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The existence and multiplicity of positive solutions for a conformable fractional boundary value problem

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Abstract. This work will investigate the existence of positive solutions for a fractional boundary value problem, including a conformable fractional differential equation with an integral boundary condition. To reach the desired results, **Krasnoselskii's, Schaefer and Legget-Williams fixed-point theorems on cones will be used.** Some examples will be given to point out the results.

1. Introduction

The theory of fractional differential equations has received much attention over the past years due to its numerous applications in a great number of areas as control theory, signal and image processing, physics, and even chemistry and economics. Such applications have been described in a number of monographs, see [2–8, 23, 25–28, 32, 33, 37].
Numerous operators such as Riemann-Liouville, Caputo, Grünwald–Letnikov and etc., have been introduced to generalize the derivative of the natural order to the derivative of the fractional order. Recently, the new definition of a conformable fractional derivative, given by [1, 24, 36, 38–40], has drawn much interest from many mathematicians and scientists. In the recent years, various papers have been presented about the fractional boundary value problems (see [9–11, 13–15, 17, 19, 30, 34, 35, 43]. However, few papers have been published about the existence and uniqueness of the solutions and positive solutions of differential equations with conformable fractional derivatives [12, 16, 18, 22, 41].

Recently, in [42], authors discussed the existence of positive solutions for the following problem (with conformable derivative):

$$\begin{aligned} &D^{\vartheta}\chi(\tau) + \varphi(\tau, \chi(\tau)) = 0, \quad \tau \in [0,1]; \vartheta \in (1,2], \\ &\chi(0) = 0; \chi(1) = \lambda \int_0^1 \chi(\tau)d\tau, \end{aligned} \tag{1}$$

where $D^{\vartheta}\chi(\tau)$ denotes the conformable fractional derivative of a function χ of order ϑ, and $\varphi : [0;1]\times[0;1) \to [0;1)$ is a continuous function. Employing a fixed point theorem in a cone, they established some criteria for the existence of at least one positive solution.

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Very recently in [21], by using some fixed point theorems on cones, Faouzi Haddouchi obtained results about the existence of positive solutions of the following boundary value problem including fractional differential equation with conformable derivative.

$$D^{\vartheta}\chi(\tau) + \varphi(\tau, \chi(\tau)) = 0, \quad \tau \in [0, 1]; \vartheta \in (1, 2],$$
$$\chi(0) = 0; \chi(1) = \lambda \int_0^{\zeta} \chi(\tau)d\tau, \quad 0 < \zeta < 1,$$

where, $\varphi : [0; 1] \times [0; 1) \to [0; 1)$ is a continuous function. **It is mentioned that the integral boundary condition of this problem depends on the parameter ζ, while the limits of the integral boundary condition of the problem (1) are constant.**

With inspiration from the above works, in this paper existence and multiplicity of solutions for the following boundary value problem will be discussed:

$$D^{\vartheta}\chi(\tau) + \theta(\tau)\varphi(\tau, \chi(\tau)) = 0, \quad \tau \in (0, 1),$$
$$\chi(0) = x'(0) = 0, \quad \chi(1) = \lambda \int_0^{\zeta} \chi(\tau)d\tau, \tag{2}$$

where $2 < \vartheta \leq 3$, $0 < \zeta < 1$, D^{ϑ} is the conformable fractional derivative of χ at τ of order ϑ, $\varphi \in C([0, 1] \times [0, \infty), [0, \infty))$, $\theta \in C([0, 1], [0, \infty))$ with some property and $\lambda < \frac{3}{\zeta^3}$.

The rest of the paper is organized as: In section 2, some preliminary facts that will be used in this paper will be presented. In section 3, the green function of (2) will be computed and some aspects of it will be proved. In section 4, by using some fixed point theorems, our existence results will be proved. Finally, In section 5, two examples will be presented to make clear the existence theorems.

2. Preliminaries

In this section, we present some definitions and properties of fractional derivatives that will be helpful throughout the paper and can be found in [1, 24].

Definition 2.1. *Let $0 < \vartheta \leq 1$, the conformable derivative of a mapping $\varphi : [0, \infty) \to \mathbb{R}$ of order ϑ is presented by*

$$D^{\vartheta}\varphi(\tau) = \lim_{\epsilon \to 0} \frac{\varphi(\tau + \epsilon\tau^{1-\vartheta}) - \varphi(\tau)}{\epsilon}. \tag{3}$$

If $D^{\vartheta}\varphi(\tau)$ exists on $(0, b)$, then $D^{\vartheta}\varphi(0) = \lim_{t\to 0} D^{\vartheta}\varphi(\tau)$.

Definition 2.2. *Let $n < \vartheta \leq n + 1$, the conformable derivative for $\varphi : [0, \infty) \to \mathbb{R}$ defined by*

$$D^{\vartheta}\varphi(\tau) = D^{\beta}\varphi^{(n)}(\tau),$$

where $\beta = \vartheta - n$.

Definition 2.3. *Let $n < \vartheta \leq n + 1$, the conformable integral for $\varphi : [0, \infty) \to \mathbb{R}$ of order ϑ is defined by*

$$I^{\vartheta}\varphi(\tau) = \frac{1}{n!} \int_0^t (\tau - \varsigma)^n \varsigma^{\vartheta-n-1} \varphi(\varsigma)d\varsigma.$$

Lemma 2.4. *Let $\vartheta \in (n, n + 1]$. If φ is a continuous function on $[0, \infty)$, then for all $t > 0$, $D^{\vartheta}I^{\vartheta}\varphi(\tau) = \varphi(\tau)$.*

Lemma 2.5. *Let $\vartheta \in (n, n + 1]$, then $D^{\vartheta}\tau^k = 0$ for $\tau \in [0, 1]$ and $k = 1, 2, \ldots, n$.*

Lemma 2.6. *(See [37]) Let $\vartheta \in (n, n+1]$. If $D^{\vartheta}\varphi(\tau)$ is continuous on $[0,\infty)$, then*

$$I^{\vartheta}D^{\vartheta}\varphi(\tau) = \varphi(\tau) + C_1 + C_2\tau^2 + \cdots + C_n\tau^n,$$

for some real numbers.

Throughout this paper, we assume that $\mathcal{B}$ is equal to Banach space of all real continuous functions on $[0,1]$ which is equipped with the following supremum norm

$$\|\chi\| = \sup_{\tau\in[0,1]} |\chi(\tau)|.$$

Definition 2.7. *Assume $\wp \subset \mathcal{B}$ be a nonempty, closed, convex set and $\lambda \geq 0$, $\wp$ is called a cone if*

(i) $\chi \in \wp$ implies $\lambda\chi \in \wp$;

(ii) $\chi \in \wp$ and $-\chi \in \wp$ implies $\chi = 0$.

Also considering c be arbitrary number we define $\wp_c$ as follows:

$$\wp_c = \{\chi \in K : \|\chi\| < c\}. \tag{4}$$

Theorem 2.8. *(Krasnoselskii's)[29] Assume $\wp \subset B$ be a cone and let Ω_1 and Ω_2 be open subsets of $\mathcal{B}$ such that $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, also let*

$$\Im : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P,$$

be a completely continuous operator with the following properties:

(i) $\|\Im\chi\| \leq \|\chi\|$, $\chi \in \wp \cap \partial\Omega_1$, and $\|\Im\chi\| \geq \|\chi\|$, $\chi \in \wp \cap \partial\Omega_2$; or

(ii) $\|\Im\chi\| \geq \|\chi\|$, $\chi \in \wp \cap \partial\Omega_1$, and $\|\Im\chi\| \leq \|\chi\|$, $\chi \in \wp \cap \partial\Omega_2$.

Then $\Im$ has a fixed point in $\wp \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.9. *(Schaefer)[20] Suppose E be a closed convex subset of Bnach space $\mathcal{B}$ and U an open subset of E and $0 \in U$. If $\Im : \overline{U} \to E$ be a continuous, compact, then either*

(i) $\Im$ has a fixed point in $\overline{U}$,

or

(ii) There exist $\chi \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda\Im(\chi)$.

Theorem 2.10. *(Legget-Williams) [31] Suppose $\wp$ is a cone and $\Im : \overline{\wp}_c \to \overline{\wp}_c$ be a completely continuous operator and ψ a nonnegative continuous concave on cone $\wp$ with $\psi(\chi) \leq \|\chi\|$ for χ in $\overline{\wp}_c$. Assume that there exist constants $0 < a < b < d \leq c$ with the following property:*

(i) $\{\chi \in P(\psi, b, d) : \psi(\chi) > b\} \neq 0$ and $\psi(\Im\chi) > b$ if $\chi \in P(\psi, b, d)$,

(ii) $\|\Im\chi\| < a$ if $\chi \in P_a$,

(iii) $\psi(\Im\chi) > b$ for $\chi \in P(\psi, b, c)$ with $\|\Im\chi\| > d$.

Then, there exist some fixed points χ_1, χ_2 and χ_3 for χ such that $\|\chi_1\| < a$, $b < \psi(\chi_2)$ and $\|\chi_3\| > a$ with $\psi(\chi_3) < b$.

3. Green Function and Bounds

Green's function plays the most fundamental role in applying fixed point theorems for differential equations operators, so we first introduce this function and investigate its characteristics.

Lemma 3.1. *Let $\varrho : [0,1] \to [0,\infty)$ be a continuous function, then the FBVP*

$$\begin{aligned} &D^{\vartheta}\chi(\tau) + \varrho(\tau) = 0, \quad \tau \in (0,1), \quad 2 < \vartheta \leq 3, \\ &\chi(0) = \chi'(0) = 0, \chi(1) = \int_0^{\zeta} \chi(\tau)d\tau, \quad 0 < \zeta < 1, \end{aligned} \tag{5}$$

is equivalent with

$$\chi(\tau) = \int_0^1 G(\tau,\varsigma)\varrho(\varsigma)d\varsigma, \tag{6}$$

where

$$G(\tau,\varsigma) = K(\tau,\varsigma) + \frac{\lambda\tau^2}{6-2\lambda\zeta^3}H(\zeta,\varsigma), \tag{7}$$

$$K(\tau,\varsigma) = \begin{cases} \frac{1}{2}[\tau^2(1-\varsigma)^2\varsigma^{\vartheta-3} - (\tau-\varsigma)^2\varsigma^{\vartheta-3}], & 0 \leq \varsigma \leq \tau \leq 1, \\ \frac{1}{2}\tau^2(1-\varsigma)^2\varsigma^{\vartheta-3}, & 0 \leq \tau \leq \varsigma \leq 1, \end{cases} \tag{8}$$

$$H(\zeta,\varsigma)\begin{cases} \zeta^3(1-\varsigma)^2\varsigma^{\vartheta-3} - (\zeta-\varsigma)^3\varsigma^{\vartheta-3}, & 0 \leq \varsigma \leq \zeta \leq 1, \\ \zeta^3(1-\varsigma)^2\varsigma^{\vartheta-3}, & 0 \leq \zeta \leq \varsigma \leq 1. \end{cases} \tag{9}$$

Proof. By integrating from equation (5), we get

$$\chi(\tau) = -I^{\vartheta}\varrho(\tau) + c_0 + c_1\tau + c_2\tau^2.$$

By applying the first and second boundary conditions one can see $c_0 = c_1 = 0$. By FBVP (3) we have

$$\chi(1) = -I^{\vartheta}h(1) + c_2 = \lambda\int_0^{\zeta}\chi(\tau)d\tau,$$

and so

$$c_2 = I^{\vartheta}h(1) + \lambda\int_0^{\zeta}\chi(\varsigma)d\varsigma = \frac{1}{2}\int_0^1(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma + \lambda\int_0^{\zeta}\chi(\varsigma)d\varsigma.$$

Consequently

$$\begin{aligned} \chi(\tau) &= -\frac{1}{2}\int_0^{\tau}(\tau-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma + \frac{1}{2}\tau^2\int_0^1(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma \\ &\quad +\lambda\tau^2\int_0^{\zeta}\chi(\varsigma)d\varsigma. \end{aligned} \tag{10}$$

Now one can integrate the relation (10) from 0 to ζ and get

$$\begin{aligned} \int_0^{\zeta}\chi(\tau)d\tau &= -\frac{1}{2}\int_0^{\zeta}\int_0^{t}(\tau-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma d\tau \\ &\quad +\frac{1}{2}\frac{\zeta^3}{3}\int_0^1(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma \\ &\quad +\frac{\lambda\zeta^3}{3}\int_0^{\zeta}\chi(\tau)d\tau. \end{aligned}$$

So

$$\begin{aligned}(1-\frac{\lambda\zeta^3}{3})\int_0^{\zeta}\chi(\tau)d\tau &= -\frac{1}{2}\int_0^{\zeta}\int_{\varsigma}^{\zeta}(\tau-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\tau d\varsigma\\ &\quad+\frac{1}{6}\zeta^3\int_0^1(1-\varsigma)^2\varsigma^{\vartheta-3}d\varsigma\\ &= -\frac{1}{6}\int_0^{\zeta}(\zeta-\varsigma)^3\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma\\ &\quad+\frac{1}{6}\zeta^3\int_0^1(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma,\end{aligned}$$

and

$$\begin{aligned}\int_0^{\zeta}\chi(\tau)d\tau &= -\frac{1}{6-2\lambda\zeta^3}\int_0^{\zeta}(\zeta-\varsigma)^3\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma\\ &\quad+\frac{1}{6-2\lambda\zeta^3}\int_0^1\zeta^3(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma.\end{aligned}$$

Hence

$$\begin{aligned}\chi(\tau) &= -\frac{1}{2}\int_0^{\tau}(\tau-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma+\frac{1}{2}\int_0^1\tau^2(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma\\ &\quad-\frac{\lambda\tau^2}{6-2\lambda\zeta^3}\int_0^{\zeta}(\zeta-\varsigma)^3\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma\\ &\quad+\frac{\lambda\tau^2}{6-2\lambda\zeta^3}\int_0^1\zeta^3(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma\\ &= \int_0^1 K(\tau,\varsigma)\varrho(\varsigma)d\varsigma+\frac{\lambda\tau^2}{6-2\lambda\zeta^3}\int_0^1 H(\zeta,\varsigma)\varrho(\varsigma)d\varsigma\\ &= \int_0^1 G(\tau,\varsigma)\varrho(\varsigma)d\varsigma.\end{aligned}$$

□

Lemma 3.2. *The functions $K(\tau,\varsigma)$ and $H(\tau,\varsigma)$ defined by relations* (8) *and* (9) *are continuous in* $[0,1]\times[0,1]$ *and also satisfies the following:*

1. $K(\tau,\varsigma)\geq 0, H(\zeta,\varsigma)\geq 0$ *for all* $\tau,\varsigma,\zeta\in[0,1]$;
2. $\frac{1}{2}\tau^2(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3}\leq K(\tau,\varsigma)\leq\varsigma^{\vartheta-3}(1-\varsigma)^2$ *for all* $(\tau,\varsigma)\in[0,1]\times[0,1]$;
3. $\zeta^3(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3}\leq H(\zeta,\varsigma)\leq 2\zeta^3(1-\varsigma)^2\varsigma^{\vartheta-3}$ *for all* $(\tau,\varsigma)\in[0,1]\times[0,1]$.

Proof. The continuity of functions $K(\tau,\varsigma)$ and $H(\tau,\varsigma)$ is obvious. For proving statement (1) at first assume $0\leq\varsigma\leq\tau\leq 1$, so

$$\begin{aligned}K(\tau,\varsigma) &= \frac{1}{2}[\tau^2(1-\varsigma)^2-(\tau-\varsigma)^2]\varsigma^{\vartheta-3}\geq\frac{1}{2}[\tau^2(1-\varsigma)^2-(\tau-\tau\varsigma)^2]\varsigma^{\vartheta-3}\\ &= \frac{1}{2}[\tau^2(1-\varsigma)^2-(1-\varsigma)^2]\varsigma^{\vartheta-3}=0.\end{aligned}$$

For $\varsigma\geq\tau$, clearly

$$K(\tau,\varsigma)=\frac{1}{2}[\tau^2(1-\varsigma)^2]\varsigma^{\vartheta-3}\geq 0,$$

so,

$$K(\tau,\varsigma)\geq 0,\quad \text{for all}\quad \tau,\varsigma\in[0,1].$$

Now let $0\leq\varsigma\leq\tau\leq 1$, we have

$$\begin{aligned}H(\zeta,\varsigma) &= [\zeta^3(1-\varsigma)^2-(\zeta-\varsigma)^3]\varsigma^{\vartheta-3}\geq[\zeta^3(1-\varsigma)^2-(\zeta-\zeta\varsigma)^3]\varsigma^{\vartheta-3}\\ &= \zeta^3[(1-\varsigma)^2-(1-\varsigma)^3]\varsigma^{\vartheta-3}=0.\end{aligned}$$

For $\varsigma\geq\zeta$, clearly

$$H(\zeta,\varsigma)=[\zeta^3(1-\varsigma)^2]\varsigma^{\vartheta-3}\geq 0.$$

So it is concluded that $H(\tau,\varsigma)\geq 0$, for all $\tau,\varsigma\in[0,1]$.
For proving statement (2) let $0<\varsigma\leq\tau$, then

$$\begin{aligned}K(\tau,\varsigma) &= \frac{1}{2}[\tau^2(1-\varsigma)^2-(\tau-\varsigma)^2]\varsigma^{\vartheta-3}\geq\frac{1}{2}[\tau^2(1-\varsigma)^2-(\tau-\varsigma)(\tau-\tau\varsigma)]\varsigma^{\vartheta-3}\\ &\geq \frac{1}{2}[\tau^2(1-\varsigma)^2-\tau(\tau-\varsigma)(1-\varsigma)]\varsigma^{\vartheta-3}=\frac{1}{2}\tau(1-\varsigma)\varsigma^{\vartheta-3}[\tau(1-\varsigma)-(\tau-\varsigma)]\\ &= \frac{1}{2}\tau(1-\tau)(1-\varsigma)\varsigma^{\vartheta-3}\geq\frac{1}{2}\tau^2(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3}.\end{aligned}$$

Also for $\tau\geq\varsigma$ clearly $K(\tau,\varsigma)\geq\frac{1}{2}\tau^2(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-2}$.

Moreover, for $0<\varsigma\leq\tau$, we get

$$\begin{aligned}K(\tau,\varsigma) &= \frac{1}{2}[\tau^2(1-\varsigma)^2-(\tau-\varsigma)^2]\varsigma^{\vartheta-3}=\frac{1}{2}[\tau^2(1-\varsigma)^2+(\tau-\tau\varsigma)^2]\varsigma^{\vartheta-3}\\ &\leq \tau^2(1-\varsigma)^2\varsigma^{\vartheta-3}\leq(1-\varsigma)^2\varsigma^{\vartheta-3},\end{aligned}$$

and clearly for $\tau\geq\varsigma$ clearly we have $K(\tau,\varsigma)\leq(1-\varsigma)^2\varsigma^{\vartheta-3}$. Consequently for all $\tau,\varsigma\in[0,1]$ we have

$$\frac{1}{2}\tau^2(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3}\leq K(\tau,\varsigma)\leq(1-\varsigma)^2\varsigma^{\vartheta-3}.$$

Next we prove statement (3), for $\varsigma\leq\zeta$ we have

$$\begin{aligned}H(\zeta,\varsigma) &= [\zeta^3(1-\varsigma)^2-(\zeta-\varsigma)^3]\varsigma^{\vartheta-3}\\ &\geq [\zeta^3(1-\varsigma)^2-(\zeta-\varsigma)(\zeta-\zeta\varsigma)^2]\varsigma^{\vartheta-3}\\ &= \zeta^2[(1-\varsigma)\zeta^2-(\zeta-\varsigma)(1-\varsigma)^2]\varsigma^{\vartheta-3}=\zeta^2(1-\varsigma)^2\varsigma^{\vartheta-3}\\ &\geq \zeta^3(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3},\end{aligned}$$

and for $\zeta\geq\varsigma$ we have

$$H(\zeta,\varsigma)=\zeta^3(1-\varsigma)^2\varsigma^{\vartheta-3}\geq\zeta^3(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3}.$$

On the other hand for $\varsigma\leq\zeta$ we have

$$\begin{aligned}H(\zeta,\varsigma) &= [\zeta^3(1-\varsigma)^2-(\zeta-\varsigma)^3]\varsigma^{\vartheta-3}=[\zeta^3(1-\varsigma)^2+(\zeta-\zeta\varsigma)^3]\varsigma^{\vartheta-3}\\ &\leq (1-\varsigma)^2 2\zeta^3\varsigma^{\vartheta-3},\end{aligned}$$

and clearly for $\zeta\geq\varsigma$ we have $H(\zeta,\varsigma)\leq(1-\varsigma)^2 2\zeta^3\varsigma^{\vartheta-3}$.

□

Lemma 3.3. *Suppose $\xi\in(0,1)$ and $G(\tau,\varsigma)$ is the function defined by (7), then*

1. $G(\tau,\varsigma)\geq 0$, *for all* $\tau,\varsigma\in[0,1]$,

2. $\tau^2(1-\tau)\left(\frac{3}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3} \le G(\tau,\varsigma) \le \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}$ *for all* $0 \le \tau,\varsigma \le 1$,
3. $\xi^2(1-\xi)\left(\frac{3}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3} \le G(\tau,\varsigma) \le \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}$, *for all* $(\tau,\varsigma) \in [0,\xi]\times[0,1]$.

Proof. Statement (1) can be proven easily and statement (3) can be concluded directly from (2). So we need olny prove statmnet (2). In view of Lemma 3.2 and relation (7), it is concluded that

$$\begin{aligned} G(\tau,\varsigma) &= K(\tau,\varsigma) + \frac{\lambda\tau^2}{6-2\lambda\zeta^3}H(\zeta,\varsigma) \\ &\le (1-\varsigma)^2\varsigma^{\vartheta-3} + \frac{2\lambda\zeta^3}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3} \\ &= \left(\frac{6}{6-2\lambda\zeta^3}\right)\varsigma^{\vartheta-3}(1-\varsigma)^2. \end{aligned}$$

Moreover from Lemma 3.2, we get

$$\begin{aligned} G(\tau,\varsigma) &= K(\tau,\varsigma) + \frac{\lambda\tau^2}{6-2\lambda\zeta^3}H(\tau,\varsigma) \\ &\ge \frac{1}{2}\tau^2(1-\varsigma)^2\varsigma^{\vartheta-3} + \frac{\lambda\tau^2}{6-2\lambda\zeta^3}\zeta^3(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-2} \\ &\ge \tau^2(1-\tau)\left[\frac{1}{2} + \frac{\lambda\zeta^3}{6-2\lambda\zeta^3}\right](1-\varsigma)^2\varsigma^{\vartheta-3} \\ &= \frac{3}{6-2\lambda\zeta^3}\tau^2(1-\tau)(1-\varsigma)^2\varsigma^{\vartheta-3}. \end{aligned}$$

□

Lemma 3.4. *Let* $\varrho : [0,1] \to [0,\infty)$ *be a continuous function and* $0 < \xi < 1$*, then FBVP* (5) *has a unique non-negative solution* χ *with*

$$\min_{\tau\in(0,\xi)} \chi(\tau) \ge \mathbf{X}\|\chi\|,$$

where $\mathbf{X} = \frac{1}{2}\xi^2(1-\xi)$.

Proof. It is concluded directly from lemma 3.1 and Lemma 3.3 that $\chi(\tau)$ is a positive. Let $\tau \in [0,1]$, we have

$$\begin{aligned} \chi(\tau) &= \int_0^1 G(\tau,\varsigma)\varrho(\varsigma)d\varsigma \\ &\le \int_0^1 \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma, \end{aligned}$$

then

$$\|u\| \le \int_0^1 \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma. \tag{11}$$

On the other hand, for any $\tau \in [0,\xi]$, in view of Lemmas 3.1, 3.3 and relation (11), we get

$$\begin{aligned} \chi(\tau) &= \int_0^1 G(\tau,\varsigma)\varrho(\varsigma)d\varsigma \\ &\ge \int_0^1 \frac{3}{6-2\lambda\zeta^3}\xi^2(1-\xi)(1-\varsigma)^2\varsigma^{\vartheta-3}d\varsigma \\ &= \frac{1}{2}\xi^2(1-\xi)\int_0^1 \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}\varrho(\varsigma)d\varsigma \\ &\ge \mathbf{X}\|u\|, \end{aligned}$$

therefore

$$\min_{\tau\in[0,\xi]} \chi(\tau) \geq \mathbf{X}\|u\|.$$

□

4. Existence and Multiplicity Results

Now we are ready to present some existence and multiplicity results about FBVP (2). In this section we assume $\xi \in (0,1)$ be fixed, and consider the set of all functions $\chi \in \mathcal{B}$ such that $\min_{\tau\in[0,\xi]} \chi(\tau) \geq \mathbf{X}\|u\|$. It is easy to see that this set is a cone, we show this set by $\wp$. Now we define the operator $\mathfrak{I} : \wp \to \wp$ by

$$\mathfrak{I}\chi(\tau) = \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma, \tag{12}$$

where $G(\tau,\varsigma)$ is the same as appeared in (12). Clearly, the fixed points of the operator $\mathfrak{I}$ in $\wp$ are the solutions to the problem (2). We have the following items for problem (2):

(A1) $\varphi \in C([0,1]\times[0,\infty),[0,\infty))$ with $\varphi(\tau,0) \not\equiv 0$;

(A2) $\theta \in C([0,1],[0,\infty)$ and $\theta(\tau) \not\equiv 0$ on $[0,1]$.

Theorem 4.1. *Considering (A1) and (A2), the operator $\mathfrak{I}$, is completely continuous and $\mathfrak{I}\wp \subset \wp$.*

Proof. Given that φ is continuous (According to condition (A1)), by Lemma 3.4 we get $\mathfrak{I}\wp \subset \wp$. By (A1) and non negativeness and continuity of $G(\tau,\varsigma)$ and utilizing Lebesgue's dominated convergence theorem, it obtains that $\mathfrak{I} : \wp \to \wp$. If Ω be a bounded set in $\wp$. Then, $\exists\, M > 0$ with $\Omega \subset \{\chi \in \wp : \|\chi\| < M\}$. Put

$$\gamma = \max\{\varphi(\tau,\chi) : \tau \in [0,1], u \in \Omega\}.$$

From Lemma 3.1 and 3.2, we obtain

$$\begin{aligned}
\mathfrak{I}\chi(\tau) &= \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\
&\leq \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\gamma d\varsigma \\
&\leq \gamma\left(\int_0^1 \theta(\varsigma)d\varsigma\right)\int_0^1 \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}d\varsigma.
\end{aligned}$$

Hence, $\mathfrak{I}(\Omega)$ is uniformly bounded. For $\chi \in \Omega$ and for $\tau_1, \tau_2 \in [0,1]$ that satisfy $\tau_1 < \tau_2$, by Lemma 3.1 and

3.3 we get

$$
\begin{aligned}
|\Im\chi(\tau_1) - \Im\chi(\tau_2)| &= \Big| \int_0^1 G(\tau_1,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\
&\quad - \int_0^1 G(\tau_2,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \Big| \\
&= \Big| \int_0^1 [K(\tau_1,\varsigma) - K(\tau_2,\varsigma)]\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\
&\quad + \frac{\lambda(\tau_1^2-\tau_2^2)}{6-2\lambda\zeta^3}\int_0^1 H(\zeta,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \Big| \\
&\leq \Big| \int_0^1 [K(\tau_1,\varsigma) - K(\tau_2,\varsigma)]\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \Big| \\
&\quad + \Big| \frac{\lambda(\tau_1^2-\tau_2^2)}{6-2\lambda\zeta^3}\int_0^1 H(\zeta,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \Big| \\
&\leq \gamma \int_0^1 |K(\tau_1,\varsigma) - K(\tau_2,\varsigma)|\theta(\varsigma)d\varsigma \\
&\quad + \frac{\lambda(\tau_1^2-\tau_2^2)\gamma}{6-2\lambda\zeta^3}\int_0^1 H(\zeta,\varsigma)\theta(\varsigma)d\varsigma.
\end{aligned}
$$

Since K is continuous for $0 \leq \tau,\varsigma \leq 1$, we can see the first integral of the right-hand side of the above relation tends to 0 when $\tau_2 \to \tau_1$. On the other hand in view of the term $(\tau_1^2 - \tau_2^2)$, the second relation of the right-hand side of the above tends to zero if $\tau_2 \to \tau_1$. So, $\Im(\Omega)$ is equicontinuous. Hence, utilizing the Arzela-Ascoli theorem, $\Im : \wp \to \wp$ is completely continuous. □

We put

$$\Lambda_1 = \left(\int_0^1 \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma\right)^{-1},$$

$$\Lambda_2 = \left(\mathbf{X}\int_0^\xi \left(\frac{3}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma\right)^{-1}.$$

We prove that $0 < \Lambda_1 < \Lambda_2$. Indeed we have

$$
\begin{aligned}
\Lambda_2^{-1} &= \mathbf{X}\int_0^\xi \left(\frac{3}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^3\theta(\varsigma)d\varsigma \\
&< \int_0^1 \left(\frac{6}{6-2\lambda\zeta^3}\right)(1-\varsigma)^2\varsigma^3\theta(\varsigma)d\varsigma \\
&= \Lambda_1^{-1}.
\end{aligned}
$$

Our first result is based on the Theorem 2.8.

Theorem 4.2. *Considering (A1)-(A2) and also assuming that there exist constants* $\rho_1 > 0$, $\rho_2 > 0$, $\mu_1 \in (0, \Lambda_1]$, *and* $\mu_2 \in [\Lambda_2, \infty)$, *such that* $\rho_1 < \rho_2$ *and* $\mu_2\rho_1 < \mu_1\rho_2$. *Furthermore,* φ *satisfies:*

- $\varphi(\tau,\chi) \leq \mu_1\rho_2$ *for all* $\chi \in [0,\rho_2]$ *and* $\tau \in [0,1]$, *and*
- $\varphi(\tau,\chi) \geq \mu_2\rho_1$ *for all* $\chi \in [\mathbf{X}\rho_1, \rho_1]$ *and* $\tau \in [0,\xi]$.

Then the problem (2) *has at least one positive solution* $\chi \in \wp$ *with property* $\rho_1 < \|\chi\| < \rho_2$.

Proof. We set

$$\Omega_2 = \{\chi \in \mathcal{B}; \|\chi\| < \rho_2\}.$$

Let $\chi \in \wp \cap \partial\Omega_2$. Then from (1) and Lemma 3.3, we have

$$\begin{aligned}\Im\chi(\tau) &= \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\ &\leq \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\mu_1\rho_2 d\varsigma \\ &\leq \mu_1\rho_2\int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma \\ &\leq \Lambda_1\Lambda_1^{-1}\rho_2 = \rho_2.\end{aligned}$$

So for all $\chi \in \wp \cap \partial\Omega_2$ we have $\|\Im\chi\| \leq \|\chi\|$.
Now we define the open set $\Omega_1 = \{\chi \in \mathcal{B} : \|\chi\| < \rho_1\}$. For $u \in P \cap \partial\Omega_1$, considering (A1)-(A2), and regarding (2) and Lemma 3.3, for $\tau \in [0,\xi]$, we get

$$\begin{aligned}\Im\chi(\tau) &= \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\ &\geq \int_0^\xi G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\ &\geq \mathbf{X}\int_0^\xi \frac{3}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\ &\geq \rho_1 N\mathbf{X}\int_0^\xi \frac{3}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma, \\ &= \rho_1\Lambda_2\Lambda_2^{-1} = \rho_1.\end{aligned}$$

So for all $\chi \in \wp \cap \partial\Omega_1$, we have $\|\Im\chi\| \geq \|\chi\|$. So by utilizing the property (ii) in Theorem 2.8, T has at least one fixed point in $\wp \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is the solution of problem (2), that means the problem (2) has at least one positive solution u with $\rho_1 < \|\chi\| < \rho_2$. □

In a similar way, we can get the following result.

Theorem 4.3. *With considering (A1)-(A2), if there exist $\rho_1 > 0, \rho_2 > 0$, $\mu_1 \in (0,\Lambda_1]$, and $\mu_2 \in [\Lambda_2,\infty)$, where $\mathbf{X}\rho_2 < \rho_1$, and $\mu_1\rho_1 > N\rho_2$, such that f satisfies*

- *$\varphi(\tau,\chi) \geq \mu_2\rho_2$ for $x \in [\mathbf{X}\rho_2, \rho_2]$, and*
- *$\varphi(\tau,\chi) \leq \mu_1\rho_1$ for $x \in [0,\rho_1]$ and $\tau \in [0,1]$.*

Then the problem (2) *has at least one positive solution $u \in P$ satisfiying $\rho_1 < \|\chi\| < \rho_2$.*

The next result is based on the Theorem 2.9.

Theorem 4.4. *Let (A1)-(A2) hold and there exists $\mu > 0$ with*

$$\mu > \gamma\left(\int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma\right), \tag{13}$$

with $\gamma = \max\{\varphi(\tau,\chi) : (\tau,\chi) \in [0,1]\times[0,\mu]\}$, then, the problem (2) *has at least one positive solution.*

Proof. Suppose **that**

$$\mathcal{X} = \{\chi \in \wp : \|\chi\| < \mu\}.$$

By Theorem 4.1, $\mathfrak{I} : \overline{\mathcal{U}} \to P$ is completely continuous. Also there exist $\chi \in \overline{\mathcal{X}}$ and $\kappa \in (0,1)$ with $\chi = \kappa\mathfrak{I}\chi$. Then,

$$\begin{aligned}
|\chi(\tau)| = |\kappa T\chi(\tau)| &= \left|\kappa \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma\right| \\
&\leq \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\
&\leq \gamma \int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma.
\end{aligned}$$

So

$$\|\chi\| \leq \gamma \int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma.$$

Now **,**(4.4) **implies** that $\|\chi\| < \mu$, that is X is bounded. So there exists no $\chi \in \partial\mathcal{X}$ with $\chi = \kappa\mathfrak{I}\chi$ for $\kappa \in (0,1)$. Hence by Theorem 2.9, the problem (2) has at least one positive solution. □

According to Leggett-Williams theorem, we provide the conditions for problem (1) that it has at least three positive solution. We consider the below set of a cone $\wp$.

$$P_c = \{\chi \in K : \|\chi\| < c\}, \quad P(\psi,b,d) = \{x \in \wp : b \leq \psi(\chi), \quad \|\chi\| \leq d\}. \tag{14}$$

Theorem 4.5. *Let (A1)-(A2) are true and there exist a,b,c satisfying $0 < a < \mathbf{X}b < b \leq c$ with*

(B1) $\varphi(\tau,\chi(\tau)) < \Lambda_1 a$, $(\tau,\chi) \in [0,1]\times[0,a]$,

(B2) $\varphi(\tau,\chi(\tau)) \geq \mathbf{X}\Lambda_2 b$, $(\tau,\chi) \in [0,\xi]\times[\mathbf{X}b,b]$,

(B3) $\varphi(\tau,\chi(\tau)) \leq \Lambda_1 c$, $(\tau,\chi) \in [0,\xi]\times[0,c]$.

Then the problem (2) *has at least three positive solution χ_1,χ_2 and χ_3, such that*

$$\|\chi_1\| < a, \quad \mathbf{X}b < \psi(\chi_2), \quad \|\chi_3\| > a \quad \text{with} \quad \psi(\chi_3) < \mathbf{X}b.$$

Proof. By Theorem 4.1 $\mathfrak{I} : \wp \to \wp$ is a completely continuous. Suppose that

$$\psi(\chi) = \min_{0\leq\tau\leq\xi} \chi(\tau).$$

Clearly, ψ is continuous and concave on $\wp$ and $\psi(\chi) \leq \|\chi\|$, for $\chi \in \overline{\wp}_c$. We check the properties of Theorem 2.10. Let $\chi \in \overline{K}_c$, with $\|\chi\| \leq c$. For $\tau \in [0,1]$ by definition of (12) and (B3), we get

$$\begin{aligned}
\|\mathfrak{I}\chi(\tau)\| &= \max_{0\leq\tau\leq1} \int_0^1 G(\tau,\varsigma)\theta(\varsigma)\varphi(\varsigma,\chi(\varsigma))d\varsigma \\
&\leq \int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)Mcd\varsigma \\
&= Mc\int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma \\
&= McM^{-1} = c.
\end{aligned}$$

This implies that $\Im : \overline{P}_c \to P_c$. In a similar way, if $\chi \in \overline{P}_a$ **(**P_a **is the same be defined in** (14)**)**, we can conclude $\|\Im\chi\| < a$ and hence (ii) of Theorem 2.10 is true. Since $\frac{\mathbf{X}b+b}{2} \in \{\chi \in P(\psi, \mathbf{X}b, b) : \psi(\chi) > \mathbf{X}b\}$, we get $\{\chi \in P(\psi, \mathbf{X}b, b) : \psi(\chi) > \mathbf{X}b\} = \emptyset$. Moreover, for $\chi \in P(\psi, \mathbf{X}b, b)$, we get

$$\mathbf{X}b \le \psi(\chi) = \min_{0\le\tau\le\xi} \le \chi(\tau) \le \|\chi\| \le b, \quad \tau \in [0, \xi].$$

That is $\psi(\Im\chi) > \mathbf{X}b$ for $\chi \in P(\psi, \mathbf{X}b, b)$. So by Lemma 3.3 and (B2), we obtain

$$\begin{aligned}
\psi(\Im\chi) &= \min_{0\le t\le\xi} \int_0^1 G(\tau, \varsigma)\theta(\varsigma)\varphi(\varsigma, \chi(\varsigma))d\varsigma \\
&\ge \int_0^\xi \mathbf{X}\frac{3}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)\varphi(\varsigma, \chi(\varsigma))d\varsigma \\
&= Nb\int_0^\xi \mathbf{X}\frac{3}{6-2\lambda\zeta^3}(1-\varsigma)^2\varsigma^{\vartheta-3}\theta(\varsigma)d\varsigma \\
&= b > \mathbf{X}b.
\end{aligned}$$

Hence, (i) of Theorem 2.10 holds. Now, we prove that if $\chi \in P(\psi, \mathbf{X}b, c)$ with $\|\Im\chi\| > b$ then $\psi(\Im\chi) > \mathbf{X}b$. Assume that $\chi \in P(\psi, \mathbf{X}b, c)$ with $\|\Im\chi\| > b$, then by Lemma 3.4, we get

$$\psi(\Im\chi) = \min_{0\le t\le 1}(\Im\chi)(\tau) \ge \mathbf{X}\|\Im\chi\| > \mathbf{X}b.$$

Therefore, the property (iii) of Theorem 2.10 hold.
So, by Theorem 2.10 the proof is complete. □

5. Examples

Now it is time to explain the results by two examples.

Example 5.1. *Consider the following FIBVP:*

$$\begin{aligned}
&D^\vartheta\chi(\tau) + \sqrt{\tau}\varphi(\tau, \chi(\tau)) = 0, \\
&\chi(0) = \chi'(0) = 0, \quad \chi(1) = \lambda\int_0^\zeta \chi(\tau)dt,
\end{aligned} \tag{15}$$

where $\varphi(\tau, \chi(\tau)) = \frac{1}{100}(40 + 5\sqrt{\chi} + \tau)$, $\theta(\tau) = \tau^{\frac{1}{2}}$, $\lambda = 20$, $\vartheta = \frac{5}{2}$ *and* $\zeta = \frac{1}{2}$. *By a simple calculation we have*

$$\Lambda_1^{-1} = \int_0^1 \frac{6}{6-2\lambda\zeta^3}(1-\varsigma)^d\varsigma = 2,$$

so $\Lambda_1 = 0.5$, *also*

$$\Lambda_2^{-1} = \delta\int_0^\xi \frac{3}{6-2\lambda\zeta^3}(1-\varsigma)^2 d\varsigma = \xi^2[(1-\xi)(1-(1-\xi)^3].$$

It is easy to see that $0.18 \le \Lambda_2 \le 0.33$ *if* $0.3 \le \xi \le 0.7$. *Now by choosing* $\mu_1 = \Lambda_1$, $\mu_2 = \Lambda_2$, $\rho_1 = 0.01$ *and* $\rho_2 = 1$ *we have*

- $\varphi(\tau, \chi(\tau)) = \frac{1}{100}(40 + 5\sqrt{\chi} + \tau) \le 0.46 = \mu_1\rho_2 = 0.5$, $[\tau, \chi] \in [0, 1] \times [0, 1] \times [0, 1]$,
- $\varphi(\tau, \chi(\tau)) = \frac{1}{100}(40 + 5\sqrt{\chi} + \tau) \ge 0.4 \ge \mu_2\rho_1 = 0.33$, $[\tau, \chi] \in [0, \xi] \times [0.01\delta, \delta]$.

So all conditions of Theorem 4.2 hold and hence FBVP 15 has at least one positive solution χ *with* $0.01 < \|\chi\| \le 1$.

Example 5.2. *Let us consider the FIBVP:*

$$\begin{aligned} &D^{\vartheta}\chi(\tau) + \theta(\tau)\varphi(\tau, \chi(\tau)) = 0, \\ &\chi(0) = \chi'(0) = 0, \quad \chi(1) = \lambda \int_0^{\zeta} \chi(\tau)d\tau, \end{aligned} \tag{16}$$

where $\vartheta = \frac{5}{2}$, $\zeta = \frac{1}{2}$, $\theta(\tau) = \tau^{\frac{1}{2}}$, $\lambda = 20$ *and*

$$\varphi(\tau, \chi) = \begin{cases} 30u^7 + 0.01\sin^2 \pi\tau, & (\tau, \chi) \in [0, 1] \times [0, 1], \\ 29 + u^{\frac{1}{4}} + 0.01\sin^2 \pi\tau & (\tau, \chi) \in [0, 1] \times [1, \infty). \end{cases}$$

In view of the previous example by choosing $\xi = 0.5$ *we have* $\Lambda_1 = 0.5$ *and* $\Lambda_2 = 18.51$. *Let* $a = \frac{1}{2}$, $b = 10$ *and* $c = 64$, *then we have*

- $\varphi(\tau, \chi(\tau)) \leq 0.243 < \Lambda_1 a = 0.25, (\tau, \chi) \in [0, 0.5] \times [0, 0.5]$;
- $\varphi(\tau, \chi(\tau)) \geq 30.11 \geq \delta\Lambda_2 b = 23.1375, (\tau, \chi) \in [0, 0.5] \times [1.25, 10]$;
- $\varphi(\tau, \chi(\tau)) \leq 31.83 \leq \Lambda_1 c = 32$, $(\tau, \chi) \in [0, 0.5] \times [0, 64]$.

Consequently the FIBVP (16) *satisfy all conditions of Theorem 4.5 and therefore the problem has at least three solutions* χ_1, χ_2 *and* χ_3 *such that*

$$\|\chi_1\| \leq \frac{1}{2}, \quad 1.25 < \psi(\chi_2), \quad \|\chi_3\| > \frac{1}{2}, \psi(\chi_3) < 1.25.$$

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