_6\partial_1. \tag{24}$$

Now, applying ψ to (24), we obtain

$$\begin{aligned}\tilde{\Phi}\partial_1 &= \tilde{\Phi}R_1\partial_1 + \tilde{\Phi}R_2\partial_1 + \tilde{\Phi}R_3\partial_1 + \tilde{\Phi}R_4\,\partial_1 \\ &\quad + \tilde{\Phi}\mathring{R}_5\partial_1 + \tilde{\Phi}\hat{R}_6\partial_1,\end{aligned} \tag{25}$$

which yields

$$\begin{aligned}\tilde{\Phi}\partial_1 &= \tilde{\Phi}R_1\partial_1 + \tilde{\Phi}R_2\partial_1 + \tilde{\Phi}R_3\partial_1 + \tilde{\Phi}R_4\,\partial_1 \\ &\quad + t\mathring{R}_5\partial_1 + n\mathring{R}_5\partial_1 + t\hat{R}_6\partial_1 + n\hat{R}_6\partial_1,\end{aligned} \tag{26}$$

where $t\mathring{R}_5\partial_1$ and $t\hat{R}_6\partial_1$ denotes the tangential component of $\tilde{\Phi}\mathring{R}_5\partial_1$and $\tilde{\Phi}\hat{R}_6\partial_1$, $n\mathring{R}_5\partial_1$ and $n\hat{R}_6\partial_1$ denotes the transversal component of $\tilde{\Phi}\mathring{R}_5\partial_1$ and $\tilde{\Phi}\hat{R}_6\partial_1$.

Also, for any $W \in \Gamma(tr(TG))$, we have

$$W = Q_1W + Q_2W. \tag{27}$$

Applying ψ to (27), we have

$$\tilde{\Phi}W = \tilde{\Phi}Q_1W + \tilde{\Phi}Q_2W,$$

which yields

$$\tilde{\Phi}W = \tilde{\Phi}Q_1W + b\mathring{Q}_2W + c\mathring{Q}_2W + b\hat{Q}_2W + c\hat{Q}_2W, \tag{28}$$

where $b\mathring{Q}_2W$ and $b\hat{Q}_2W$ denotes the tangential component of $\tilde{\Phi}Q_2W$, $c\mathring{Q}_2W$ and $c\hat{Q}_2W$ denotes the transversal component of $\tilde{\Phi}Q_2W$. Thus we arrive at

$$\begin{aligned}&\tilde{\Phi}Q_1W \in \Gamma(ltr(TG)), \quad && b\mathring{Q}_2W \in \Gamma(\mathring{\gamma}), \\ &b\hat{Q}_2W \in \Gamma(\hat{\gamma}), && c\mathring{Q}_2W, c\hat{Q}_2W \in \Gamma(S(TG^\perp)).\end{aligned}$$

4. MAIN RESULTS

Now, we give the main results of our article:

Theorem 4.1. *Let G be a bi-slant submanifold of a PNsR-manifold* $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. *Then RadTG is integrable if and only if*

i) $\tilde{g}(h^l(\zeta_1, \tilde{\Phi}\zeta_2), \zeta_3) = \tilde{g}(h^l(\zeta_2, \tilde{\Phi}\zeta_1), \zeta_3)$,

ii) $\tilde{g}(h^*(\zeta_1, \tilde{\Phi}\zeta_2), N) = \tilde{g}(h^*(\zeta_2, \tilde{\Phi}\zeta_1), N)$,

iii) $\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 - \sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1, \tilde{\Phi}\partial_1) = \omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 - \sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1, \partial_1)$,

iv) $\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 - \sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1, t\partial_2) + \tilde{g}(h^s(\zeta_1, \tilde{\Phi}\zeta_2) - h^s(\zeta_2, \tilde{\Phi}\zeta_1), n\partial_2) = \omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 - \sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1, \partial_1)$,

v) $\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 - \sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1, t\partial_3) + \tilde{g}(h^s(\zeta_1, \tilde{\Phi}\zeta_2) - h^s(\zeta_2, \tilde{\Phi}\zeta_1), n\partial_3) = \omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 - \sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1, \partial_3)$,

for all $\zeta_i \in \Gamma(RadTG)$, $(i = 1, 2, 3)$, $\partial_1 \in \Gamma(\gamma)$, $\partial_2 \in \Gamma(\mathring{\gamma})$, $\partial_3 \in \Gamma(\hat{\gamma})$ *and* $N \in \Gamma(ltr(TG))$.

Proof. We know that the the distribution *RadTG* is integrable iff

$$\tilde{g}([\zeta_1, \zeta_2], \tilde{\Phi}\zeta_3) = \tilde{g}([\zeta_1, \zeta_2], \tilde{\Phi}N) = \tilde{g}([\zeta_1, \zeta_2], \partial_1) = \tilde{g}([\zeta_1, \zeta_2], \partial_2) = \tilde{g}([\zeta_1, \zeta_2], \partial_3) = 0$$

for any $\zeta_i \in \Gamma(RadTG)$, $(i = 1, 2, 3)$, $\partial_1 \in \Gamma(\gamma)$, $\partial_2 \in \Gamma(\mathring{\gamma})$, $\partial_3 \in \Gamma(\hat{\gamma})$ and $N \in \Gamma(ltr(TG))$. Because of $\tilde{\sharp}$ is a metric connection, from (3), (11), (16) with (23), we have

$$\begin{aligned}
\tilde{g}([\zeta_1, \zeta_2], \tilde{\Phi}\zeta_3) &= \tilde{g}(\tilde{\sharp}_{\zeta_1}\zeta_2 - \tilde{\sharp}_{\zeta_2}\zeta_1, \tilde{\Phi}\zeta_3) \\
&= \tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1, \zeta_3) \\
&= \tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2 + h^l(\zeta_1, \tilde{\Phi}\zeta_2) + h^s(\zeta_1, \tilde{\Phi}\zeta_2), \zeta_3) \\
&\quad - \tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1 + h^l(\zeta_2, \tilde{\Phi}\zeta_1) + h^s(\zeta_2, \tilde{\Phi}\zeta_1), \zeta_3) \\
&= \tilde{g}(h^l(\zeta_1, \tilde{\Phi}\zeta_2), \zeta_3) - \tilde{g}(h^l(\zeta_2, \tilde{\Phi}\zeta_1), \zeta_3), \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\zeta_1, \zeta_2], \tilde{\Phi}N) &= \tilde{g}(\tilde{\sharp}_{\zeta_1}\zeta_2 - \tilde{\sharp}_{\zeta_2}\zeta_1, \tilde{\Phi}N) \\
&= \tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1, N) \\
&= \tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2 + h^l(\zeta_1, \tilde{\Phi}\zeta_2) + h^s(\zeta_1, \tilde{\Phi}\zeta_2), N) \\
&\quad - \tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1 + h^l(\zeta_2, \tilde{\Phi}\zeta_1) + h^s(\zeta_2, \tilde{\Phi}\zeta_1), N) \\
&= \tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2, N) - \tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1, N), \\
&= \tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2 + h^*(\zeta_1, \tilde{\Phi}\zeta_2), N) \\
&\quad - \tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1 + h^*(\zeta_2, \tilde{\Phi}\zeta_1), N) \\
&= \tilde{g}(h^*(\zeta_1, \tilde{\Phi}\zeta_2) - h^*(\zeta_2, \tilde{\Phi}\zeta_1), N), \qquad (30)
\end{aligned}$$

$$
\begin{aligned}
\tilde{g}([\zeta_1,\zeta_2],\partial_1) &= -\tilde{g}(\tilde{\Phi}[\zeta_1,\zeta_2],\tilde{\Phi}\partial_1)+\omega\tilde{g}(\tilde{\Phi}[\zeta_1,\zeta_2],\partial_1)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\tilde{\Phi}\partial_1)\\
&\quad +\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_1)\\
&= -\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2+h^l(\zeta_1,\tilde{\Phi}\zeta_2)+h^s(\zeta_1,\tilde{\Phi}\zeta_2),\tilde{\Phi}\partial_1)\\
&\quad +\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1+h^l(\zeta_2,\tilde{\Phi}\zeta_1)+h^s(\zeta_2,\tilde{\Phi}\zeta_1),\tilde{\Phi}\partial_1)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2+h^l(\zeta_1,\tilde{\Phi}\zeta_2)+h^s(\zeta_1,\tilde{\Phi}\zeta_2),\partial_1)\\
&\quad -\omega\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1+h^l(\zeta_2,\tilde{\Phi}\zeta_1)+h^s(\zeta_2,\tilde{\Phi}\zeta_1),\partial_1)\\
&= -\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2,\tilde{\Phi}\partial_1)+\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1,\tilde{\Phi}\partial_1)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2,\partial_1)-\omega\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_1)\\
&= -\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2+h^*(\zeta_1,\tilde{\Phi}\zeta_2),\tilde{\Phi}\partial_1)\\
&\quad +\tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1+h^*(\zeta_2,\tilde{\Phi}\zeta_1),\tilde{\Phi}\partial_1)\\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2+h^*(\zeta_1,\tilde{\Phi}\zeta_2),\partial_1)\\
&\quad -\omega\tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1+h^*(\zeta_2,\tilde{\Phi}\zeta_1),\partial_1)\\
&= \tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1-\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2,\tilde{\Phi}\partial_1)\\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_1),
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\tilde{g}([\zeta_1,\zeta_2],\partial_2) &= -\tilde{g}(\tilde{\Phi}[\zeta_1,\zeta_2],\tilde{\Phi}\partial_2)+\omega\tilde{g}(\tilde{\Phi}[\zeta_1,\zeta_2],\partial_2)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\tilde{\Phi}\partial_2)\\
&\quad +\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_2)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,t\partial_2+n\partial_2)\\
&\quad +\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_2)\\
&= -\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2+h^l(\zeta_1,\tilde{\Phi}\zeta_2)+h^s(\zeta_1,\tilde{\Phi}\zeta_2),t\partial_2+n\partial_2)\\
&\quad +\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1+h^l(\zeta_2,\tilde{\Phi}\zeta_1)+h^s(\zeta_2,\tilde{\Phi}\zeta_1),t\partial_2+n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2+h^l(\zeta_1,\tilde{\Phi}\zeta_2)+h^s(\zeta_1,\tilde{\Phi}\zeta_2),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1+h^l(\zeta_2,\tilde{\Phi}\zeta_1)+h^s(\zeta_2,\tilde{\Phi}\zeta_1),\partial_2)\\
&= -\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp_{\zeta_2}\tilde{\Phi}\zeta_1,t\partial_2)\\
&\quad -\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_2)\\
&= -\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2+h^*(\zeta_1,\tilde{\Phi}\zeta_2),t\partial_2)\\
&\quad +\tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1+h^*(\zeta_2,\tilde{\Phi}\zeta_1),t\partial_2)\\
&\quad -\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2+h^*(\zeta_1,\tilde{\Phi}\zeta_2),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1+h^*(\zeta_2,\tilde{\Phi}\zeta_1),\partial_2)\\
&= -\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1,t\partial_2)\\
&\quad -\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_2),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\tilde{g}([\zeta_1,\zeta_2],\partial_3) &= -\tilde{g}(\tilde{\Phi}[\zeta_1,\zeta_2],\tilde{\Phi}\partial_3)+\omega\tilde{g}(\tilde{\Phi}[\zeta_1,\zeta_2],\partial_3)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\tilde{\Phi}\partial_3)\\
&\quad+\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_3)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,t\partial_3+n\partial_3)\\
&\quad+\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\tilde{\Phi}\zeta_2-\tilde{\sharp}_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_3)\\
&= -\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2+h^l(\zeta_1,\tilde{\Phi}\zeta_2)+h^s(\zeta_1,\tilde{\Phi}\zeta_2),t\partial_3+n\partial_3)\\
&\quad+\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1+h^l(\zeta_2,\tilde{\Phi}\zeta_1)+h^s(\zeta_2,\tilde{\Phi}\zeta_1),t\partial_3+n\partial_3)\\
&\quad+\omega\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2+h^l(\zeta_1,\tilde{\Phi}\zeta_2)+h^s(\zeta_1,\tilde{\Phi}\zeta_2),\partial_3)\\
&\quad-\omega\tilde{g}(\sharp_{\zeta_2}\tilde{\Phi}\zeta_1+h^l(\zeta_2,\tilde{\Phi}\zeta_1)+h^s(\zeta_2,\tilde{\Phi}\zeta_1),\partial_3)\\
&= -\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp_{\zeta_2}\tilde{\Phi}\zeta_1,t\partial_3)\\
&\quad-\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_3)\\
&\quad+\omega\tilde{g}(\sharp_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_3)\\
&= -\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2+h^*(\zeta_1,\tilde{\Phi}\zeta_2),t\partial_3)\\
&\quad+\tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1+h^*(\zeta_2,\tilde{\Phi}\zeta_1),t\partial_3)\\
&\quad-\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_3)\\
&\quad+\omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2+h^*(\zeta_1,\tilde{\Phi}\zeta_2),\partial_3)\\
&\quad-\omega\tilde{g}(\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1+h^*(\zeta_2,\tilde{\Phi}\zeta_1),\partial_3)\\
&= -\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1,t\partial_3)\\
&\quad-\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_3)\\
&\quad+\omega\tilde{g}(\sharp^*_{\zeta_1}\tilde{\Phi}\zeta_2-\sharp^*_{\zeta_2}\tilde{\Phi}\zeta_1,\partial_3).
\end{aligned}
\tag{33}
$$

So, we obtain the required equations with (29)∼(33). □

Theorem 4.2. *Let G be a bi-slant submanifold of a PNsR-manifold $(\tilde{G},\tilde{\Phi},\tilde{g})$. Then $\tilde{\Phi}(RadTG)$ is integrable if and only if*

i) $\tilde{g}(h^l(\tilde{\Phi}\zeta_1,\zeta_2),\zeta_3)=\tilde{g}(h^l(\zeta_1,\tilde{\Phi}\zeta_2),\zeta_3)$,

ii) $\tilde{g}(A^*_{\zeta_1}\tilde{\Phi}\zeta_2,\tilde{\Phi}\partial_1)=\tilde{g}(A^*_{\zeta_2}\tilde{\Phi}\zeta_1,\tilde{\Phi}\partial_1)$,

iii) $\tilde{g}(A^*_{\zeta_2}\tilde{\Phi}\zeta_1-A^*_{\zeta_1}\tilde{\Phi}\zeta_2,t\partial_2)=\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_2)$,

iv) $\tilde{g}(A^*_{\zeta_2}\tilde{\Phi}\zeta_1-A^*_{\zeta_1}\tilde{\Phi}\zeta_2,t\partial_3)=\tilde{g}(h^s(\zeta_1,\tilde{\Phi}\zeta_2)-h^s(\zeta_2,\tilde{\Phi}\zeta_1),n\partial_3)$,

v) $\tilde{g}(A_N\tilde{\Phi}\zeta_1,\tilde{\Phi}\zeta_2)=\tilde{g}(A_N\tilde{\Phi}\zeta_2,\tilde{\Phi}\zeta_1)$,

for all $\zeta_i\in\Gamma(RadTG)$, $(i=1,2,3)$, $\partial_1\in\Gamma(\gamma)$, $\partial_2\in\Gamma(\mathring{\gamma})$, $\partial_3\in\Gamma(\hat{\gamma})$ and $N\in\Gamma(ltr(TG))$.

Proof. In view of the definition of bi-slant lightlike submanifold then the distribution $\tilde{\Phi}(RadTG)$ is integrable iff

$$\tilde{g}([\tilde{\Phi}\zeta_1,\tilde{\Phi}\zeta_2],\tilde{\Phi}\zeta_3)=\tilde{g}([\tilde{\Phi}\zeta_1,\tilde{\Phi}\zeta_2],\partial_1)=\tilde{g}([\tilde{\Phi}\zeta_1,\tilde{\Phi}\zeta_2],\partial_2)=\tilde{g}([\tilde{\Phi}\zeta_1,\tilde{\Phi}\zeta_2],\partial_3)=\tilde{g}([\tilde{\Phi}\zeta_1,\tilde{\Phi}\zeta_2],N)=0,$$

for any $\zeta_i\in\Gamma(RadTG)$, $(i=1,2,3)$, $\partial_1\in\Gamma(\gamma)$, $\partial_2\in\Gamma(\mathring{\gamma})$, $\partial_3\in\Gamma(\hat{\gamma})$ and $N\in\Gamma(ltr(TG))$. In view of (3), (11), (12),

(17) with (23) and $\tilde{\sharp}$ being a metric connection, we get

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_2], \tilde{\Phi}\zeta_3) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_3) \\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1), \tilde{\Phi}\zeta_3) \\
&= \omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1), \zeta_3) - \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, \zeta_3) \\
&= \omega\tilde{g}((\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1), \tilde{\Phi}\zeta_3) - \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, \zeta_3) \\
&= \omega\tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 + h^l(\tilde{\Phi}\zeta_1, \zeta_2) + h^s(\tilde{\Phi}\zeta_1, \zeta_2), \tilde{\Phi}\zeta_3) \\
&\quad -\omega\tilde{g}(\sharp_{\tilde{\Phi}\zeta_2}\zeta_1 + h^l(\tilde{\Phi}\zeta_2, \zeta_1) + h^s(\tilde{\Phi}\zeta_2, \zeta_1), \tilde{\Phi}\zeta_3) \\
&\quad -\tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 + h^l(\tilde{\Phi}\zeta_1, \zeta_2) + h^s(\tilde{\Phi}\zeta_1, \zeta_2), \zeta_3) \\
&\quad +\tilde{g}(\sharp_{\tilde{\Phi}\zeta_2}\zeta_1 + h^l(\tilde{\Phi}\zeta_2, \zeta_1) + h^s(\tilde{\Phi}\zeta_2, \zeta_1), \zeta_3) \\
&= \tilde{g}(h^l(\tilde{\Phi}\zeta_2, \zeta_1) - h^l(\tilde{\Phi}\zeta_1, \zeta_2), \zeta_3),
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_2], \partial_1) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\tilde{\Phi}\zeta_1, \partial_1) \\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1), \partial_1) \\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, \tilde{\Phi}\partial_1) \\
&= \tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 + h^l(\tilde{\Phi}\zeta_1, \zeta_2) + h^s(\tilde{\Phi}\zeta_1, \zeta_2), \tilde{\Phi}\partial_1) \\
&\quad -\tilde{g}(\sharp_{\tilde{\Phi}\zeta_2}\zeta_1 + h^l(\tilde{\Phi}\zeta_2, \zeta_1) + h^s(\tilde{\Phi}\zeta_2, \zeta_1), \tilde{\Phi}\partial_1) \\
&= \tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2, \tilde{\Phi}\partial_1) - \tilde{g}(\sharp_{\tilde{\Phi}\zeta_2}\zeta_1, \tilde{\Phi}\partial_1) \\
&= \tilde{g}(-A^*_{\zeta_2}\tilde{\Phi}\zeta_1 + \sharp^{*t}_{\tilde{\Phi}\zeta_1}\zeta_2, \tilde{\Phi}\partial_1) - \tilde{g}(-A^*_{\zeta_1}\tilde{\Phi}\zeta_2 + \sharp^{*t}_{\tilde{\Phi}\zeta_2}\zeta_1, \tilde{\Phi}\partial_1) \\
&= (A^*_{\zeta_1}\tilde{\Phi}\zeta_2 - A^*_{\zeta_2}\tilde{\Phi}\zeta_1, \tilde{\Phi}\partial_1),
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_2], \partial_2) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\tilde{\Phi}\zeta_1, \partial_2) \\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1), \partial_2) \\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, \tilde{\Phi}\partial_2) \\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, t\partial_2 + n\partial_2) \\
&= \tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 + h^l(\tilde{\Phi}\zeta_1, \zeta_2) + h^s(\tilde{\Phi}\zeta_1, \zeta_2), t\partial_2 + n\partial_2) \\
&\quad -\tilde{g}(\sharp_{\tilde{\Phi}\zeta_2}\zeta_1 + h^l(\tilde{\Phi}\zeta_2, \zeta_1) + h^s(\tilde{\Phi}\zeta_2, \zeta_1), t\partial_2 + n\partial_2) \\
&= \tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 - \sharp_{\tilde{\Phi}\zeta_2}\zeta_1, t\partial_2) \\
&\quad +\tilde{g}(h^s(\tilde{\Phi}\zeta_1, \zeta_2) - h^s(\tilde{\Phi}\zeta_2, \zeta_1), n\partial_2) \\
&= \tilde{g}(-A^*_{\zeta_2}\tilde{\Phi}\zeta_1 + \sharp^{*t}_{\tilde{\Phi}\zeta_1}\zeta_2, t\partial_2) - \tilde{g}(-A^*_{\zeta_1}\tilde{\Phi}\zeta_2 + \sharp^{*t}_{\tilde{\Phi}\zeta_2}\zeta_1, t\partial_2) \\
&\quad +\tilde{g}(h^s(\tilde{\Phi}\zeta_1, \zeta_2) - h^s(\tilde{\Phi}\zeta_2, \zeta_1), n\partial_2) \\
&= \tilde{g}(A^*_{\zeta_1}\tilde{\Phi}\zeta_2 - A^*_{\zeta_2}\tilde{\Phi}\zeta_1, t\partial_2) \\
&\quad +\tilde{g}(h^s(\tilde{\Phi}\zeta_1, \zeta_2) - h^s(\tilde{\Phi}\zeta_2, \zeta_1), n\partial_2),
\end{aligned} \tag{36}
$$

$$\begin{aligned}
\tilde{g}([\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_2], \partial_3) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\tilde{\Phi}\zeta_1, \partial_3) \\
&\quad \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1), \partial_3) \\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, \tilde{\Phi}\partial_3) \\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\zeta_1, t\partial_3 + n\partial_3) \\
&= \tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 + h^l(\tilde{\Phi}\zeta_1, \zeta_2) + h^s(\tilde{\Phi}\zeta_1, \zeta_2), t\partial_3 + n\partial_3) \\
&\quad -\tilde{g}(\sharp_{\tilde{\Phi}\zeta_2}\zeta_1 + h^l(\tilde{\Phi}\zeta_2, \zeta_1) + h^s(\tilde{\Phi}\zeta_2, \zeta_1), t\partial_3 + n\partial_3) \\
&= \tilde{g}(\sharp_{\tilde{\Phi}\zeta_1}\zeta_2 - \sharp_{\tilde{\Phi}\zeta_2}\zeta_1, t\partial_3) + \tilde{g}(h^s(\tilde{\Phi}\zeta_1, \zeta_2) - h^s(\tilde{\Phi}\zeta_2, \zeta_1), n\partial_3) \\
&= \tilde{g}(-A^*_{\zeta_2}\tilde{\Phi}\zeta_1 + \sharp^{*t}_{\tilde{\Phi}\zeta_1}\zeta_2, t\partial_3) - \tilde{g}(-A^*_{\zeta_1}\tilde{\Phi}\zeta_2 + \sharp^{*t}_{\tilde{\Phi}\zeta_2}\zeta_1, t\partial_3) \\
&\quad +\tilde{g}(h^s(\tilde{\Phi}\zeta_1, \zeta_2) - h^s(\tilde{\Phi}\zeta_2, \zeta_1), n\partial_3) \\
&= \tilde{g}(A^*_{\zeta_1}\tilde{\Phi}\zeta_2 - A^*_{\zeta_2}\tilde{\Phi}\zeta_1, t\partial_3) \\
&\quad +\tilde{g}(h^s(\tilde{\Phi}\zeta_1, \zeta_2) - h^s(\tilde{\Phi}\zeta_2, \zeta_1), n\partial_3),
\end{aligned} \tag{37}$$

$$\begin{aligned}
\tilde{g}([\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_2], N) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}\zeta_1}\tilde{\Phi}\zeta_2 - \tilde{\sharp}_{\tilde{\Phi}\zeta_2}\tilde{\Phi}\zeta_1, N) \\
&= -\tilde{g}(\tilde{\Phi}\zeta_2, \tilde{\sharp}_{\tilde{\Phi}\zeta_1}N) + \tilde{g}(\tilde{\Phi}\zeta_1, \tilde{\sharp}_{\tilde{\Phi}\zeta_2}N) \\
&= -\tilde{g}(-A_N\tilde{\Phi}\zeta_1 + \sharp^l_{\tilde{\Phi}\zeta_1}N + D^s(\tilde{\Phi}\zeta_1, N), \tilde{\Phi}\zeta_2) \\
&\quad +\tilde{g}(-A_N\tilde{\Phi}\zeta_2 + \sharp^l_{\tilde{\Phi}\zeta_2}N + D^s(\tilde{\Phi}\zeta_2, N), \tilde{\Phi}\zeta_1) \\
&= \tilde{g}(A_N\tilde{\Phi}\zeta_1, \tilde{\Phi}\zeta_2) - \tilde{g}(A_N\tilde{\Phi}\zeta_2, \tilde{\Phi}\zeta_1).
\end{aligned} \tag{38}$$

So, proof follows from (34)~(38). □

Theorem 4.3. *Let G be a bi-slant submanifold of a PNsR-manifold $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then $\tilde{\Phi}(ltr(TG))$ is integrable if and only if*

i) $\tilde{g}(A_{N_1}\tilde{\Phi}N_2, N_3) = \tilde{g}(A_{N_2}\tilde{\Phi}N_1, N_3)$,
ii) $\tilde{g}(A_{N_1}\tilde{\Phi}N_2, \tilde{\Phi}\partial_1) = \tilde{g}(A_{N_2}\tilde{\Phi}N_1, \tilde{\Phi}\partial_1)$,
iii) $\tilde{g}(A_{N_1}\tilde{\Phi}N_2 - A_{N_2}\tilde{\Phi}N_1, t\partial_2) = \tilde{g}(D^s(\tilde{\Phi}N_2, N_1) - D^s(\tilde{\Phi}N_1, N_2), n\partial_2)$,
iv) $\tilde{g}(A_{N_1}\tilde{\Phi}N_2 - A_{N_2}\tilde{\Phi}N_1, t\partial_3) = \tilde{g}(D^s(\tilde{\Phi}N_2, N_1) - D^s(\tilde{\Phi}N_1, N_2), n\partial_3)$,
v) $\tilde{g}(A_{N_3}\tilde{\Phi}N_1, \tilde{\Phi}N_2) = \tilde{g}(A_{N_3}\tilde{\Phi}N_2, \tilde{\Phi}N_1)$,
for all $N_i \in \Gamma(\tilde{\Phi}(ltr(TG)))$, $(i = 1, 2, 3)$, $\partial_1 \in \Gamma(\gamma)$, $\partial_2 \in \Gamma(\mathring{\gamma})$, $\partial_3 \in \Gamma(\hat{\gamma})$.

Proof. We know that the the distribution $\tilde{\Phi}(ltr(TG))$ is integrable iff

$$\tilde{g}([\tilde{\Phi}N_1, \tilde{\Phi}N_2], \tilde{\Phi}N_3) = \tilde{g}([\tilde{\Phi}N_1, \tilde{\Phi}N_2], \partial_1) = \tilde{g}([\tilde{\Phi}N_1, \tilde{\Phi}N_2], \partial_2) = \tilde{g}([\tilde{\Phi}N_1, \tilde{\Phi}N_2], \partial_3) = \tilde{g}([\tilde{\Phi}N_1, \tilde{\Phi}N_2], N_3) = 0,$$

for any $N_i \in \Gamma(\tilde{\Phi}(ltr(TG)))$, $(i = 1, 2, 3)$, $\partial_1 \in \Gamma(\gamma)$, $\partial_2 \in \Gamma(\mathring{\gamma})$, $\partial_3 \in \Gamma(\hat{\gamma})$. Using (3), (11), (12), (17) with (23) and $\tilde{\sharp}$ being a metric connection, we have

$$\begin{aligned}
\tilde{g}([\tilde{\Phi}N_1, \tilde{\Phi}N_2], \tilde{\Phi}N_3) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}\tilde{\Phi}N_2 - \tilde{\sharp}_{\tilde{\Phi}N_2}\tilde{\Phi}N_1, \tilde{\Phi}N_3) \\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2 - \tilde{\sharp}_{\tilde{\Phi}N_2}N_1), \tilde{\Phi}N_3) \\
&= \omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2 - \tilde{\sharp}_{\tilde{\Phi}N_2}N_1), N_3) - \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2 - \tilde{\sharp}_{\tilde{\Phi}N_2}N_1, N_3) \\
&= \omega\tilde{g}((\tilde{\sharp}_{\tilde{\Phi}N_1}N_2 - \tilde{\sharp}_{\tilde{\Phi}N_2}N_1), \tilde{\Phi}N_3) - \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2 - \tilde{\sharp}_{\tilde{\Phi}N_2}N_1, N_3) \\
&= \omega\tilde{g}(-A_{N_2}\tilde{\Phi}N_1 + \sharp^l_{\tilde{\Phi}N_1}N_2 + D^s(\tilde{\Phi}N_1, N_2), \tilde{\Phi}N_3) \\
&\quad -\omega\tilde{g}(-A_{N_1}\tilde{\Phi}N_2 + \sharp^l_{\tilde{\Phi}N_2}N_1 + D^s(\tilde{\Phi}N_2, N_1), \tilde{\Phi}N_3) \\
&\quad -\tilde{g}(-A_{N_2}\tilde{\Phi}N_1 + \sharp^l_{\tilde{\Phi}N_1}N_2 + D^s(\tilde{\Phi}N_1, N_2), N_3) \\
&\quad +\tilde{g}(-A_{N_1}\tilde{\Phi}N_2 + \sharp^l_{\tilde{\Phi}N_2}N_1 + D^s(\tilde{\Phi}N_2, N_1), N_3) \\
&= \tilde{g}(A_{N_2}\tilde{\Phi}N_1 - A_{N_1}\tilde{\Phi}N_2, N_3),
\end{aligned} \tag{39}$$

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}N_1,\tilde{\Phi}N_2],\partial_1) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}\tilde{\Phi}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}\tilde{\Phi}N_1,\partial_1)\\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1),\partial_1)\\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1,\tilde{\Phi}\partial_1)\\
&= \tilde{g}(-A_{N_2}\tilde{\Phi}N_1+\sharp^{l}_{\tilde{\Phi}N_1}N_2+D^{s}(\tilde{\Phi}N_1,N_2),\tilde{\Phi}\partial_1)\\
&\quad -\tilde{g}(-A_{N_1}\tilde{\Phi}N_2+\sharp^{l}_{\tilde{\Phi}N_2}N_1+D^{s}(\tilde{\Phi}N_2,N_1),\tilde{\Phi}\partial_1)\\
&= \tilde{g}(A_{N_1}\tilde{\Phi}N_2-A_{N_2}\tilde{\Phi}N_1,\tilde{\Phi}\partial_1),
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}N_1,\tilde{\Phi}N_2],\partial_2) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}\tilde{\Phi}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}\tilde{\Phi}N_1,\partial_2)\\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1),\partial_2)\\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1,\tilde{\Phi}\partial_2)\\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1,t\partial_2+n\partial_2)\\
&= \tilde{g}(-A_{N_2}\tilde{\Phi}N_1+\sharp^{l}_{\tilde{\Phi}N_1}N_2+D^{s}(\tilde{\Phi}N_1,N_2),t\partial_2+n\partial_2)\\
&\quad -\tilde{g}(-A_{N_1}\tilde{\Phi}N_2+\sharp^{l}_{\tilde{\Phi}N_2}N_1+D^{s}(\tilde{\Phi}N_2,N_1),t\partial_2+n\partial_2)\\
&= \tilde{g}(A_{N_1}\tilde{\Phi}N_2-A_{N_2}\tilde{\Phi}N_1,t\partial_2)\\
&\quad +\tilde{g}(D^{s}(\tilde{\Phi}N_1,N_2)-D^{s}(\tilde{\Phi}N_2,N_1),n\partial_2),
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}N_1,\tilde{\Phi}N_2],\partial_3) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}\tilde{\Phi}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}\tilde{\Phi}N_1,\partial_3)\\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1),\partial_3)\\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1,\tilde{\Phi}\partial_3)\\
&= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}N_1,t\partial_3+n\partial_3)\\
&= \tilde{g}(-A_{N_2}\tilde{\Phi}N_1+\sharp^{l}_{\tilde{\Phi}N_1}N_2+D^{s}(\tilde{\Phi}N_1,N_2),t\partial_3+n\partial_3)\\
&\quad -\tilde{g}(-A_{N_1}\tilde{\Phi}N_2+\sharp^{l}_{\tilde{\Phi}N_2}N_1+D^{s}(\tilde{\Phi}N_2,N_1),t\partial_3+n\partial_3)\\
&= \tilde{g}(A_{N_1}\tilde{\Phi}N_2-A_{N_2}\tilde{\Phi}N_1,t\partial_3)\\
&\quad +\tilde{g}(D^{s}(\tilde{\Phi}N_1,N_2)-D^{s}(\tilde{\Phi}N_2,N_1),n\partial_3),
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\tilde{g}([\tilde{\Phi}N_1,\tilde{\Phi}N_2],N_3) &= \tilde{g}(\tilde{\sharp}_{\tilde{\Phi}N_1}\tilde{\Phi}N_2-\tilde{\sharp}_{\tilde{\Phi}N_2}\tilde{\Phi}N_1,N_3)\\
&= -\tilde{g}(\tilde{\Phi}N_2,\tilde{\sharp}_{\tilde{\Phi}N_1}N_3)+\tilde{g}(\tilde{\Phi}N_1,\tilde{\sharp}_{\tilde{\Phi}N_2}N_3)\\
&= -\tilde{g}(-A_{N}\tilde{\Phi}N_1+\sharp^{l}_{\tilde{\Phi}N_1}N+D^{s}(\tilde{\Phi}N_1,N_3),\tilde{\Phi}N_2)\\
&\quad +\tilde{g}(-A_{N}\tilde{\Phi}N_2+\sharp^{l}_{\tilde{\Phi}\zeta_2}N+D^{s}(\tilde{\Phi}N_2,N_3),\tilde{\Phi}N_1)\\
&= \tilde{g}(A_{N_3}\tilde{\Phi}N_1,\tilde{\Phi}N_2)-\tilde{g}(A_{N_3}\tilde{\Phi}N_2,\tilde{\Phi}N_1).
\end{aligned}
\tag{43}
$$

The proof follows from (39)~(43). □

Theorem 4.4. *Let G be a bi-slant submanifold of a PNsR-manifold $(\tilde{G},\tilde{\Phi},\tilde{g})$. Then γ is integrable if and only if*
i) $\tilde{g}(\sharp^{*}_{\partial_4}\tilde{\Phi}\partial_1-\sharp^{*}_{\partial_1}\tilde{\Phi}\partial_4,t\partial_2)+\tilde{g}(h^{s}(\partial_4,\tilde{\Phi}\partial_1)-h^{s}(\partial_1,\tilde{\Phi}\partial_4),n\partial_2)=\omega\tilde{g}(\sharp^{*}_{\partial_4}\tilde{\Phi}\partial_1-\sharp^{*}_{\partial_1}\tilde{\Phi}\partial_4,\partial_2)$,
ii) $\tilde{g}(\sharp^{*}_{\partial_4}\tilde{\Phi}\partial_1-\sharp^{*}_{\partial_1}\tilde{\Phi}\partial_4,t\partial_3)+\tilde{g}(h^{s}(\partial_4,\tilde{\Phi}\partial_1)-h^{s}(\partial_1,\tilde{\Phi}\partial_4),n\partial_3)=\omega\tilde{g}(\sharp^{*}_{\partial_4}\tilde{\Phi}\partial_1-\sharp^{*}_{\partial_1}\tilde{\Phi}\partial_4,\partial_3)$,
iii) $\tilde{g}(\sharp^{*}_{\partial_4}\tilde{\Phi}\partial_1-\sharp^{*}_{\partial_1}\tilde{\Phi}\partial_4,\tilde{\Phi}N)=\tilde{g}(h^{*}(\partial_4,\tilde{\Phi}\partial_1)-h^{*}(\partial_1,\tilde{\Phi}\partial_4),N)$,
iv) $\tilde{g}(A_N\partial_4,\tilde{\Phi}\partial_1)=\tilde{g}(A_N\partial_1,\tilde{\Phi}\partial_4)$,
for all $\partial_1,\partial_4\in\Gamma(\gamma)$, $\partial_2\in\Gamma(\mathring{\gamma})$, $\partial_3\in\Gamma(\hat{\gamma})$, $N\in\Gamma(ltr(TG))$.

Proof. If we consider the definition of the bi-slant lightlike submanifolds then the distribution γ is integrable iff

$$\tilde{g}([\partial_4,\partial_1],\partial_2)=\tilde{g}([\partial_4,\partial_1],\partial_3)=\tilde{g}([\partial_4,\partial_1],N)=\tilde{g}([\partial_4,\partial_1],\tilde{\Phi}N)=0,$$

for any $\partial_1,\partial_4\in\Gamma(\gamma)$, $\partial_2\in\Gamma(\mathring{\gamma})$, $\partial_3\in\Gamma(\hat{\gamma})$, $N\in\Gamma(ltr(TG))$. By use of (3), (11), (12), (16) with (23) and $\tilde{\sharp}$ being a metric connection, we find

$$\begin{aligned}
\tilde{g}([\partial_4,\partial_1],\partial_2) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4,\partial_2)\\
&= -\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4),\tilde{\Phi}\partial_2)+\omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4),\partial_2)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1+h^l(\partial_4,\tilde{\Phi}\partial_1)+h^s(\partial_4,\tilde{\Phi}\partial_1),t\partial_2+n\partial_2)\\
&\quad +\tilde{g}(\sharp_{\partial_1}\tilde{\Phi}\partial_4+h^l(\partial_1,\tilde{\Phi}\partial_4)+h^s(\partial_1,\tilde{\Phi}\partial_4),t\partial_2+n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1+h^l(\partial_4,\tilde{\Phi}\partial_1)+h^s(\partial_4,\tilde{\Phi}\partial_1),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp_{\partial_1}\tilde{\Phi}\partial_4+h^l(\partial_1,\tilde{\Phi}\partial_4)+h^s(\partial_1,\tilde{\Phi}\partial_4),\partial_2)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1-\sharp_{\partial_1}\tilde{\Phi}\partial_4,t\partial_2)\\
&\quad -\tilde{g}(h^s(\partial_4,\tilde{\Phi}\partial_1)-h^s(\partial_1,\tilde{\Phi}\partial_4),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1-\sharp_{\partial_1}\tilde{\Phi}\partial_4,\partial_2)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1+h^*(\partial_4,\tilde{\Phi}\partial_1),t\partial_2)\\
&\quad +\tilde{g}(\sharp^*_{\partial_1}\tilde{\Phi}\partial_4+h^*(\partial_1,\tilde{\Phi}\partial_4),t\partial_2)\\
&\quad -\tilde{g}(h^s(\partial_4,\tilde{\Phi}\partial_1)-h^s(\partial_1,\tilde{\Phi}\partial_4),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1+h^*(\partial_4,\tilde{\Phi}\partial_1),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp^*_{\partial_1}\tilde{\Phi}\partial_4+h^*(\partial_1,\tilde{\Phi}\partial_4),\partial_2)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1-\sharp^*_{\partial_1}\tilde{\Phi}\partial_4,t\partial_2)\\
&\quad -\tilde{g}(h^s(\partial_4,\tilde{\Phi}\partial_1)-h^s(\partial_1,\tilde{\Phi}\partial_4),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1+\sharp^*_{\partial_1}\tilde{\Phi}\partial_4,\partial_2),
\end{aligned}\tag{44}$$

$$
\begin{aligned}
\tilde{g}([\partial_4,\partial_1],\partial_3) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4, \partial_3)\\
&= -\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4), \tilde{\Phi}\partial_3) + \omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4), \partial_3)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4,\tilde{\Phi}\partial_1) + h^s(\partial_4,\tilde{\Phi}\partial_1), t\partial_3 + n\partial_3)\\
&\quad +\tilde{g}(\sharp_{\partial_1}\tilde{\Phi}\partial_4 + h^l(\partial_1,\tilde{\Phi}\partial_4) + h^s(\partial_1,\tilde{\Phi}\partial_4), t\partial_3 + n\partial_3)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4,\tilde{\Phi}\partial_1) + h^s(\partial_4,\tilde{\Phi}\partial_1), \partial_3)\\
&\quad -\omega\tilde{g}(\sharp_{\partial_1}\tilde{\Phi}\partial_4 + h^l(\partial_1,\tilde{\Phi}\partial_4) + h^s(\partial_1,\tilde{\Phi}\partial_4), \partial_3)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 - \sharp_{\partial_1}\tilde{\Phi}\partial_4, t\partial_3)\\
&\quad -\tilde{g}(h^s(\partial_4,\tilde{\Phi}\partial_1) - h^s(\partial_1,\tilde{\Phi}\partial_4), n\partial_3)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 - \sharp_{\partial_1}\tilde{\Phi}\partial_4, \partial_3)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4,\tilde{\Phi}\partial_1), t\partial_3)\\
&\quad +\tilde{g}(\sharp^*_{\partial_1}\tilde{\Phi}\partial_4 + h^*(\partial_1,\tilde{\Phi}\partial_4), t\partial_3)\\
&\quad -\tilde{g}(h^s(\partial_4,\tilde{\Phi}\partial_1) - h^s(\partial_1,\tilde{\Phi}\partial_4), n\partial_3)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4,\tilde{\Phi}\partial_1), \partial_3)\\
&\quad -\omega\tilde{g}(\sharp^*_{\partial_1}\tilde{\Phi}\partial_4 + h^*(\partial_1,\tilde{\Phi}\partial_4), \partial_3)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 - \sharp^*_{\partial_1}\tilde{\Phi}\partial_4, t\partial_3)\\
&\quad -\tilde{g}(h^s(\partial_4,\tilde{\Phi}\partial_1) - h^s(\partial_1,\tilde{\Phi}\partial_4), n\partial_3)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + \sharp^*_{\partial_1}\tilde{\Phi}\partial_4, \partial_3), \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}([\partial_4,\partial_1],N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4, N)\\
&= -\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4), \tilde{\Phi}N) + \omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4), N)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4,\tilde{\Phi}\partial_1) + h^s(\partial_4,\tilde{\Phi}\partial_1), \tilde{\Phi}N)\\
&\quad +\tilde{g}(\sharp_{\partial_1}\tilde{\Phi}\partial_4 + h^l(\partial_1,\tilde{\Phi}\partial_4) + h^s(\partial_1,\tilde{\Phi}\partial_4), \tilde{\Phi}N)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4,\tilde{\Phi}\partial_1) + h^s(\partial_4,\tilde{\Phi}\partial_1), N)\\
&\quad -\omega\tilde{g}(\sharp_{\partial_1}\tilde{\Phi}\partial_4 + h^l(\partial_1,\tilde{\Phi}\partial_4) + h^s(\partial_1,\tilde{\Phi}\partial_4), N)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 - \sharp_{\partial_1}\tilde{\Phi}\partial_4, \tilde{\Phi}N) + \omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 - \sharp_{\partial_1}\tilde{\Phi}\partial_4, N)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4,\tilde{\Phi}\partial_1), \tilde{\Phi}N)\\
&\quad +\tilde{g}(\sharp^*_{\partial_1}\tilde{\Phi}\partial_4 + h^*(\partial_1,\tilde{\Phi}\partial_4), \tilde{\Phi}N)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4,\tilde{\Phi}\partial_1), N)\\
&\quad -\omega\tilde{g}(\sharp^*_{\partial_1}\tilde{\Phi}\partial_4 + h^*(\partial_1,\tilde{\Phi}\partial_4), N)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 - \sharp^*_{\partial_1}\tilde{\Phi}\partial_4, \tilde{\Phi}N) \qquad (46)\\
&\quad +\omega\tilde{g}(h^*(\partial_4,\tilde{\Phi}\partial_1) + h^*(\partial_1,\tilde{\Phi}\partial_4), N),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}([\partial_4,\partial_1],\tilde{\Phi}N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1 - \tilde{\sharp}_{\partial_1}\partial_4, \tilde{\Phi}N)\\
&= -\tilde{g}(\tilde{\Phi}\partial_1, \tilde{\sharp}_{\partial_4}N) + \tilde{g}(\tilde{\Phi}\partial_4, \tilde{\sharp}_{\partial_1}N)\\
&= -\tilde{g}(-A_N\partial_4 + \sharp^l_{\partial_4}N + D^s(\partial_4,N), \tilde{\Phi}\partial_1)\\
&\quad +\tilde{g}(-A_N\partial_1 + \sharp^l_{\partial_1}N + D^s(\partial_1,N), \tilde{\Phi}\partial_4)\\
&= \tilde{g}(A_N\partial_4, \tilde{\Phi}\partial_1) - \tilde{g}(A_N\partial_1, \tilde{\Phi}\partial_4). \qquad (47)
\end{aligned}
$$

So, we arrive at the proof from (44)~(47). □

Theorem 4.5. *Let G be a bi-slant submanifold of a PNsR-manifold $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then $\tilde{\gamma}$ is integrable if and only if*
i) $\tilde{g}(\sharp_{\partial_2} t\partial_3 - A_{n\partial_3}\partial_2, \tilde{\Phi}\partial_1) + \omega\tilde{g}(\sharp_{\partial_3} t\partial_2 - A_{n\partial_2}\partial_3, \partial_1) = \omega\tilde{g}(\sharp_{\partial_2} t\partial_3 - A_{n\partial_3}\partial_2, \partial_1) + \tilde{g}(\sharp_{\partial_3} t\partial_2 - A_{n\partial_2}\partial_3, \tilde{\Phi}\partial_1)$,
ii) $\tilde{g}(\sharp_{\partial_2} t\partial_3 - A_{n\partial_3}\partial_2, \tilde{\Phi}N) + \omega\tilde{g}(\sharp_{\partial_3} t\partial_2 - A_{n\partial_2}\partial_3, N) = \omega\tilde{g}(\sharp_{\partial_2} t\partial_3 - A_{n\partial_3}\partial_2, N) + \tilde{g}(\sharp_{\partial_3} t\partial_2 - A_{n\partial_2}\partial_3, \tilde{\Phi}N)$,
iii) $\tilde{g}(\sharp_{\partial_2} t\partial_3 - A_{n\partial_3}\partial_2, N) = \tilde{g}(\sharp_{\partial_3} t\partial_2 - A_{n\partial_2}\partial_3, N)$,
for all $\partial_2, \partial_3 \in \Gamma(\tilde{\gamma})$ (where $\tilde{\gamma}$ is any slant ditribution), $\partial_1, \partial_4 \in \Gamma(\gamma)$ and $N \in \Gamma(ltr(TG))$.

Proof. If we consider the definition of the slant distributions on submanifold G, then the distribution $\tilde{\gamma}$ is integrable iff

$$\tilde{g}([\partial_2, \partial_3], \partial_1) = \tilde{g}([\partial_2, \partial_3], N) = \tilde{g}([\partial_2, \partial_3], \tilde{\Phi}N) = 0,$$

for any $\partial_2, \partial_3 \in \Gamma(\tilde{\gamma})$, $\partial_1, \partial_4 \in \Gamma(\gamma)$ and $N \in \Gamma(ltr(TG))$. From (3), (11), (13) and (23), we get

$$\begin{aligned}
\tilde{g}([\partial_2, \partial_3], \partial_1) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2, \partial_1) \\
&= -\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2), \tilde{\Phi}\partial_1) + \omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2), \partial_1) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), \tilde{\Phi}\partial_1) + \tilde{g}(\tilde{\sharp}_{\partial_3}(t\partial_2 + n\partial_2), \tilde{\Phi}\partial_1) \\
&\quad + \omega\tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), \partial_1) - \omega\tilde{g}(\tilde{\sharp}_{\partial_3}(t\partial_2 + n\partial_2), \partial_1) \\
&= -\tilde{g}(\sharp_{\partial_2} t\partial_3), \tilde{\Phi}\partial_1) - \tilde{g}(-A_{n\partial_3}\partial_2, \tilde{\Phi}\partial_1) \\
&\quad + \tilde{g}(\sharp_{\partial_3} t\partial_2), \tilde{\Phi}\partial_1) + \tilde{g}(-A_{n\partial_2}\partial_3, \tilde{\Phi}\partial_1) \\
&\quad + \omega\tilde{g}(\sharp_{\partial_2} t\partial_3), \partial_1) + \omega\tilde{g}(-A_{n\partial_3}\partial_2, \partial_1) \\
&\quad - \omega\tilde{g}(\sharp_{\partial_3} t\partial_2), \partial_1) - \omega\tilde{g}(-A_{n\partial_2}\partial_3, \partial_1),
\end{aligned} \tag{48}$$

$$\begin{aligned}
\tilde{g}([\partial_2, \partial_3], N) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2, N) \\
&= -\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2), \tilde{\Phi}N) + \omega\tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2), N) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), \tilde{\Phi}N) + \tilde{g}(\tilde{\sharp}_{\partial_3}(t\partial_2 + n\partial_2), \tilde{\Phi}N) \\
&\quad + \omega\tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), N) - \omega\tilde{g}(\tilde{\sharp}_{\partial_3}(t\partial_2 + n\partial_2), N) \\
&= -\tilde{g}(\sharp_{\partial_2} t\partial_3), \tilde{\Phi}N) - \tilde{g}(-A_{n\partial_3}\partial_2, \tilde{\Phi}N) \\
&\quad + \tilde{g}(\sharp_{\partial_3} t\partial_2), \tilde{\Phi}N) + \tilde{g}(-A_{n\partial_2}\partial_3, \tilde{\Phi}N) \\
&\quad + \omega\tilde{g}(\sharp_{\partial_2} t\partial_3), N) + \omega\tilde{g}(-A_{n\partial_3}\partial_2, N) \\
&\quad - \omega\tilde{g}(\sharp_{\partial_3} t\partial_2), N) - \omega\tilde{g}(-A_{n\partial_2}\partial_3, N),
\end{aligned} \tag{49}$$

$$\begin{aligned}
\tilde{g}([\partial_2, \partial_3], \tilde{\Phi}N) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2, \tilde{\Phi}N) \\
&= \tilde{g}(\tilde{\Phi}(\tilde{\sharp}_{\partial_2}\partial_3 - \tilde{\sharp}_{\partial_3}\partial_2), N) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), N) - \tilde{g}(\tilde{\sharp}_{\partial_3}(t\partial_2 + n\partial_2), N \\
&= \tilde{g}(\sharp_{\partial_2} t\partial_3 - \sharp_{\partial_3} t\partial_2, N) + \tilde{g}(A_{n\partial_3}\partial_2 - A_{n\partial_2}\partial_3, N).
\end{aligned} \tag{50}$$

The proof follows from (48)~(50). □

Now, we obtain the necessary and sufficient conditions for foliations determined by distribution on a bi-slant lightlike submanifolds of a *PNsR*-manifold to be totally geodesic.

Theorem 4.6. *Let G be a bi-slant submanifold of a PNsR-manifold $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then γ defines totally geodesic foliation if and only if*
i) $\tilde{g}(\sharp_{\partial_4} t\partial_2 - A_{n\partial_2}\partial_4, \tilde{\Phi}\partial_1) = \omega\tilde{g}(\sharp_{\partial_4} t\partial_2 - A_{n\partial_2}\partial_4, \partial_1)$,
ii) $\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) = \omega\tilde{g}(h^*(\partial_4, \tilde{\Phi}\partial_1), N)$,
iii) $h^*(\partial_4, \tilde{\Phi}\partial_1)$ *has no component in* $\Gamma(RadTG)$,
for all $\partial_1, \partial_4 \in \Gamma(\gamma)$, $\partial_2 \in \Gamma(\tilde{\gamma})$ and $N \in \Gamma(ltr(TG))$.

Proof. The distribution γ defines totally geodesic foliation iff $\sharp_{\partial_4}\partial_1 \in \Gamma(\gamma)$ for all $\partial_1, \partial_4 \in \Gamma(\gamma)$. $\tilde{\sharp}$ being a metric connection and from (3), (11), (13) and (23), we get

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_4}\partial_1, \partial_2) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1, \partial_2) \\
&= -\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4}\partial_2) \\
&= \tilde{g}(\tilde{\Phi}\partial_1, \tilde{\sharp}_{\partial_4}(t\partial_2 + n\partial_2)) - \omega\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4}(t\partial_2 + n\partial_2)) \\
&= \tilde{g}(\tilde{\Phi}\partial_1, \tilde{\sharp}_{\partial_4}t\partial_2) + \tilde{g}(\tilde{\Phi}\partial_1, \tilde{\sharp}_{\partial_4}n\partial_2) \\
&\quad -\omega\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4}t\partial_2) - \omega\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4}n\partial_2) \\
&= \tilde{g}(\tilde{\Phi}\partial_1, \sharp_{\partial_4}t\partial_2 + h^l(\partial_4, t\partial_2) + h^s(\partial_4, t\partial_2) \\
&\quad +\tilde{g}(\tilde{\Phi}\partial_1, -A_{n\partial_2}\partial_4 + \sharp^l_{\partial_4}n\partial_2 + D^s(\partial_4, n\partial_2)) \\
&\quad -\omega\tilde{g}(\partial_1, \sharp_{\partial_4}t\partial_2 + h^l(\partial_4, t\partial_2) + h^s(\partial_4, t\partial_2)) \\
&\quad -\omega\tilde{g}(\partial_1, -A_{n\partial_2}\partial_4 + \sharp^l_{\partial_4}n\partial_2 + D^s(\partial_4, n\partial_2)) \\
&= \tilde{g}(\tilde{\Phi}\partial_1, \sharp_{\partial_4}t\partial_2 - A_{n\partial_2}\partial_4) - \omega\tilde{g}(\partial_1, \sharp_{\partial_4}t\partial_2 - A_{n\partial_2}\partial_4).
\end{aligned}$$

Similarly, from (3), (11) and (16), we have

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_4}\partial_1, N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1, N) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) + \omega\tilde{g}(\tilde{\sharp}_{\partial_4}\tilde{\Phi}\partial_1, N) \\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4, \tilde{\Phi}\partial_1) + h^s(\partial_4, \tilde{\Phi}\partial_1), \tilde{\Phi}N) \\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4, \tilde{\Phi}\partial_1) + h^s(\partial_4, \tilde{\Phi}\partial_1), N) \\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) + \omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1, N) \\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4, \tilde{\Phi}\partial_1), \tilde{\Phi}N) \\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4, \tilde{\Phi}\partial_1), N) \\
&= \tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) - \omega\tilde{g}(h^*(\partial_4, \tilde{\Phi}\partial_1), N).
\end{aligned}$$

Furthermore, using (3), (11) and (16), we obtain

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_4}\partial_1, \tilde{\Phi}N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\tilde{\Phi}\partial_1, N) \\
&= \tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4, \tilde{\Phi}\partial_1) + h^s(\partial_4, \tilde{\Phi}\partial_1), N) \\
&= \tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4, \tilde{\Phi}\partial_1), N) \\
&= \tilde{g}(h^*(\partial_4, \tilde{\Phi}\partial_1), N).
\end{aligned}$$

So, the proof is completed. □

Theorem 4.7. *Let G be a bi-slant submanifold of a PNsR-manifold* $(\tilde{G}, \tilde{\Phi}, \tilde{g})$*. Then* $\tilde{\gamma}$ *defines totally geodesic foliation if and only if*

i) $\tilde{g}(t\partial_3, \sharp_{\partial_2}\tilde{\Phi}\partial_1) + \tilde{g}(n\partial_3, h^s(\partial_2, \tilde{\Phi}\partial_1)) = \omega\tilde{g}(\sharp_{\partial_2}\tilde{\Phi}\partial_1, \partial_3)$,
ii) $\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, \tilde{\Phi}N) = \omega\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, N)$,
iii) $\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2$ *has no component in* $\Gamma(RadTG)$,
for all $\partial_1 \in \Gamma(\gamma)$, $\partial_2, \partial_3 \in \Gamma(\tilde{\gamma})$ *and* $N \in \Gamma(ltr(TG))$.

Proof. The distribution $\tilde{\gamma}$ defines totally geodesic foliation iff $\sharp_{\partial_2}\partial_3 \in \Gamma(\tilde{\gamma})$ for all $\partial_2, \partial_3 \in \Gamma(\tilde{\gamma})$. In view of (3),

(11) and (23) with the properties of the connection $\tilde{\sharp}$, we get

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_2}\partial_3, \partial_1) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3, \partial_1) \\
&= -\tilde{g}(\partial_3, \tilde{\sharp}_{\partial_2}\partial_1) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}\tilde{\Phi}\partial_1, \tilde{\Phi}\partial_3) - \omega\tilde{g}(\tilde{\sharp}_{\partial_2}\tilde{\Phi}\partial_1, \partial_3) \\
&= \tilde{g}(\sharp_{\partial_2}\tilde{\Phi}\partial_1 + h^l(\partial_2, \tilde{\Phi}\partial_1) + h^s(\partial_2, \tilde{\Phi}\partial_1), t\partial_3 + n\partial_3) \\
&\quad -\omega\tilde{g}(\sharp_{\partial_2}\tilde{\Phi}\partial_1 + h^l(\partial_2, \tilde{\Phi}\partial_1) + h^s(\partial_2, \tilde{\Phi}\partial_1), \partial_3) \\
&= \tilde{g}(\sharp_{\partial_2}\tilde{\Phi}\partial_1, t\partial_3) + \tilde{g}(h^s(\partial_2, \tilde{\Phi}\partial_1), n\partial_3) \\
&\quad -\omega\tilde{g}(\sharp_{\partial_2}\tilde{\Phi}\partial_1, \partial_3).
\end{aligned}$$

Similarly, from (3), (11), (13) and (23), we have

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_2}\partial_3, N) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3, N) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_2}\tilde{\Phi}\partial_3, \tilde{\Phi}N) + \omega\tilde{g}(\tilde{\sharp}_{\partial_2}\tilde{\Phi}\partial_3, N) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), \tilde{\Phi}N) + \omega\tilde{g}((t\partial_3 + n\partial_3), N) \\
&= -\tilde{g}(\sharp_{\partial_2}t\partial_3 + h^l(\partial_2, t\partial_3) + h^s(\partial_2, t\partial_3), \tilde{\Phi}N) \\
&\quad -\tilde{g}(-A_{n\partial_3}\partial_2 + \sharp^l_{\partial_2}n\partial_3 + D^s(\partial_2, n\partial_3), \tilde{\Phi}N) \\
&\quad +\omega\tilde{g}(\sharp_{\partial_2}t\partial_3 + h^l(\partial_2, t\partial_3) + h^s(\partial_2, t\partial_3), N) \\
&\quad +\omega\tilde{g}(-A_{n\partial_3}\partial_2 + \sharp^l_{\partial_2}n\partial_3 + D^s(\partial_2, n\partial_3), N) \\
&= -\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, \tilde{\Phi}N) + \omega\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, N).
\end{aligned}$$

Also, from (3), (11), (13) and (23), we get

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_2}\partial_3, N) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3, \tilde{\Phi}N) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}\tilde{\Phi}\partial_3, N) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), N) \\
&= \tilde{g}(\sharp_{\partial_2}t\partial_3 + h^l(\partial_2, t\partial_3) + h^s(\partial_2, t\partial_3), N) \\
&\quad +\tilde{g}(-A_{n\partial_3}\partial_2 + \sharp^l_{\partial_2}n\partial_3 + D^s(\partial_2, n\partial_3), N) \\
&= \tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, N).
\end{aligned}$$

which gives proof of our assertion. □

Theorem 4.8. *Let G be a bi-slant submanifold of a PNsR-manifold $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then G is mixed geodesic (i.e., $h^l(\partial_2, \partial_3) = 0$, $h^s(\partial_2, \partial_3) = 0$ for all $\partial_2 \in \Gamma(\mathring{\gamma})$, $\partial_3 \in \Gamma(\hat{\gamma})$) if and only if*

i) $n(\sharp_{\partial_1}t\partial_2 - A_{n\partial_2}\partial_1) = -c(h^s(\partial_1, t\partial_2) + \sharp^s_{\partial_1}n\partial_2)$,

ii) $h^l(\partial_1, t\partial_2) + D^l(\partial_1, n\partial_2) = h^s(\partial_1, t\partial_2) + \sharp^s_{\partial_1}n\partial_2$,

for all $\partial_1 \in \Gamma(\gamma)$, $\partial_2 \in \Gamma(\tilde{\gamma})$.

Proof. Using (9), (11), (13), (23), (26) with (28), we have

$$\begin{aligned} h(\partial_1,\partial_2) &= \tilde{\sharp}_{\partial_1}\partial_2 - \sharp_{\partial_1}\partial_2 \\ &= \tilde{\Phi}(\tilde{\Phi}\tilde{\sharp}_{\partial_1}\partial_2) - 3\tilde{\Phi}\tilde{\sharp}_{\partial_1}\partial_2 - \sharp_{\partial_1}\partial_2 \\ &= \tilde{\Phi}(\tilde{\sharp}_{\partial_1}\tilde{\Phi}\partial_2) - 3\tilde{\sharp}_{\partial_1}\tilde{\Phi}\partial_2 - \sharp_{\partial_1}\partial_2 \\ &= \tilde{\Phi}(\tilde{\sharp}_{\partial_1}(t\partial_2 + n\partial_2)) - 3(\tilde{\sharp}_{\partial_1}(t\partial_2 + n\partial_2)) - \sharp_{\partial_1}\partial_2 \\ &= \tilde{\Phi}\begin{pmatrix} \sharp_{\partial_1}t\partial_2 + h^l(\partial_1,t\partial_2) + h^s(\partial_1,t\partial_2) \\ -A_{n\partial_2}\partial_1 + \sharp^s_{\partial_1}n\partial_2 + D^l(\partial_1,n\partial_2) \end{pmatrix} \\ &\quad -3\begin{pmatrix} \sharp_{\partial_1}t\partial_2 + h^l(\partial_1,t\partial_2) + h^s(\partial_1,t\partial_2) \\ -A_{n\partial_2}\partial_1 + \sharp^s_{\partial_1}n\partial_2 + D^l(\partial_1,n\partial_2) \end{pmatrix} \\ &\quad -\sharp_{\partial_1}\partial_2. \end{aligned}$$

If we consider the transversal part of above equation, we arrive at

$$\begin{aligned} h(\partial_1,\partial_2) &= n(\sharp_{\partial_1}t\partial_2 - A_{n\partial_2}\partial_1) + c(h^s(\partial_1,t\partial_2) + \sharp^s_{\partial_1}n\partial_2) \\ &\quad -3(h^l(\partial_1,t\partial_2) + h^s(\partial_1,t\partial_2) - \sharp^s_{\partial_1}n\partial_2 - D^l(\partial_1,n\partial_2)), \end{aligned}$$

which completes the proof. □

References

[1] A.M. Blaga, C.E. Hretcanu, *Invariant, anti-invariant and slant submanifolds of metallic Riemannian manifolds*, Novi Sad J. Math. **48** (2018), 55–80.

[2] B.E. Acet, *Lightlike hypersurfaces of metallic semi-Riemannian manifolds*, Int. J. Geo. Meth. Modern Phys. **15** (2018), 185–201.

[3] B.E. Acet, S. Yüksel Perktaş, *Screen slant radical transversal null submanifolds of para-Sasakian manifolds*, Facta Univ Ser: Math and Inform. **31** (2016), 543–557.

[4] B. Şahin, *Slant lightlike submanifolds of indefinite Hermitian manifolds*, Balkan J Geo Appl. **13** (2008), 107–119.

[5] B. Şahin, *Slant submanifolds of almost product Riemannian manifolds*, J. Korean Math. Soc. **43** (2006), 717–732.

[6] B. Şahin, *Almost poly-Norden manifolds*, Int. J. Maps in Math. **1** (2018), 68–79.

[7] B.-Y. Chen, *Geometry of slant submanifolds*, Katholike Universiteit Leuven, Louvain, 123pp, 1990.

[8] C.E. Hretcanu, A.M. Blaga, *Submanifolds in metallic semi-Riemannian manifolds*, Differ. Geom. Dynm. Syst. **20** (2018), 83–97.

[9] C.E. Hretcanu, M.C. Crasmareanu, *Metallic structure on Riemannian manifolds*, Rev. Un. Mat. Argentina. **54** (2013), 15–27.

[10] E. Kılıç, T. Acet, S. Yüksel Perktaş, *Lightlike hypersurfaces of poly-Norden semi-Riemannian manifolds*, Turk. J. Sci. **7** (2022), 21–30.

[11] F.E. Erdoğan, S. Yüksel Perktaş, B.E. Acet, A.M. Blaga, *Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds*, J. Geom. Phys. **142** (2019), 111–120.

[12] J.L. Cabrerizo, A. Carriazo, L.M Fernandez, and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J. **42** (2000), 125–138.

[13] K.L. Duggal, A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and Its Applications. Kluwer Publisher, 1996.

[14] K.L. Duggal, B. Şahin, *Differential geometry of lightlike submanifolds*, Frontiers in Mathematics, 2010.

[15] K.L. Duggal, B. Şahin, *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Math Hungar. **112** (2006), 107–130.

[16] M. Ahmad, M. Ahmad, F. Mofarreh, *Bi-slant lightlike submanifolds of golden semi-Riemannian manifolds*, AIMS Mathematics. **8** (2023), 19526–19545.

[17] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi Sect. I a Mat. **40** (1994), 55–61.

[18] P. Alegre, A. Carriazo, *Bi-slant submanifolds of para-Hermitian manifolds*, Mathematics. **7** (2019), 1–15.

[19] S. Kalia, *The generalizations of the golden ratio, their powers, continued fractions and convergents*, http://math.mit.edu/research/highschool/primes/papers.php.

[20] S. Yüksel Perktaş, F.E. Erdoğan, B.E. Acet, *Lightlike submanifolds of metallic semi-Riemannian manifolds*, Filomat. **34** (2020), 1781–1794.

[21] S. Yüksel Perktaş, *Submanifolds of poly-Norden Riemannian manifolds*, Turk. J. Math. **44** (2020), 31–49.

[22] S. Yüksel Perktaş, T. Acet, E. Kılıç, *On lightlike submanifolds of poly-Norden semi-Riemannian manifolds*, Turk. J. Math. Comp. Sci. **15** (2023), 1–11.

[23] T. Acet, S. Yüksel Perktaş, E. Kılıç, *On some types of lightlike submanifolds of poly-Norden semi-Riemannian manifolds*, Filomat. **37** (2023), 3725–3740.

[24] V.W. Spinadel, *The metallic means family and multifractal spectra*, Nonlinear Analy Ser B: Real World Appl. **36** (1999), 721–745.

[25] V.W. Spinadel, *The metallic means family and forbidden symmetries*, Rev. Un. Mat. Argentina. **54** (2013), 15–27.