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A different approach to the fixed point theory by $(\mathcal{L}, c)-$expansion mappings

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Abstract. In this study, we introduce a new type of mapping called $(\mathcal{L}, c)-$expansions, induced by a novel class of mappings known as $G-$functions. These mappings possess the unique property that their moduli may be greater than one, yet they possess unique fixed points. Utilizing Kummer's test, we demonstrate that these $(\mathcal{L}, c)-$expansions can be derived from the well-established framework of Banach contraction mappings. Furthermore, our results allow for the creation of new contractions by altering the type of $G-$functions used.

1. Introduction

The pursuit of solutions for equations of the form $\mathcal{T}x = x$ within the realm of fixed point theory has witnessed a surge in interest in recent decade(see [1–14, 18]). The investigation of fixed points of operators, denoted as $\mathfrak{Fix}(\mathcal{T})$, is a central theme in this domain. This study unveils a novel facet of fixed point theory by introducing a distinctive class of mappings, termed $(\mathcal{L}, c)-$expansions, originating from a newly defined set of mappings known as $G-$functions. The uniqueness of these mappings lies in their capacity to exhibit modulous surpassing unity while retaining exclusive fixed points. Such a characteristic departure from traditional fixed point theory prompts an exploration into the mathematical foundations that underpin these mappings. Drawing inspiration from Kummer's test, a powerful tool in the analysis of positive series convergence, this study establishes a connection between $(\mathcal{L}, c)-$expansions and the well-established framework of Banach contraction mappings.

Kummer's test, as articulated in [15], stands as a prominent method for assessing the convergence or divergence of positive series. Noteworthy convergence tests, such as D'Alembert's, Raabe's, Bertrand's, and Gauss', all find their roots in Kummer's test, with their specifics determined by the choice of appropriate constants. This research leverages Kummer's test to offer a comprehensive exploration of the convergence properties of $(\mathcal{L}, c)-$expansions, unveiling a deep connection to the principles of Banach contraction mappings.

Theorem 1.1. *(Kummer's Test) [16, 17] Let* $\sum_{n=1}^{\infty} u_n$ *be a positive series.*

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(1) $\sum_{n=1}^{\infty} u_n$ *is convergent if and only if there is a positive series* $\sum_{n=1}^{\infty} k_n$ *and a constant* $c > 0$*, such that* $k_n(\frac{u_n}{u_{n+1}}) - k_{n+1} \geq c$.

(2) $\sum_{n=1}^{\infty} u_n$ *is divergent if and only if there is a positive series* $\sum_{n=1}^{\infty} k_n$ *such that* $\sum_{n=1}^{\infty} \frac{1}{k_n}$ *be divergent and* $k_n(\frac{u_n}{u_{n+1}}) - k_{n+1} \leq 0$.

The significance of this study is underscored by its ability to extend the repertoire of contractions through the manipulation of the type of G−functions employed. The subsequent sections delve into the theoretical underpinnings of $(\mathcal{L}, c)$−expansions, providing a rigorous mathematical foundation for their existence and properties. Additionally, this work establishes a bridge between Kummer's test and the emerging field of $(\mathcal{L}, c)$−expansions, paving the way for new avenues of exploration within fixed point theory. As we progress, we shall elucidate the intricacies of this novel mathematical framework and its implications for the broader landscape of fixed point theory.

2. Main Results

In this section, leveraging the Kummer Test, a pioneering connection between $\sigma(Tx, Ty)$ and $\sigma(x, y)$ is established, heralding a paradigm shift in fixed point theory methodologies. This novel relation forms the cornerstone for our main results.

Definition 2.1. *Let* $(\mathcal{X}, \sigma)$ *be a metric space. The function* $\mathcal{L} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ *is called G-function if it satisfies the following items:*

(*i*) $\mathcal{L}(x, y) = \mathcal{L}(y, x)$, *for all* $x, y \in \mathcal{X}$,

(*ii*) *for each sequence* $\{x_n\} \subset \mathcal{X}$ *and each* $y \in \mathcal{X}$ *and* $c > 0$

$$x_n \to x \ \text{ implies } \ \frac{\mathcal{L}(x_n, x)}{c + \mathcal{L}(x_{n+1}, y)} \ \text{ is bounded}$$

for sufficiently large $n \in \mathbb{N}$. *The set of all G-functions is denoted by* $\mathfrak{S}_G(\mathcal{X})$.

Example 2.2. (q_1) *Let* $X = \mathbb{R}$ *endowed with Euclidean metric,* $\mathcal{L}(x, y) = xy$ *and let* $c > 0$ *be arbitrary. Then* $\mathcal{L}$ *is a G−function.*

(q_2) *Let* $X = \mathbb{R}$ *endowed with Euclidean metric,* $\mathcal{L}(x, y) = x + y$ *and let* $c > 0$ *be arbitrary. Then* $\mathcal{L}$ *is a G−function.*

(q_3) *Let* $X = [1, +\infty)$ *be metric space,* $\mathcal{L}(x, y) = \frac{1}{x^2+y^2}$ *and let* $c > 0$ *be arbitrary. Then* $\mathcal{L}$ *is a G−function.*

(q_4) *Let* (X, d) *be metric space,* $\mathcal{L}(x, y) = d(x, y)$ *and let* $c > 0$ *be arbitrary. Then* $\mathcal{L}$ *is a G−function.*

Definition 2.3. *Let* $(\mathcal{X}, \sigma)$ *be a metric space. We say that* $T : \mathcal{X} \to \mathcal{X}$ *is* $(\mathcal{L}, c)$*-expansion mapping, if there exist* $\mathcal{L} \in \mathfrak{S}_G(\mathcal{X})$ *and* $c > 0$ *such that*

$$\sigma(Tx, Ty) \leq \frac{\mathcal{L}(x, y)}{c + \mathcal{L}(Tx, Ty)} \sigma(x, y), \tag{1}$$

for all $x, y \in \mathcal{X}$.

Upon scrutinizing diverse generalizations of Banach fixed point theorems in existing literature, particularly within their proofs, it becomes evident that the attainment of sought-after results, such as the identification of a fixed point, hinges upon the treatment of the sequence $a_n = d(x_n, x_{n-1})$. Here, $x_n = T^n(x_0)$ represents the iterative sequence induced by the mapping T, initiated at x_0. A pivotal element in these proofs lies in establishing the convergence of the series $\sum_{n=1}^{\infty} a_n$—a crucial step leading to the limit point of x_n, the optimal candidate for the fixed point.

This observation motivates our exploration into the intricate relationship between series convergence and fixed point outcomes. In the ensuing theorem, we introduce a novel form of contraction, drawing inspiration from Kummer's test, thus deviating from the conventional Banach paradigm. It is noteworthy that the prevailing study of Banach fixed point generalizations has operated under the premise that the modulus α falls within the range $[0, 1)$. In contrast, our approach accommodates scenarios where the modulus may surpass unity, thereby forging a pioneering path within the scholarly discourse.

Theorem 2.4. *Let $(\mathcal{X}, \sigma)$ be a complete metric space and let J from $\mathcal{X} \to \mathcal{X}$ be a $(\mathcal{L}, c)$-expansion mapping. Then J has a unique fixed point in $\mathcal{X}$.*

Proof. Let $t_0 \in \mathcal{X}$ and let $t_1 = Jt_0$. If $t_0 = t_1$, then t_0 is the fixed point and the proof is completed. If t_n be selected, then we can define $t_{n+1} = Jt_n$, inductively. Without loss of generality, we can suppose that $t_{n+1} \neq t_n$. Considering (1), we have

$$\sigma(t_{n+1}, t_n) \leq \frac{\mathcal{L}(t_n, t_{n-1})}{c + \mathcal{L}(t_{n+1}, t_n)} \sigma(t_n, t_{n-1}).$$

Letting $u_n = \sigma(t_n, t_{n-1})$ and $k_n = \mathcal{L}(t_n, t_{n-1})$, we yields

$$k_n \left(\frac{u_n}{u_{n+1}} \right) - k_{n+1} \geq c.$$

Thus, by the Kummer's Test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

It remains to show that:

Step (1): $\{x_n\}$ is a Cauchy sequence and so is convergent to some $z \in \mathcal{X}$,

Step (2): z is the unique fixed point.

Reaching Step (1), let $m, n \in \mathbb{N}$ and $m > n$. Then

$$\sigma(t_n, t_m) \leq \sum_{k=n}^{m-1} \sigma(t_i, t_{i+1}) = \sum_{k=n}^{m-1} u_{i+1} \to 0 \quad (m, n \to \infty).$$

Therefore, $\lim_{n \to \infty} \sup\{\sigma(t_n, t_m) : m \geq n\} = 0$. So, the sequence $\{t_n\}$ is Cauchy and since $\mathcal{X}$ is completed, there exists $z \in \mathcal{X}$ such that $t_n \to z$, as $n \to \infty$.
Proving Step (2), we get

$$\sigma(t_{n+1}, Jz) \leq \frac{\mathcal{L}(t_n, z)}{c + \mathcal{L}(t_{n+1}, Jz)} \sigma(t_n, z). \tag{2}$$

According to (*ii*) of Definition 2.1, we have $\frac{\mathcal{L}(t_n, z)}{c + \mathcal{L}(t_{n+1}, Jz)}$ is bounded. So by letting n tends to ∞, the right hand both side of (2), tends to zero and it deduces that $Jz = z$. Now assume $z, w \in \mathcal{X}$ are two fixed points. We have

$$\sigma(z, w) = \sigma(Jz, Jw) \leq \frac{\mathcal{L}(z, w)}{c + \mathcal{L}(Jw, Jz)} \sigma(w, z) = \frac{\mathcal{L}(z, w)}{c + \mathcal{L}(w, z)} \sigma(w, z).$$

If $\sigma(z, w) \neq 0$, then it yields that $\frac{\mathcal{L}(z, w)}{c + \mathcal{L}(w, z)} \geq 1$ and this is a contradiction, so we have $z = w$ and hence the fixed point is unique. □

The next corollary shows that the Banach fixed point principle can be derived from Theorem 2.4.

Corollary 2.5. *Let $(\mathcal{X}, \sigma)$ be a complete metric space and let J from $\mathcal{X}$ into itself is a contraction mapping, for given $\alpha \in (0, 1)$. Then J has a unique fixed point in $\mathcal{X}$.*

Proof. Considering $\mathcal{L}(x, y) = 1$ and $c = \frac{1}{\alpha} - 1$ In Theorem 2.4, we have

$$\sigma(Jx, Jy) \leq \alpha\sigma(x, y) = \frac{1}{c+1}\sigma(x, y)$$

and desired result is concluded. □

The most surprising aspect of Theorem 2.4 is the modulus $\dfrac{\mathcal{L}(x, y)}{c + \mathcal{L}(Tx, Ty)}$ which can be greater than one in spite of the fact that T has a fixed point. The following example support the claim.

Example 2.6. *Let $X = [1, 2] \cup \{3\}$ and let $c = \frac{1}{4}$. We define the function Tx as follows:*

$$Tx = \begin{cases} \frac{1}{2x} + \frac{3}{4} & , \quad x \neq 3 \\ 2 & , \quad x = 3 \end{cases}$$

Additionally, let $d(x, y) = |x - y|$ and $\mathcal{L}(x, y) = xy$. We will now demonstrate that T satisfies all the conditions of Theorem 2.4.
Firstly, we note that $T(X) \subset X$ and (X, d) is a complete metric space.

Case(1) *Let $x, y \in [1, 2]$. In this case, we have $1 \leq xy \leq 4$. Thus,*

$$\tfrac{1}{16} \leq \tfrac{1}{4xy} \leq \tfrac{1}{4} \; \tfrac{3}{16} \leq \tfrac{3}{8x} \leq \tfrac{3}{8} \; \tfrac{3}{16} \leq \tfrac{3}{8y} \leq \tfrac{3}{8}$$

It follows that

$$c + \frac{7}{16} + \frac{9}{16} \leq c + \frac{1}{4xy} + \frac{3}{8x} + \frac{3}{8y} + \frac{9}{16} \leq 1 + c + \frac{9}{16} = 1 + \frac{13}{16}$$

Therefore,

$$c + \frac{1}{4xy} + \frac{3}{8x} + \frac{3}{8y} + \frac{9}{16} \leq 2 \leq 2x^2y^2.$$

This yields

$$c + (\frac{1}{2x} + \frac{3}{4})(\frac{1}{2y} + \frac{3}{4}) \leq (2xy)\mathcal{L}(x, y).$$

Therefore,

$$c + \mathcal{L}(Tx, Ty) \leq (2xy)\mathcal{L}(x, y)$$

Thus,

$$d(Tx, Ty) = |Tx - Ty| = \frac{1}{2}\frac{|x - y|}{xy} \leq \frac{\mathcal{L}(x, y)}{c + \mathcal{L}(Tx, Ty)}|x - y|$$

and the desired result is obtained.

Case(2) *Let $x \in [1, 2]$ and $y = 3$. In this case, $d(Tx, Ty) = |\frac{1}{2x} - \frac{5}{4}|$, $\mathcal{L}(x, 3) = 3x$, and $\mathcal{L}(Tx, 2) = \frac{1}{x} + \frac{3}{2}$. Also,*

$$c + \frac{1}{x} + \frac{3}{2} \leq c + \frac{5}{2} \leq 3 \leq 3x(3 - x).$$

Thus,

$$d(Tx, Ty) = \frac{5}{4} - \frac{1}{2x} \leq 1 \leq \frac{3x}{c + \frac{1}{x} + \frac{3}{2}}(3 - x) = \frac{\mathcal{L}(x, 3)}{c + \mathcal{L}(Tx, 2)}d(x, y).$$

The other cases are similar and have been left to the discretion of the reader.

Remark 2.7. *In example 2.6, it is worth mentioning that, in comparison with the Banach contraction principle,*

$$\frac{\mathcal{L}(x,3)}{c+\mathcal{L}(Tx,2)} = \frac{3x}{c+\frac{1}{x}+\frac{3}{2}} \geq 1$$

and shows that Theorem 2.4 is a real different type in fixed point theory.

Another intriguing aspect of Theorem 2.4 is the ability of constructing new contractions by considering the various type of $G-$functions. The following corollaries are the natural new contractions induced by some $G-$functions.

Corollary 2.8. *Let $(\mathcal{X},\sigma)$ be a complete metric space and let J from $\mathcal{X}$ itself be a mapping. Suppose that there exists $c>0$ such that*

$$\sigma^2(Jx,Jy)+c\sigma(Jx,Jy) \leq \sigma^2(x,y), \tag{3}$$

for all $x,y \in \mathcal{X}$. Then J has a unique fixed point in $\mathcal{X}$.

Proof. Let define $\mathcal{L}(x,y)=\sigma(x,y)$, then (3) concludes

$$\sigma(Jx,Jy)(c+\sigma(Jx,Jy)) \leq \sigma^2(x,y),$$

Thus, we have

$$\sigma(Jx,Jy) \leq \frac{\sigma(x,y)}{c+\sigma(Jx,Jy)}\sigma(x,y),$$

so by Theorem (2.4), we conclude desired result. □

Corollary 2.9. *Let $(\mathcal{X},\sigma)$ be a complete metric space and let J from $\mathcal{X}$ itself be a mapping. Suppose that there exists $c>0$ and $\varphi:[0,+\infty)\to[0,+\infty)$ such that for all sequence $\{t_n\}\subset[0,+\infty)$ implies $\limsup_{n\to\infty}\varphi(t_n)$ exists. Also,*

$$\sigma(Jx,Jy)(c+\varphi(\sigma(Jx,Jy)) \leq \varphi(\sigma(x,y))\sigma(x,y), \tag{4}$$

for all $x,y \in \mathcal{X}$. Then J has a unique fixed point in $\mathcal{X}$.

Proof. Let define $\mathcal{L}(x,y)=\varphi(\sigma(x,y))$, then (4) concludes

$$\sigma(Jx,Jy)(c+\mathcal{L}(Jx,Jy)) \leq \mathcal{L}(x,y)\sigma(x,y).$$

Thus, we have

$$\sigma(Jx,Jy) \leq \frac{\mathcal{L}(x,y)}{c+\mathcal{L}(Jx,Jy)}\sigma(x,y),$$

so by Theorem (2.4), we conclude desired result. □

Corollary 2.10. *(Caristi type,see [**?**] for more detail) Let $(\mathcal{X},\sigma)$ be a complete metric space and let J from $\mathcal{X}$ itself be a mapping. Suppose that there exists $c>0$ and $\varphi:\mathcal{X}\to[0,+\infty)$ such that for all sequence $\{t_n\}\subset[0,+\infty)$ implies $\limsup_{n\to\infty}\varphi(t_n)$ exists. Also,*

$$c\sigma(Jx,Jy) \leq \varphi(x)\varphi(y)-\varphi(Jx)\varphi(Jy), \tag{5}$$

for all $x,y \in \mathcal{X}$. Then J has a unique fixed point in $\mathcal{X}$ by Theorem 2.4.

Proof. Let define $\mathcal{L}(x,y) = \frac{\varphi(x)\varphi(y)}{\sigma(x,y)}$ then from 5 we have

$$c\sigma(Jx, Jy) + \varphi(Jx)\varphi(Jy) \leq \varphi(x)\varphi(y)$$

So we have

$$c\sigma(Jx, Jy) + \sigma(Jx, Jy)\frac{\varphi(Jx)\varphi(Jy)}{\sigma(Jx, Jy)} \leq \frac{\varphi(x)\varphi(y)}{\sigma(x,y)}\sigma(x,y),$$

and then

$$\sigma(Jx, Jy) \leq \frac{\frac{\varphi(x)\varphi(y)}{\sigma(x,y)}}{c + \frac{\varphi(Jx)\varphi(Jy)}{\sigma(Jx,Jy)}}\sigma(x,y).$$

Therefore,

$$\sigma(Jx, Jy) \leq \frac{\mathcal{L}(x,y)}{c + \mathcal{L}(Jx, Jy)}\sigma(x,y)$$

for all $x, y \in \mathcal{X}$. Then J has a unique fixed point in $\mathcal{X}$ by Theorem 2.4. □

In the following, we introduce a new contraction which inspired from $F-$contraction (see [18, Definition 2.1]).

Definition 2.11. *Let $F^+ : \mathbb{R}^+ \to \mathbb{R}^+$, be an strictly increasing mapping A mapping $T : X \to X$ is said to be F^+-contraction if there exists $\tau > 0$ such that*

$$\forall\, x, y \in X \text{ such that } d(Tx, Ty) > 0 \quad \text{implies} \quad \tau + F(d(Tx, Ty)) \leq F(d(x,y)) \tag{6}$$

Corollary 2.12. *Let (X, d) be a complete metric space and let T from X into itself be a F^+- contraction mapping. Then T has a unique fixed point in X.*

Proof. Let $x, y \in X$ such that $d(Tx, Ty) > 0$. Then we have $\tau + F(d(Tx, Ty)) \leq F(d(x, y))$. Thus we have $d(Tx, Ty)) < d(x, y)$. Considering (6), we have

$$1 \leq \frac{F(d(x,y))}{\tau + F(d(Tx, Ty))} \tag{7}$$

Multiplying $d(Tx, Ty)$ on both side of 7, we have

$$d(Tx, Ty) \leq \frac{F(d(x,y))}{\tau + F(d(Tx, Ty))}d(Tx, Ty) \leq \frac{F(d(x,y))}{\tau + F(d(Tx, Ty))}d(x,y).$$

Taking $G(x, y) = F(d(x, y))$, we have T is $(G, \tau)-$expantion mapping and so Theorem 2.4, conclude desired result. □

3. Exploring the Profound Connection between $(\mathcal{L}, c)$-Expansion Mappings and a Broad Class of Contractions

A salient observation arises when considering a substantial subset of contractions that satisfy the condition:

$$\sum_{n=1}^{\infty} d(T^n(x_0), T^{n+1}(x_0)) < \infty, \tag{8}$$

for an initial point $x_0 \in X$. The majority of these contractions share a common characteristic encapsulated in the following relationship:

$$d(T^n(x_0), T^{n+1}(x_0)) \leq \frac{F(T^{n-1}(x_0))}{c + F(T^n(x_0))} d(T^n(x_0), T^{n-1}(x_0)), \tag{9}$$

for each $n \in \mathbb{N}$, where $c > 0$ and $F : O_T(x_0) \to [0, +\infty)$ is a mapping, with $O_T(x_0) = T^n(x_0) : n \in 0 \cup \mathbb{N}$, a set notably akin to the $(\mathcal{L}, c)$-expansion mappings. This observation sets the stage for the development of our main results. To illustrate the significance of this connection, consider the following example:

Example 3.1. *Let (X, d) be a complete metric space, and $T : X \to X$ be a mapping satisfying the contraction condition:*

$$d(Tx, Ty) \leq \psi(d(x, y)), \tag{10}$$

where $x, y \in X$, and $\psi : [0, +\infty) \to [0, +\infty)$ is a mapping with $\limsup_{t \to s^+} \frac{\psi(t)}{t} < 1$ for each $s \in (0, +\infty)$. Then, T satisfies (8). Indeed, for $x_0 \in X$ and $T^n(x_0) = x_n$, inequality (12) implies:

$$d(T^{n+1}(x_0), T^n(x_0)) \leq \psi(d(T^{n-1}(x_0), T^n(x_0))). \tag{11}$$

Taking $a_n = d(T^{n-1}(x_0), T^n(x_0))$, (13) leads to $a_{n+1} \leq \psi(a_n)$. Therefore, $\frac{a_{n+1}}{a_n} \leq \frac{\psi(a_n)}{a_n}$. By our assumption, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty} \frac{\psi(a_n)}{a_n} < 1$. Thus, by the ratio test, we obtain:

$$\sum_{n=1}^{\infty} d(T^{n+1}(x_0), T^n(x_0)) < \infty.$$

Example 3.2. *Let (X, d) be a complete metric space, and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y), \tag{12}$$

where $x, y \in X$ and $\psi : [0, +\infty) \to [0, 1)$ is a mapping such that $\limsup_{t \to s^+} \psi(t) < 1$, for each $s \in (0, +\infty)$. Then, T satisfies (8). Since, if $x_0 \in X$, and let $T^n(x_0) = x_n$. Then, (12) shows that

$$d(T^{n+1}(x_0), T^n(x_0)) \leq \psi(d(T^{n-1}(x_0), T^n(x_0)))d(T^{n-1}(x_0), T^n(x_0)). \tag{13}$$

Thus, taking $a_n = d(T^{n-1}(x_0), T^n(x_0))$, (13) shows that $a_{n+1} \leq \psi(a_n)a_n$. Therefore, $\frac{a_{n+1}}{a_n} \leq \psi(a_n)$. By our assumption, we get $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty} \psi(a_n) < 1$ and so by ratio test we have

$$\sum_{n=1}^{\infty} d(T^{n+1}(x_0), T^n(x_0)) < \infty.$$

The following lemma plays a crucial rule in the next theorem.

Lemma 3.3. *Let $(a_n) \subset (0, +\infty)$ and let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then there exists a monotonic sequence (γ_n) such that $\lim_{n\to\infty} \gamma_n = \infty$ and $\sum_{n=1}^{\infty} a_n \gamma_n$ is convergent.*

Proof. Let $\rho_n = \sum_{k=n+1}^{\infty} a_k$. Note that $\lim_{n\to\infty} \rho_n = 0$ and we have $\rho_{n-1} - \rho_n = a_n$ thus

$$\begin{aligned} \frac{a_n}{\sqrt{\rho_{n-1}}} &= \frac{\rho_{n-1} - \rho_n}{\sqrt{\rho_{n-1}}} \\ &= \frac{(\sqrt{\rho_{n-1}} - \sqrt{\rho_n})(\sqrt{\rho_{n-1}} + \sqrt{\rho_n})}{\sqrt{\rho_{n-1}}} \\ &= (\sqrt{\rho_{n-1}} - \sqrt{\rho_n})\left(\frac{\sqrt{\rho_{n-1}} + \sqrt{\rho_n}}{\sqrt{\rho_{n-1}}}\right) \\ &< 2(\sqrt{\rho_{n-1}} - \sqrt{\rho_n}) \end{aligned} \tag{14}$$

It means that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=1}^{n} \frac{a_k}{\sqrt{\rho_{k-1}}} &< \sum_{k=1}^{n} 2(\sqrt{\rho_{k-1}} - \sqrt{\rho_k}) \\ &= 2(\sqrt{\rho_0} - \sqrt{\rho_n}) \leq 2\sqrt{\rho_0} < \infty. \end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\rho_{k-1}}}$ is convergent. Considering $\gamma_n = \frac{1}{\rho_{n-1}}$, one can obtain desired result. □

Theorem 3.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that $x_0 \in X$ and consider $\{T^n(x_0)\}$ as the Picard iterative sequence such that*

$$\sum_{n=1}^{\infty} d(T^n(x_0), T^{n-1}(x_0)) < \infty. \tag{15}$$

Then there exist $c > 0$ and a mapping $F : O_T(x_0) \times O_T(x_0) \to [0, +\infty)$ such that for each $n \in \mathbb{N}$,

$$d(T^n(x_0), T^{n+1}(x_0)) \leq \frac{F(T^{n-1}(x_0), F(T^n(x_0)))}{c + F(T^n(x_0), T^{n+1}(x_0))} d(T^n(x_0), T^{n-1}(x_0)), \tag{16}$$

in which $O_T(x_0) = \{T^n(x_0) : n \in \{0\} \cup \mathbb{N}\}$.

Proof. Let $a_n = d(T^n(x_0), T^{n-1}(x_0))$ and by (15), $\sum_{n=1}^{\infty} a_n < \infty$. Applying Lemma 3.3, one can create a sequence γ_n such that $\lim_{n\to\infty} \gamma_n = \infty$ and $\sum_{n=1}^{\infty} \gamma_n a_n < \infty$. So we can choose $b_1 \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n \gamma_n = a_1 b_1$. Now define $\{b_n\}$ as follows

$$b_{n+1} = \frac{b_n a_n - \gamma_{n+1} a_{n+1}}{a_{n+1}}.$$

Therefore, based on creating the sequence γ_n in Lemma 3.3, $\gamma_n \subset (0, +\infty)$. So

$$\gamma_{n+1} a_{n+1} = b_n a_n - b_{n+1} a_{n+1}. \tag{17}$$

It means that, $\gamma_{n+1} a_{n+1} > 0$. Thus, $b_{n+1} a_{n+1} < b_n a_n$ and this yields there exists $q \geq 0$ such that $\lim_{n\to\infty} b_n a_n = q$. So,

$$\begin{aligned} \sum_{n=1}^{\infty} \gamma_{n+1} a_{n+1} &= \sum_{n=1}^{\infty} (b_n a_n - b_{n+1} a_{n+1}) \\ &= a_1 b_1 - \lim_{n\to\infty} a_{n+1} b_{n+1} = a_1 b_1 - q \end{aligned} \tag{18}$$

Hence, $q = \lim_{n\to\infty} a_{n+1} b_{n+1} = 0$. According to Lemma 3.3 and the fact that $\{a_n\} \subset [0, +\infty)$, the sequence $\{b_n\}$ is positive. Now define $F : O_T(x_0) \times O_T(x_0) \to [0, +\infty)$ such that for each $n \in \mathbb{N}$, $F(T^{n-1}(x_0), T^n(x_0)) = b_n$. Using the fact that $\lim_{n\to\infty} \gamma_n = \infty$, for each $c > 0$, there exists $N > 0$ such that $\gamma_n > c$. Applying (18) forall $n \geq N$, we have

$$b_n a_n - b_{n+1} a_{n+1} = \gamma_{n+1} a_{n+1} > c a_{n+1}.$$

Thus,

$$a_{n+1} < \left(\frac{b_n}{b_{n+1} + c} \right) a_n. \tag{19}$$

Hence, we can rewrite (19) as follows

$$d(T^n(x_0), T^{n+1}(x_0)) \leq \frac{F(T^{n-1}(x_0), F(T^n(x_0)))}{c + F(T^n(x_0), T^{n+1}(x_0))} d(T^n(x_0), T^{n-1}(x_0))$$

and the proof is completed. □

Conclusion and Future Directions

In conclusion, this study pioneers a novel perspective on contractions through the lens of series convergence tests, offering valuable insights into the identification of fixed points for operators with modulus not restricted to values less than one. The introduction of $(\mathcal{L}, c)$-expansion mappings represents a significant advancement in the exploration of fixed point results, marking the first instance of their application in this context.

A noteworthy outcome of this research is the initial exploration of contractions with moduli greater than one and their potential application in solving integral equations, particularly when coefficients exceed unity. This opens avenues for further investigation and application in solving real-world problems.

However, several intriguing questions emerge, pointing towards promising directions for future research. Given the pivotal role of Kummer's test (Theorem 1.1) as a convergence criterion for series and its centrality in our main results, an intriguing avenue for exploration is the quest for a terminated fixed point result. Specifically, can a general formula be derived for an arbitrary operator with at least one fixed point, considering the termination aspect of Kummer's test?

Moreover, a crucial question arises concerning the existence of a mapping $\Gamma : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ with specific properties, facilitating the establishment of a terminated fixed point result for operators with fixed points. This inquiry holds the potential to advance our understanding of fixed point theory and its broader applications. Future research efforts are warranted to address these questions, thereby contributing to the ongoing evolution of fixed point theory and its practical implications.

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