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Geometrical eigenproblem of various types higher order tensors in Riemannian space

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Abstract. The geometrical eigenproblem (G-eigenproblem) is validated as an extension of the classical eigenproblem of a linear operator within Euclidean vector space. It is homogeneous and invariant, making it efficient and advantageous for tensors describing real phenomena, regardless of the representation, while determining their eigenspaces. The G-eigenproblem in Riemannian space will be defined not only for covariant tensors but also for mixed and contravariant ones. Achievements will be discussed and related to higher-order unit tensors.

1. Introduction

Higher Order Tensor Spectral Analysis (HOTSA) has been approved as a highly applicable and valuable theory in various fields, including image processing [9, 20, 25], data analysis [8, 13, 14], physics [7, 21], and stability theory of nonlinear autonomous systems [12, 24]. The core concept involves generalizing the classical eigenproblem $Ax = \lambda x$, from the matrix to the tensor framework. Tensors are commonly treated as multidimensional arrays or multilinear operators in vector space with Euclidean metric. Only the researches presented in [1, 2, 22, 23] consider eigenproblems of tensors as multilinear operators in non-Euclidean vector spaces.

The generalizations are motivated and approached in different ways, leading to diverse definitions. They all involve a real m-th order totally symmetric covariant tensor $\mathcal{A}$ in n-dimensional space, and its contraction with a rank-one tensor $z^{m-1} = z \otimes z \otimes \ldots \otimes z$ of order $(m-1)$ produced by a vector z, i.e. an incomplete action of the tensor, $\mathcal{A}(\cdot, z, z, \ldots, z)$. However, the resulting expression $\mathcal{A}z^{m-1}$ is compared to various objects depending on the specific generalization.

The following are generalizations consistent with this research:
- Z-eigenvalue problem [3, 4, 16, 18, 21, 24]
- Generalized tensor eigenvalue problem [5, 6, 10, 11, 15, 26]
- D-eigenvalue problem [17, 19, 21, 27]
- Geometrical eigenproblem [23].

These are listed together with the main references that provide comprehensive insights into the topic.

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This work is a continuation of the research presented in [23], building upon the foundational notions introduced in the previous study. The main aim is to further discuss the geometrical eigenproblem concerning different types of tensors, and to position it in relation to the other listed eigenproblems. A comparative overview of the listed eigenproblems is provided, with a focus on highlighting both the shared concepts and the distinct differences between them. Specifically, the geometrical eigenproblem is a generalization of the Z-eigenproblem, it also encompasses the D-eigenproblem, and it is a special case of the generalized tensor eigenvalue problem.

A key contribution of this work is the confirmation that the framework of the geometrical eigenproblem can be adjusted to involve tensors of a more general type than totally symmetric covariant ones. This adjustment broadens the scope of the geometrical eigenproblem, making it applicable to a wider range of mathematical and physical contexts.

Furthermore, the work includes specific observations on higher-order unit tensors, which are integral to the discussion of the geometrical eigenproblem. In the course of observing geometrical eigenproblem of even order, a particular tensor has been identified that shares many analogous properties with the identity mapping/unit matrix/Kronecker delta tensor. It is therefore justified to refer to it as the higher-order unit tensor or the higher-order Kronecker delta tensor. These observations provide additional insights into the structure of the geometrical eigenproblem, contribute to its global consideration over a Riemannian manifold, and enable the use of tensor analysis.

This research reveals several open questions and potential directions for further work.

2. Various eigenproblems in flat space

Eigenproblems in flat space have been discussed and explored extensively over the last two decades. These efforts have resulted in significant developments in theory, the discovery of new application domains, and the development of innovative solving procedures and algorithms. Only the details that are of current interest are presented here. A comprehensive theory can be found in the book [21] and the references therein.

2.1. Z-eigenvalue problem

It is defined generally and covers a wide range of applications, with many solving techniques having been developed. It concerns a tensor in a flat vector space, meaning that all objects are treated equally, with no distinction between covariant and contravariant nature. The resulting vector $\mathcal{A}z^{m-1}$ is required to be proportional to the initial vector, with an additional condition of normality:

$$\begin{cases} \mathcal{A}z^{m-1} = \lambda z \\ z^T z = 1. \end{cases} \tag{1}$$

Real solutions are of greater interest and are referred to as Z-eigenvalue λ and Z-eigenvector z, but the occurrence of complex solutions is possible, in which case they are referred to as E-eigenvalue and E-eigenvector. The Z-eigenvalue problem depends on the representation (on the chosen base in the vector space), but the corresponding Z-eigenvalues do not (they are invariant with respect to orthogonal transformations), and the corresponding Z-eigenvectors have practical significance. It is of interest to observe Z-spectra $\sigma_Z(\mathcal{A}) = \{\lambda \mid \mathcal{A}z^{m-1} = \lambda z\}$ and Z-spectral radius $\rho_Z(\mathcal{A}) = \max\limits_{\lambda \in \sigma_Z(\mathcal{A})} |\lambda|$. Inclusion sets specify localization of the Z-eigenvalues, making them very useful in solving procedures. However, the definition does not satisfy the homogeneity property, which means that eigenspaces cannot be considered.

2.2. Generalized tensor eigenvalue problem

It has in focus two higher order tensors in the same n-dimensional vector space, $\mathcal{A}$ of order m and $\mathcal{B}$ of order m'. The problem under consideration is defined by:

$$\begin{cases} \mathcal{A}z^{m-1} = \lambda \mathcal{B}z^{m'-1} \\ \mathcal{B}z^{m'} = 1, \end{cases} \tag{2}$$

with the assumption that both tensors $\mathcal{A}$ and $\mathcal{B}$ are regular, meaning that $\mathcal{A}z^{m-1} = 0$ and $\mathcal{B}z^{m'-1} = 0$ have no nonzero complex solutions. The requirement $\mathcal{B}z^{m'} = 1$ is optional, and may be omitted. The solutions are referred to as B-eigenvalue λ and B-eigenvector z. The generalized tensor eigenvalue problem is a vector equation, and it is invariant if tensors are treated as multilinear transformations. Hence, B-spectra $\sigma_B(\mathcal{A})$ and B-spectral radius $\rho_B(\mathcal{A})$ are determined in analogously with the Z-eigenproblem. In the case $m = m'$ the eigenproblem (2) is homogeneous and one-dimensional B-eigenspaces are determined as invariant subset spanned by B-eigenvectors, $S_\lambda = \{\alpha z \,|\, \alpha \in \mathbb{R}\}$.

2.3. D-eigenvalue problem

It is defined in accordance with the particular problem of spin-echo signal attenuation in an anisotropic biological medium (brain or neural tissue). The signal attenuation is determined by a totally symmetric fourth order diffusion kurtosis tensor, $\mathcal{W}$, and a diffusion tensor $\mathcal{D}$ (of second order, symmetric and positive definite) which describes structure of the tissue. The D-eigenvalue problem is given by:

$$\begin{cases} \mathcal{W}z^3 = \lambda \mathcal{D}z \\ \mathcal{D}z^2 = 1 \end{cases} \tag{3}$$

The D-eigenvalues λ and D-eigenvectors z of the diffusion kurtosis tensor are important parameters in medical imaging and in diagnosing neurological conditions through MRI-based techniques.

3. Eigenproblem in Riemannian space

Due to the invariance of the Z-eigenvalues and the widespread applicability of the Z-eigenproblem, it has been adjusted to non-Euclidean vector spaces. The geometrical eigenproblem also enables the observation over Riannian manifolds.

The content is consistent with the findings in [23]. After the briefly presented basics, the theory is expanded by exploring the characteristic polynomial of tensor and various equivalent forms of the geometric eigenproblem suitable for both local and global analysis and solving. As the framework of the geometrical eigenproblem is well established, it allows definition of geometrical eigenvalues, eigenvectors and eigencovectors of a higher order tensor of any type. Unit tensor of (higher) even order as a multilinear operator will be discussed.

3.1. Geometrical eigenproblem of covariant totally symmetric tensor

The geometrical eigenproblem is developed as an invariant and homogeneous extension of the Z-eigenproblem, extending from a flat vector space toward a Riemannian manifold (M, g). It is the only definition that distinguishes objects by nature, whether they are covariant or contravariant.

A framework for the geometrical eigenproblem, which will be the default in the sequel, consists of:

- an n-dimensional Riemannian manifold (M, g) with the tangent bundle TM and the cotangent bundle T^*M;
- the metric tensor field $g \in \mathcal{T}^0_2(M)$ having the corresponding inverse metric tensor field $g^{-1} \in \mathcal{T}^2_0(M)$;
- a collection of the symmetric bilinear forms - the metric g defines at each point $x \in M$ the scalar product $g : T_xM \times T_xM \to \mathbb{R}$, $(u, v) \mapsto \langle u, v\rangle = guv = g_{ij}u^iv^j$ where $g_{ij}(x)$ are the components of the metric tensor at the point x;
- a collection of norms - a vector $v \in T_xM$ has the norm $\|v\| = \sqrt{\langle v, v\rangle}$, i.e. $\|v\|^2 = gv^2 = g_{ij}v^iv^j$;
- natural isomorphism between the tangent and cotangent bundle - for a vector field $z \in \mathcal{T}^1_0(M)$ the contraction $z^\flat = gz$ produces corresponding covector field $z^\flat \in \mathcal{T}^0_1(M)$;
- musical isomorphism between tensor bundles - lowering ($\flat$) and raising ($\sharp$) indices of a tensor field by contraction with g and contraction with g^{-1} , respectively.

Geometrical eigenproblem is defined globally, over the manifold.

Definition 3.1. *The geometrical eigenproblem (G-eigenproblem) for an m-th order totally symmetric covariant tensor field $\mathcal{A} \in \mathcal{T}^0_m(M)$ is defined by :*

$$\mathcal{A}^\sharp z^{m-1} = \lambda \, ||z||^{m-2} \, z,$$

where $\mathcal{A}^\sharp = g^{-1}\mathcal{A}$, or equivalently

$$\mathcal{A}z^{m-1} = \lambda \, ||z||^{m-2} \, z^\flat. \tag{4}$$

Since the tensor field depends on the points of the manifold, solving the geometrical eigenproblem requires a local consideration. Let a point $x \in M$ be covered by a local chart where a tangent vector z on the manifold M at the point x has the coordinates $z = (z^1, z^2, \ldots, z^n)$, the metric tensor has components g_{ij} (and its inverse g^{ij}), and the tensor $\mathcal{A}$ has the components $A_{i_1 i_2 \cdots i_m}$, where $i_1, i_2 \cdots i_m \in \{1, 2, \cdots, n\}$. The multilinear mapping determined by the tensor at a point $x \in M$,

$$\mathcal{A} : T_xM \otimes \ldots \otimes T_xM \to \mathbb{R}, \quad \mathcal{A}(v_1, \ldots, v_m) = A_{i_1 i_2 \cdots i_m} v_1^{i_1} \cdots v_m^{i_m} \in \mathbb{R}$$

has the associated endomorphism

$$\mathcal{A}^\sharp : T_xM \to T_xM, \quad \mathcal{A}^\sharp(v) = g^{-1}\mathcal{A}v^{m-1} = \left(g^{ii_1} A_{i_1 i_2 \cdots i_m} v^{i_2} \cdots v^{i_m}\right).$$

The coordinate representation of the geometrical eigenproblem (4) within a Riemannian space $(T_xM, g(x))$ can be expressed in the covectorial form

$$A_{i_1 i_2 \cdots i_m} z^{i_2} \cdots z^{i_m} = \lambda \, ||z||^{m-2} \, g_{i_1 j} z^j, \tag{5}$$

or in the vectorial form

$$g^{i_1 j} A_{i_1 i_2 \cdots i_m} z^{i_2} \cdots z^{i_m} = \lambda \, ||z||^{m-2} \, z^j. \tag{6}$$

Definition 3.2. *The real solutions λ and z to the geometrical eigenproblem (5) and (6) are respectively called G-eigenvalue and associated G-eigenvector. The pair (λ, z) is referred to as G-eigenpair or G-eigendata of the tensor $\mathcal{A}$ at the point x.*

Spectra and spectral radius are defined analogously with the Z-case. In the even case ($m = 2l$), the geometrical eigenproblem is homogeneous, while in the odd case it is positively homogeneous, with eigenvalues occurring in pairs of mutually opposite values. Consequently, eigenspaces exist and are spanned by G-eigenvectors. Since the geometrical eigenproblem is homogeneous, it can be supplemented with a normalizing condition and further reduced to:

$$\begin{cases} \mathcal{A}z^{m-1} = \lambda z^\flat \\ ||z||^2 = 1. \end{cases} \tag{7}$$

By choosing a local representation such that the metric is Euclidean, the geometrical eigenproblem obviously reduces to the Z-eigenproblem (1). The equivalence of the Z-eigenproblem and the geometrical eigenproblem in the appropriate local representation directly leads to the concept of the E-characteristic polynomial.

Definition 3.3. *An univariate polynomial in λ is the E-characteristic polynomial of the tensor $\mathcal{A}$ in a Riemannian space with a metric tensor g if it is the resultant of the following homogeneous polynomial system:*

- *m-even:* $\mathcal{A}z^{m-1} - \lambda \, ||z||^{m-2} \, z^\flat = 0;$
- *m-odd:* $\begin{cases} \mathcal{A}z^{m-1} - \lambda t^{m-2} z^\flat = 0, \\ ||z||^2 - t^2 = 0. \end{cases}$

A real number is a zero of the E-characteristic polynomial if and only if it is a G-eigenvalue of the geometrical eigenproblem. The theory of resultants within algebraic geometry confirms the existence of the G-eigenvalues. The fundamental properties concerning invariance and homogeneity of the geometrical eigenproblem are omitted for brevity. They are exposed in [23], where the correspondence between the geometrical and the Z- eigenproblem is also given in detail.

There is a full analogy between (3) and (7). The diffusion tensor $\mathcal{D}$ represent the spatial feature, so it is reasonable to treat $(\mathbb{R}^3, \mathcal{D})$ as a Riemannian space. Therefore, D-eigenproblem is a proper geometrical eigenproblem of the fourth order diffusion kurtosis tensor $\mathcal{W}$ in the Riemannian space $(S, \mathcal{D})$, where $S \subset \mathbb{R}^3$ is an open domain.

The tensor equation that defines the geometrical eigenproblem, whether considered locally or globally, has several useful and favorable equivalent forms. These are presented in the following theorem, which is aligned with the previous notation.

Theorem 3.4. *Let $\mathcal{A}$ be an even order $(m = 2l)$ totally symmetric tensor field in a Riemannian manifold (M, g). Then:*

1. *The geometrical eigenproblem considered locally at a point (5) and (6), is equivalent with the two following homogeneous polynomial systems:*

$$g^{i_1 j}\left(A_{i_1 i_2 \cdots i_m} - \lambda g_{i_1 i_2} \cdots g_{i_{m-1} i_m}\right) z^{i_2} \cdots z^{i_m} = 0, \tag{8}$$

$$\left(A_{i_1 i_2 \cdots i_m} - \lambda g_{i_1 i_2} \cdots g_{i_{m-1} i_m}\right) z^{i_2} \cdots z^{i_m} = 0, \tag{9}$$

2. *The geometrical eigenproblem (4) has the following equivalent coordinate-independent forms:*

   a) *global vector equation*

$$\left(\mathcal{A}^\sharp - \lambda\, \mathcal{U}\right) z^{m-1} = 0, \tag{10}$$

   b) *global covector equation*

$$\left(\mathcal{A} - \lambda\, \flat\mathcal{U}\right) z^{m-1} = 0, \tag{11}$$

   where $\mathcal{U} \in \mathcal{T}^1_{m-1}(M)$ is the tensor field obtained by the tensor product of the Kronecker delta tensor[1)] $\delta \in \mathcal{T}^1_1(M)$ and multiple metric tensor fields g, and $\flat\mathcal{U} = g\mathcal{U}$, and the components are $U^j_{i_2 \cdots i_m} = \delta^j_{i_2} g_{i_3 i_4} \cdots g_{i_{m-1} i_m}$ and $(\flat U)_{i_1 i_2 \cdots i_m} = g_{i_1 i_2} \cdots g_{i_{m-1} i_m}$.

3. *The E-characteristic polynomial of the geometrical eigenproblem at a point is the resultant of the homogeneous polynomial systems (8) and (9).*

Proof. The proof relies on the local expression of the righthand side term

$$\|z\|^{m-2} = \left(g_{ij} z^i z^j\right)^{m-2} = g_{i_3 i_4} z^{i_3} z^{i_4} \cdots g_{i_{m-1} i_m} z^{i_{m-1}} z^{i_m} = g_{i_3 i_4} \cdots g_{i_{m-1} i_m} z^{i_3} z^{i_4} \cdots z^{i_{m-1}} z^{i_m}. \tag{12}$$

The substitution into (6) yields

$$g^{i_1 j} A_{i_1 i_2 \cdots i_m} z^{i_2} \cdots z^{i_m} - \lambda z^j g_{i_3 i_4} \cdots g_{i_{m-1} i_m} z^{i_3} z^{i_4} \cdots z^{i_{m-1}} z^{i_m} = 0.$$

The fact $z^j = \delta^j_{i_2} z^{i_2}$ further produces

$$g^{i_1 j} A_{i_1 i_2 \cdots i_m} z^{i_2} \cdots z^{i_m} - \lambda \delta^j_{i_2} g_{i_3 i_4} \cdots g_{i_{m-1} i_m} z^{i_2} z^{i_3} \cdots z^{i_m} = 0. \tag{13}$$

[1)] Kronecker delta tensor is a constant tensor field over the manifold

By replacing $\delta^j_{i_2} = g^{i_1 j} g_{i_1 i_2}$ into (13), one obtains (8). Associated coordinate-independent form of (13) is $\left(\mathcal{A}^\sharp - \lambda \mathcal{U}\right)(\cdot, z, \dots, z) = 0$, i.e. the relation (10).

The substitution of (12) into (5) gives

$$A_{i_1 i_2 \cdots i_m} z^{i_2} \cdots z^{i_m} - \lambda g_{i_1 j} z^j g_{i_3 i_4} \cdots g_{i_{m-1} i_m} z^{i_3} z^{i_4} \cdots z^{i_{m-1}} z^{i_m} g_{i i_2} z^{i_2} = 0.$$

By use of the equality $g_{i_1 j} z^j = g_{i_1 i_2} z^{i_2}$, the previous equation transforms to (9), whose coordinate-independent form is $(\mathcal{A} - \lambda \flat \mathcal{U})(\cdot, z, \dots, z) = 0$, i.e. the relation (11).

The third statement of the theorem is direct consequence of the first one. □

The distinguished tensor fields $\mathcal{U}$ and $\flat\mathcal{U}$ are additionally considered in the last subsection. The smoothness of the solutions is not considered at this stage of the research; therefore, the equations (10) and (11) cannot be referred to as vector and covector field equations. Tensor analysis can be applied to the equations (11) and (10), providing insights into the global solution to the geometrical eigenproblem. It is of interest to explore under what conditions the G-eigendata form scalar and vector fields over the manifold.

Relation (11) confirms that the geometrical eigenproblem is a specific case of generalized tensor eigenvalue problem (2) with $\mathcal{B} = \flat\mathcal{U}$.

3.2. *Geometrical eigenproblem of contravariant and mixed tensors*

A tensor $\mathcal{A}$ of any type in Riemannian space has a unique associate covariant tensor, denoted $\flat\mathcal{A}$, and obtained by the multiple contraction of the tensor products with the metric tensor. For example, if $\mathcal{A} \in \mathcal{T}^3_2$, then $\flat\mathcal{A} = ggg\mathcal{A}$ is covariant tensor of the fifth order. Hence, all further content refers to a tensor $\mathcal{A}$ of any (higher) order and any type, and its corresponding $\flat\mathcal{A}$. (If $\mathcal{A}$ is covariant, then $\flat\mathcal{A} = \mathcal{A}$.) Due to the natural isomorphism between tangent and cotangent bundles of a Riemannian manifold, the geometrical eigenproblem has a flexible approach to dual objects.

Definition 3.5. *Let $\mathcal{A}$ be a tensor in a Riemannian space with metric g, such that $\flat\mathcal{A}$ is totally symmetric. If (λ, z) is a G-eigenpair of the tensor $\flat\mathcal{A}$, then:*

- *the covector $\omega = z^\flat = gz$ is called a G-eigencovector of the tensor $\flat\mathcal{A}$.*
- *the scalar λ is called a G-eigenvalue, and z and ω are called the corresponding G-eigenvector and G-eigencovector of the tensor $\mathcal{A}$.*

Therefore, if (λ, z, ω) is a G-eigendata of the tensor $\mathcal{A}$, one can write:

- for the covariant tensor: $\mathcal{A}z^{m-1} = \lambda\omega$ and $\mathcal{A}z^m = \lambda$
- for the contravariant tensor: $\mathcal{A}\omega^{m-1} = \lambda z$ and $\mathcal{A}\omega^m = \lambda$
- for the tensor that is p times contravariant and $m-p$ times covariant: $\mathcal{A}z^{m-p-1}\omega^p = \lambda z$ and $\mathcal{A}z^{m-p}\omega^{p-1} = \lambda\omega$ and $\mathcal{A}z^{m-p}\omega^p = \lambda$.

The geometrical eigenproblem also extends to cases where $\flat\mathcal{A}$ is not totally symmetric. The following definition is consistent with the notion of the k-mode Z-eigenvalues in flat space as presented in [16], and generalizes it.

Definition 3.6. *The k-mode geometrical eigenproblem of an arbitrary tensor $\mathcal{A}$ of order m and any type in a Riemannian space with metric tensor g is given by:*

$$(\flat A)_{i_1 i_2 \cdots i_m} z^{i_1} \cdots z^{i_{k-1}} z^{i_{k+1}} \cdots z^{i_m} = \lambda \, \|z\|^{m-2} \, z^a g_{a i_k}. \tag{14}$$

The solution to this problem is the k-mode G-eigendata that consists of the k-mode G-eigenvalue λ and the k-mode G-eigenvector z. The corresponding covector $\omega = gz$ is the k-mode G-eigencovector. If (λ, z) is the k-mode G-eigendata for all modes $k \in \{1, 2, \dots, m\}$, then (λ, z) is called the G-eigendata of the tensor $\mathcal{A}$.

3.3. On the higher order unit tensor

This consideration is motivated by the analogy between the classical eigenproblem related with matrices and the relation (10). To increase the consistency of the subsection, detailed definition of the tensor field $\mathcal{U}$ mentioned in Theorem 3.4 is provided.

Definition 3.7. *Let (M, g) be a Riemannian manifold, and $m = 2l$ for $l \geq 1$. The tensor fields of the m-th order obtained by the tensor products of the Kronecker delta tensor and the metric tensor field g, will be denoted as follows:*

$$\mathcal{U} = \delta \otimes \underbrace{g \otimes \cdots \otimes g}_{l-1}, \qquad \mathcal{U} \in \mathcal{T}^1_{m-1}(M),$$

$$\flat\mathcal{U} = \underbrace{g \otimes \cdots \otimes g}_{l}, \qquad \flat\mathcal{U} \in \mathcal{T}_m(M).$$

The tensor field $\mathcal{U}$ and its corresponding flat tensor field $\flat\mathcal{U}$ smoothly depend on the points of the Riemaniann manifold. The components of the associated tensors at a point $x \in M$ are:

$$U^i_{i_2 \cdots i_m} = \delta^i_{i_2} g_{i_3 i_4} \cdot \ldots \cdot g_{i_{m-1} i_m} \quad \text{and} \quad (\flat U)_{i_1 i_2 \cdots i_m} = g_{i_1 i_2} \cdot \ldots \cdot g_{i_{m-1} i_m}.$$

The multilinear mapping and the associated endomorphism determined at a point $x \in M$ by the tensor fields $\flat\mathcal{U}$ and $\mathcal{U}$ are given by:

$$\flat\mathcal{U} : T_xM \otimes \ldots \otimes T_xM \to \mathbb{R}, \quad \flat\mathcal{U}(v_1, \ldots, v_m) = g_{i_1 i_2} v_1^{i_1} v_2^{i_2} \cdot \ldots \cdot g_{i_{m-1} i_m} v_{m-1}^{i_{m-1}} v_m^{i_m} \in \mathbb{R} \tag{15}$$

and

$$\mathcal{U} : T_xM \to T_xM, \quad \mathcal{U}(v) = \mathcal{U}v^{m-1} = \mathcal{U}(\cdot, v, \ldots, v) = \delta^i_{i_2} v^{i_2} \cdot g_{i_3 i_4} v^{i_3} v^{i_4} \cdot \ldots \cdot g_{i_{m-1} i_m} v^{i_{m-1}} v^{i_m} = \|v\|^{m-2} \cdot v. \tag{16}$$

Important features of the tensor field $\mathcal{U}$ and its associated covariant $\flat\mathcal{U}$ are outlined in the following theorem, consistent with the previous notation.

Theorem 3.8. *Let (M, g) be a Riemannian manifold with the tensor field $\mathcal{U}$ of even order $m = 2l$, $l > 1$. Then:*

1. *The tensor $\flat\mathcal{U}$ is not totally symmetric, but it is symmetric in pairs.*
2. *The multilinear mapping $\flat\mathcal{U}$ is a multilpe scalar product:*

$$\flat\mathcal{U}(v_1, v_2, \ldots, v_m) = \langle v_1, v_2 \rangle \cdots \langle v_{m-1}, v_m \rangle.$$

For a single argument it produces powered norm,

$$\flat\mathcal{U}(v, v, \ldots, v) = \mathcal{U}v^m = \|v\|^m$$

3. *The restriction of the endomorphism $\mathcal{U}$ on the unit sphere $S = \{v \mid \|v\| = 1\} \subset T_x(M)$ coincides with the identical mapping $I : S \to S$, $I(v) = (\delta^i_j v^j) = v$.*
4. *The tensors $\mathcal{U}$ and $\flat\mathcal{U}$ have the G-eigenvalue $\lambda = 1$ and all vectors are corresponding G-eigenvectors.*
5. *In the local representation where the metric is Euclidean, the components of the both tensors $\mathcal{U}$ and $\flat\mathcal{U}$ form a hypermatrix with 1 on the main diagonal and 0 elsewhere*[2)]*.*

[2)] This hypermatrix is commonly used in the literature as the higher-order unit tensor.

6. *Let $\mathcal{A}$ be a covariant totally symmetric tensor field of the same order m. Then the E-characteristic polynomial of $\mathcal{A}$ is influenced by $\mathcal{U}$, it coincides with the resultant of the homogeneous polynomial system $\left(\mathcal{A}^\sharp - \lambda\mathcal{U}\right) z^{m-1} = 0$. A scalar λ is G-eigenvalue of $\mathcal{A}$ if and only if $\left(\mathcal{A}^\sharp - \lambda\mathcal{U}\right) z^{m-1}$ has a nonzero solution.*

Proof.

1. The symmetry within each pair of indices arises from the symmetry of the metric tensor. The right-hand side of equation (15) is determined by the multiplication of k terms, which is commutative, yielding symmetry between pairs.
2. Both equalities are immediate consequences of relation (15) when considering the definitions of scalar product and norm.
3. The statement is a direct consequence of the relation (16).
4. It is evident that any vector z is a solution to (10) when $\mathcal{A}^\sharp = \mathcal{U}$ (i.e. $\mathcal{A} = \flat\mathcal{U}$) and $\lambda = 1$.
5. If $g_{ij} = \delta_{ij}$, then
$$U^j_{i_2\cdots i_m} = \delta^j_{i_2} g_{i_3 i_4} \cdots g_{i_{m-1} i_m} = \begin{cases} 1, & j = i_2 = \cdots = i_m, \\ 0, & \text{otherwise}, \end{cases}$$
and
$$(\flat U)_{i_1 i_2 \cdots i_m} = g_{i_1 i_2} \cdots g_{i_{m-1} i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}$$
6. This is simply a reformulation of statements 1 and 3 in Theorem 3.4.

□

Given the obvious benefits of the tensor $\mathcal{U}$ and its analogies to the identity mapping and the Kronecker delta tensor, it could be referred to as the higher-order unit tensor or the higher-order Kronecker delta tensor.

Just as the characteristic matrix $A - \lambda I$ captures essential properties of a matrix A, the tensor field $\mathcal{A}^\sharp - \lambda\mathcal{U}$ (and $\mathcal{A} - \lambda\flat\mathcal{U}$) could provide valuable information on G-eigendata of the tensor field $\mathcal{A}$, or of the tensor at each point on the manifold. Tensor analysis tools contribute for the computation of G-eigendata in a coordinate-independent manner, ensuring that the results are intrinsic to the manifold. Furthermore, G-eigendata are closely related with the best rank-one approximation and tensor decomposition. Therefore, it is beneficial to use tensor analysis within higher order tensor spectral analysis for a comprehensive understanding of a tensor field.

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