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Tripled system of fractional Langevin equations with tripled multipoint boundary conditions

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Abstract. This research focuses on examining the tripled system of nonlinear fractional Langevin equations with coupled multipoint boundary conditions. By utilizing the Banach contraction mapping, one can obtain the result of existence and uniqueness. The existence of a solution is validated through the use of Krasnoselskii's fixed point theorem. Furthermore, The Ulam-Hyers stability of the mentioned system is studied. In the end, we present two examples to validate the effectiveness of our analysis.

1. Introduction

Fractional differential equations have gained a lot of significance and attention because of their various applications in applied fields, like biology, physics, and engineering, etc see [1–4]. In this regard, the various modeling can be seen in electrical circuits [5] coronas-virus [6], population growth [7], aerodynamic [8] and the references cited.
In particular, the fractional Langevin equations are a significant subject due to their rich history, see [9–21] .

On the other hand, fractional differential systems can be employed to describe a variety of physical phenomena, such as ecological effects [22], chaotic synchronization [23], anomaly diffusion [24]. Particularly, tripled fractional differential equations were examined by many authors [25–32]. For instance, in [26] the authors are proving the existence and uniqueness of a tripled system of fractional pantograph differential equations. In [27], the nonlinear coupled system of three fractional differential equations with nonlocal coupled boundary conditions has been investigated.

So, in this current work, we develop a tripled system of fractional Langevin equations with nonlocal multipoint tripled boundary conditions of the form:

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1)x(t) = f(t, x(t), y(t), z(t)), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2)y(t) = g(t, x(t), y(t), z(t)), & t \in [0,1], \\ {}^cD^{\beta_3}({}^cD^{\alpha_3} + \lambda_3)y(t) = k(t, x(t), y(t), z(t)), & t \in [0,1], \end{cases} \tag{1}$$

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subject to tripled multipoint boundary conditions

$$\begin{cases} x(0) = 0; \quad x(a_1) = 0; \quad x(1) = \sum_{i=1}^{n} \gamma_i y(s_i), \\ y(0) = 0; \quad y(b_1) = 0; \quad y(1) = \sum_{j=1}^{m} \delta_i z(u_j), \\ z(0) = 0; \quad z(c_1) = 0; \quad z(1) = \sum_{k=1}^{p} \sigma_k x(v_k), \\ 0 < a_1 < b_1 < c_1 < s_1 < s_2 < ... < s_n < u_1 < u_2 < ... < u_m < v_1 < v_2 < ... < v_p < 1. \end{cases} \tag{2}$$

Where $0 < \alpha_{k'} < 1, \quad 1 < \beta_{k'} \leq 2$, **for** $k' = 1,2,3, \gamma_i, \delta_j, \sigma_k, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^*$ for $i = 1,..,n$; $j = 1,2,..,m$; $k = 1,2,..,p$, ${}^cD^{\beta_k}$, ${}^cD^{\alpha_k}$ are the Caputo's fractional derivatives, and $f, g, k : [0;1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions.
To our knowledge, tripled fractional Langevin equations via tripled multipoint boundary conditions have not been extensively investigated yet.

The structure of this paper is as follows: the second section provides some definitions and lemmas for fractional calculus that will be helpful throughout the work.The main results are discussed in the third section using fixed point theory. In the fourth section, we established that the problem (1) - (2) is Ulam-Hyers stability. The last section, we give some examples to illustrate the results.

2. Preliminaries and notations

In this section, we present some notation, definitions and lemma that we use in our proofs later.

Definition 2.1. *[3] The gamma function is defined by :*$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. *This integral is convergent for all complex* $z \in \mathbb{C}$ *(*$Re(z) > 0$*).*

Definition 2.2. *[3] The fractional integral of order* $\alpha > 0$ *with the lower limit zero for a function* f *can be defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Definition 2.3. *[3] The Caputo derivative of order* $\alpha > 0$ *with the lower limit zero for a function* f *can be defined as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

Where $n \in \mathbb{N}$, $0 \leq n-1 < \alpha < n$, $t > 0$.

Theorem 2.4. *[33] Let M be a bounded, closed, convex and nonempty subset of a Banach space X. Let A and B be operators such that:*

(i) $Ax + By \in M$ *whenever* $x, y \in M$.

(ii) A is compact and continuous.

(iii) B is a contraction mapping.
Then there exists $z \in M$ *such that* $z = Az + Bz$.

Lemma 2.5. *[3] Let* $\alpha, \beta \geq 0$, *then the following relations hold:*

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

Lemma 2.6. *[3] Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^{\alpha} \quad {}^{c}D^{\alpha} f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + ... + a_{n-1} t^{n-1}.$$

Lemma 2.7. *Let $x, y, z \in C([0,1], \mathbb{R})$ and $\Delta \neq 0$. Then the tripled system*

$$\begin{cases} {}^{c}D^{\beta_1}({}^{c}D^{\alpha_1} + \lambda_1)x(t) = h_1(t), & t \in [0,1], \\ {}^{c}D^{\beta_2}({}^{c}D^{\alpha_2} + \lambda_2)y(t) = h_2(t), & t \in [0,1], \\ {}^{c}D^{\beta_3}({}^{c}D^{\alpha_3} + \lambda_3)z(t) = h_3(t), & t \in [0,1], \end{cases}$$

subject to the boundary conditions (2), has a solution given by

$$\begin{aligned}
x(t) \quad = \quad & \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1} h_1(s)ds - \lambda_1 \frac{\displaystyle\int_0^t (t-s)^{\alpha_1-1} x(s)ds}{\Gamma(\alpha_1)} \\
& + A_1(t) \Bigg[\lambda_1 \frac{\displaystyle\int_0^1 (1-s)^{\alpha_1-1} x(s)ds}{\Gamma(\alpha_1)} - \frac{\displaystyle\sum_{i=1}^{n} \gamma_i \lambda_2 \int_0^{s_i} (s_i-s)^{\alpha_2-1} y(s)ds}{\Gamma(\alpha_2)} \\
& + \frac{\displaystyle\sum_{i=1}^{n} \gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2+\beta_2-1} h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\displaystyle\int_0^1 (1-s)^{\alpha_1+\beta_1-1} h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} \Bigg] \\
& + A_2(t) \Bigg[\frac{\lambda_2 \displaystyle\int_0^1 (1-s)^{\alpha_2-1} y(s)ds}{\Gamma(\alpha_2)} - \frac{\displaystyle\sum_{j=1}^{m} \delta_j \lambda_3 \int_0^{u_j} (u_j-s)^{\alpha_3-1} z(s)ds}{\Gamma(\alpha_3)} \\
& + \frac{\displaystyle\sum_{j=1}^{m} \delta_j \int_0^{u_j} (u_j-s)^{\alpha_3+\beta_3-1} h_3(s)ds}{\Gamma(\alpha_3+\beta_3)} - \frac{\displaystyle\int_0^1 (1-s)^{\alpha_2+\beta_2-1} h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} \Bigg] \\
& + A_3(t) \Bigg[\frac{\displaystyle\int_0^1 (1-s)^{\alpha_3-1} \lambda_3 z(s)ds}{\Gamma(\alpha_3)} - \frac{\displaystyle\int_0^1 (1-s)^{\alpha_3+\beta_3-1} h_3(s)ds}{\Gamma(\alpha_3+\beta_3)} \\
& - \frac{\displaystyle\sum_{k=1}^{p} \sigma_k \lambda_1 \int_0^{v_k} (v_k-s)^{\alpha_1-1} x(s)ds}{\Gamma(\alpha_1)} + \frac{\displaystyle\sum_{k=1}^{p} \sigma_k \int_0^{v_k} (v_k-s)^{\alpha_1+\beta_1-1} h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} \Bigg] \\
& + A_4(t) \Bigg[\frac{\lambda_1 \displaystyle\int_0^{a_1} (a_1-s)^{\alpha_1-1} x(s)ds}{\Gamma(\alpha_1)} - \frac{\displaystyle\int_0^{a_1} (a_1-s)^{\alpha_1+\beta_1-1} h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} \Bigg] \\
& + A_5(t) \Bigg[\frac{\lambda_2 \displaystyle\int_0^{b_1} (b_1-s)^{\alpha_2-1} y(s)ds}{\Gamma(\alpha_2)} - \frac{\displaystyle\int_0^{b_1} (b_1-s)^{\alpha_2+\beta_2-1} h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} \Bigg] \\
& + A_6(t) \Bigg[\frac{\lambda_3 \displaystyle\int_0^{c_1} (c_1-s)^{\alpha_3-1} z(s)ds}{\Gamma(\alpha_3)} - \frac{\displaystyle\int_0^{c_1} (c_1-s)^{\alpha_3+\beta_3-1} h_3(s)ds}{\Gamma(\alpha_3+\beta_3)} \Bigg],
\end{aligned}$$

$$
\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t (t-s)^{\alpha_2+\beta_2-1}h_2(s)ds - \lambda_2\frac{\int_0^t (t-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} \\
&+B_1(t)\Bigg[\lambda_2\frac{\int_0^1 (1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} \\
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)} - \frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\
&+B_2(t)\Bigg[\frac{\lambda_3\int_0^1(1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} - \frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} \\
&+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Bigg] \\
&+B_3(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} \\
&-\frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} + \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\
&+B_4(t)\Bigg[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\
&+B_5(t)\Bigg[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} - \frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Bigg] \\
&+B_6(t)\Bigg[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
z(t) &= \frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^t (t-s)^{\alpha_3+\beta_3-1}h_3(s)ds - \lambda_3\frac{\int_0^t (t-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} \\
&+C_1(t)\Bigg[\lambda_3\frac{\int_0^1 (1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} - \frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} \\
&+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]
\end{aligned}
$$

$$+C_2(t)\Big[\frac{\lambda_1\int_0^1(1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^n\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}$$

$$+\frac{\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$+C_3(t)\Big[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}$$

$$-\frac{\sum_{j=1}^m\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}+\frac{\sum_{j=1}^m\delta_j\int_0^{s_i}(s_i-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_4(t)\Big[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_5(t)\Big[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$+C_6(t)\Big[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Big]$$

where

$$\Upsilon_1=\frac{(1-a_1)}{\Gamma(\alpha_1+2)},\Upsilon_2=\frac{\sum_{i=1}^n\gamma_is_i^{\alpha_2}(b_1-s_i)}{\Gamma(\alpha_2+2)},\Upsilon_3=\frac{(1-b_1)}{\Gamma(\alpha_2+2)},\Upsilon_4=\frac{\sum_{j=1}^m\delta_ju_j^{\alpha_3}(c_1-u_j)}{\Gamma(\alpha_3+2)},$$

$$\Upsilon_5=\frac{(1-c_1)}{\Gamma(\alpha_3+2)},\quad \Upsilon_6=\frac{\sum_{k=1}^p\sigma_kv_k^{\alpha_1}(a_1-v_k)}{\Gamma(\alpha_1+2)},\Delta=\Upsilon_1\Upsilon_3\Upsilon_5+\Upsilon_2\Upsilon_4\Upsilon_6,$$

$$A(t)=\frac{(-a_1t^{\alpha_1}+t^{\alpha_1+1})}{\Delta\Gamma(2+\alpha_1)},\quad A_1(t)=A(t)\Upsilon_3\Upsilon_5,\quad A_2(t)=A(t)(-\Upsilon_2\Upsilon_5),$$

$$A_3(t)=A(t)\Upsilon_2\Upsilon_4,\quad A_4(t)=\frac{t^{\alpha_1}}{a_1^{\alpha_1}}+\frac{A(t)(-\Upsilon_3\Upsilon_5+\Upsilon_2\Upsilon_4\sum_{k=1}^p\sigma_kv_k^{\alpha_1})}{a_1^{\alpha_1}},$$

$$A_5(t)=\frac{A(t)(\Upsilon_2\Upsilon_5+\Upsilon_3\Upsilon_5\sum_{i=1}^n\gamma_is_i^{\alpha_2})}{b_1^{\alpha_2}},\quad A_6(t)=\frac{A(t)(-\Upsilon_2\Upsilon_4-\Upsilon_2\Upsilon_5\sum_{j=1}^m\delta_ju_j^{\alpha_3})}{c_1^{\alpha_3}},$$

$$B(t) = \frac{(-b_1 t^{\alpha_2} + t^{\alpha_2+1})}{\Delta\Gamma(2+\alpha_2)}, \quad B_1(t) = B(t)\Upsilon_1\Upsilon_5, \quad B_2(t) = B(t)(-\Upsilon_1\Upsilon_4),$$

$$B_3(t) = B(t)\Upsilon_4\Upsilon_6, \quad B_4(t) = \frac{t^{\alpha_2}}{b_1^{\alpha_2}} + \frac{B(t)(-\Upsilon_1\Upsilon_5 + \Upsilon_4\Upsilon_6 \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2})}{b_1^{\alpha_2}},$$

$$B_5(t) = \frac{B(t)(\Upsilon_1\Upsilon_4 + \Upsilon_1\Upsilon_5 \sum_{j=1}^{m} \delta_j u_j^{\alpha_3})}{c_1^{\alpha_3}}, \quad B_6(t) = \frac{B(t)(-\Upsilon_4\Upsilon_6 - \Upsilon_1\Upsilon_5 \sum_{k=1}^{p} \sigma_k v_k^{\alpha_1})}{a_1^{\alpha_1}},$$

$$C(t) = \frac{(-c_1 t^{\alpha_3} + t^{\alpha_3+1})}{\Delta\Gamma(2+\alpha_3)}, \quad C_1(t) = C(t)\Upsilon_1\Upsilon_3, \quad C_2(t) = C(t)(-\Upsilon_3\Upsilon_6),$$

$$C_3(t) = C(t)\Upsilon_2\Upsilon_6, \quad C_4(t) = \frac{t^{\alpha_3}}{c_1^{\alpha_3}} + \frac{C(t)(-\Upsilon_1\Upsilon_3 + \Upsilon_2\Upsilon_6 \sum_{j=1}^{m} \delta_j u_j^{\alpha_3})}{c_1^{\alpha_3}},$$

$$C_5(t) = \frac{C(t)(-\Upsilon_3\Upsilon_6 + \Upsilon_1\Upsilon_3 \sum_{k=1}^{p} \sigma_k v_k^{\alpha_1})}{a_1^{\alpha_1}}, \quad C_6(t) = \frac{C(t)(-\Upsilon_2\Upsilon_6 - \Upsilon_1\Upsilon_3 \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2})}{b_1^{\alpha_2}}.$$

Proof. Using lemma 2.6, we obtain
$x(t) = I^{\alpha_1+\beta_1}h_1(t) + I^{\alpha_1}a_{01} + I^{\alpha_1}a_{11}t - I^{\alpha_1}\lambda_1 x(t) + a_{21}$,
$y(t) = I^{\alpha_2+\beta_2}h_2(t) + I^{\alpha_2}a_{02} + I^{\alpha_2}a_{12}t - I^{\alpha_2}\lambda_2 y(t) + a_{22}$,
and, $z(t) = I^{\alpha_3+\beta_3}h_3(t) + I^{\alpha_3}a_{03} + I^{\alpha_3}a_{13}t - I^{\alpha_3}\lambda_3 z(t) + a_{23}$,
where $a_{01}, a_{11}, a_{21}, a_{02}, a_{12}, a_{22}, a_{03}, a_{13}, a_{23} \in \mathbb{R}$.
Using the facts that $x(0) = 0, \quad y((0) = 0, \quad z((0) = 0$, we get $a_{21} = a_{22} = a_{23} = 0$.
According to the condition $x(a_1) = y(b_1) = z(c_1) = 0$, we obtain

$$\begin{cases} a_{01} = \eta_1 + \theta_1 a_{11}, \\ a_{02} = \eta_2 + \theta_2 a_{12}, \\ a_{03} = \eta_3 + \theta_3 a_{13}, \end{cases} \tag{3}$$

where

$$\begin{cases} \eta_1 = \dfrac{\Gamma(\alpha_1+1)}{a_1^{\alpha_1}}\Big(\dfrac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)} - \dfrac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big), \\ \eta_2 = \dfrac{\Gamma(\alpha_2+1)}{b_1^{\alpha_2}}\Big(\dfrac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)} - \dfrac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Big), \\ \eta_3 = \dfrac{\Gamma(\alpha_3+1)}{c_1^{\alpha_3}}\Big(\dfrac{\int_0^{c_1}(c_1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)} - \dfrac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big), \\ \theta_1 = -\dfrac{a_1}{1+\alpha_1}, \quad \theta_2 = -\dfrac{b_1}{1+\alpha_2}, \quad \theta_3 = -\dfrac{c_1}{1+\alpha_3}. \end{cases}$$

By the conditions $x(1)=\sum_{i=1}^{n}\gamma_i y(s_i)$, $y(1)=\sum_{j=1}^{m}\delta_j z(u_j)$, $z(1)=\sum_{k=1}^{p}\sigma_k x(v_k)$ and (3), we have

$$\begin{cases}\Upsilon_1 a_{11}+\Upsilon_2 a_{12}=\Lambda_1,\\ \Upsilon_3 a_{12}+\Upsilon_4 a_{13}=\Lambda_2,\\ \Upsilon_5 a_{13}+\Upsilon_6 a_{11}=\Lambda_3,\end{cases}\tag{4}$$

where

$$\begin{aligned}
\Lambda_1 &= -\frac{1}{a_1^{\alpha_1}}\Big(\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big)\\
&+\frac{\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}}{b_1^{\alpha_2}}\Big(\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Big)\\
&+\frac{\lambda_1\int_0^{1}(1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\lambda_2\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}\\
&+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^{1}(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)},\\
\Lambda_2 &= -\frac{1}{b_1^{\alpha_2}}\Big(\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds\Big)\\
&+\frac{\sum_{i=1}^{n}\delta_j u_j^{\alpha_1}}{c_1^{\alpha_3}}\Big(\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big)\\
&+\frac{\lambda_2\int_0^{1}(1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\lambda_3\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3-1}(s)z(s)ds}{\Gamma(\alpha_3)}\\
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{1}(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds.\\
\Lambda_3 &= -\frac{1}{c_1^{\alpha_3}}\Big(\frac{\int_0^{c_1}(c_1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)}-\frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds\Big)\\
&+\frac{\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}}{a_1^{\alpha_1}}\Big(\frac{\int_0^{a_1}(a_1-s)^{\alpha_-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big)
\end{aligned}$$

$$+\frac{\int_0^1 (1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)} - \frac{\sum_{k=1}^{p}\sigma_k \int_0^{v_k} (v_k-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}$$
$$+\frac{\sum_{k=1}^{p}\sigma_k \int_0^{v_k} (v_k-s)^{\alpha_1+\beta_1-1} h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^1 (1-s)^{\alpha_3+\beta_3-1} h_3(s)ds.$$

By solving the systhem (4), we find that

$$a_{11} = \frac{1}{\Delta}\Big(\Lambda_1\Upsilon_3\Upsilon_5 - \Lambda_2\Upsilon_5\Upsilon_2 + \Lambda_3\Upsilon_4\Upsilon_2\Big),$$
$$a_{12} = \frac{1}{\Delta}\Big(\Lambda_1\Upsilon_4\Upsilon_6 + \Lambda_2\Upsilon_5\Upsilon_1 - \Lambda_3\Upsilon_4\Upsilon_1\Big),$$
$$a_{13} = \frac{1}{\Delta}\Big(-\Lambda_1\Upsilon_3\Upsilon_6 + \Lambda_2\Upsilon_6\Upsilon_2 + \Lambda_3\Upsilon_1\Upsilon_3\Big).$$

Substituting the values of a_{11}, a_{12} and a_{13} in (3), we have

$$a_{01} = \eta_1 + \frac{\theta_1}{\Delta}\Big(\Lambda_1\Upsilon_3\Upsilon_5 - \Lambda_2\Upsilon_5\Upsilon_2 + \Lambda_3\Upsilon_4\Upsilon_2\Big),$$
$$a_{02} = \eta_2 + \frac{\theta_2}{\Delta}\Big(\Lambda_1\Upsilon_4\Upsilon_6 + \Lambda_2\Upsilon_5\Upsilon_1 - \Lambda_3\Upsilon_4\Upsilon_1\Big),$$
$$a_{03} = \eta_3 + \frac{\theta_3}{\Delta}\Big(-\Lambda_1\Upsilon_3\Upsilon_6 + \Lambda_2\Upsilon_6\Upsilon_2 + \Lambda_3\Upsilon_1\Upsilon_3\Big).$$

Substituting the value of a_{01}, a_{02}, a_{03}, a_{11}, a_{12} and a_{13}, we obtain

$$x(t) = \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1} h_1(s)ds - \lambda_1\frac{\int_0^t (t-s)^{\alpha_1-1} x(s)ds}{\Gamma(\alpha_1)}$$
$$+A_1(t)\Bigg[\lambda_1\frac{\int_0^1 (1-s)^{\alpha_1-1} x(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^{n}\gamma_i\lambda_2 \int_0^{s_i} (s_i-s)^{\alpha_2-1} y(s)ds}{\Gamma(\alpha_2)}$$
$$+\frac{\sum_{i=1}^{n}\gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2+\beta_2-1} h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1} h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$
$$+A_2(t)\Bigg[\frac{\lambda_2\int_0^1 (1-s)^{\alpha_2-1} y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\lambda_3 \int_0^{u_j} (u_j-s)^{\alpha_3-1} z(s)ds}{\Gamma(\alpha_3)}$$
$$+\frac{\sum_{j=1}^{m}\delta_j \int_0^{u_j} (u_j-s)^{\alpha_3+\beta_3-1} h_3(s)ds}{\Gamma(\alpha_3+\beta_3)} - \frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1} h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$
$$+A_3(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)} - \frac{\int_0^1 (1-s)^{\alpha_3+\beta_3-1} h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}$$

$$
\begin{aligned}
& -\frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
& +A_4(t)\Bigg[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
& +A_5(t)\Bigg[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
& +A_6(t)\Bigg[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
y(t) \;=\; & \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}h_2(s)ds-\lambda_2\frac{\int_0^t(t-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}\\
& +B_1(t)\Bigg[\lambda_2\frac{\int_0^1(1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}\\
& +\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
& +B_2(t)\Bigg[\frac{\lambda_3\int_0^1(1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}\\
& +\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]\\
& +B_3(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\\
& -\frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
& +B_4(t)\Bigg[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
& +B_5(t)\Bigg[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]
\end{aligned}
$$

$$+B_6(t)\Big[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$z(t) = \frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^t(t-s)^{\alpha_3+\beta_3-1}h_3(s)ds-\lambda_3\frac{\int_0^t(t-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}$$

$$+C_1(t)\Big[\lambda_3\frac{\int_0^1(1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}$$

$$+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_2(t)\Big[\frac{\lambda_1\int_0^1(1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}$$

$$+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$+C_3(t)\Big[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}$$

$$-\frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}+\frac{\sum_{j=1}^{m}\delta_j\int_0^{s_i}(s_i-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_4(t)\Big[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}h_3(s)ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_5(t)\Big[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$+C_6(t)\Big[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Big].$$

By direct computation, it can easily be verified the converse of the lemma. □

3. Main results

Let X be a Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with norm $\| x \|= sup\{|x(t)| : t\in[0,1]\}$. Then, the product space $(X\times X\times X, \|(x;y;z)\|)$ is also a Banach space equipped with the norm $\|(x;y;z)\| = \|x\| + \|y\| + \|z\|$.
In view of lemma 2.7, we define the operator $U : X\times X\times X \longrightarrow X\times X\times X$

by $U(x,y,z)=(U_1(x,y,z),U_2(x,y,z),U_3(x,y,z))$.
Here

$$
\begin{aligned}
U_1(x,y,z)(t) &= \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),z(s))ds - \lambda_1 \\
&\times \frac{\int_0^t (t-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} + A_1(t)\Bigg[\lambda_1 \frac{\int_0^1 (1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} \\
&+ \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \\
&+ A_2(t)\Bigg[\frac{\lambda_2\int_0^1 (1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} \\
&+ \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)} - \frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\
&+ A_3(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)} - \frac{\int_0^1 (1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)} \\
&- \frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} + \frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \\
&+ A_4(t)\Bigg[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \\
&+ A_5(t)\Bigg[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))dsds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\
&+ A_6(t)\Bigg[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)} - \frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg], \\
U_2(x,y,z)(t) &= \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t (t-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds - \lambda_2\frac{\int_0^t (t-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} \\
&+ B_1(t)\Bigg[\lambda_2\frac{\int_0^1 (1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}
\end{aligned}
$$

$$+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$+B_2(t)\Bigg[\frac{\lambda_3\int_0^1(1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}$$

$$+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+B_3(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}$$

$$-\frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))dsds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$+B_4(t)\Bigg[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$+B_5(t)\Bigg[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+B_6(t)\Bigg[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg],$$

$$U_3(x,y,z)(t) = \frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^t(t-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds-\lambda_3\frac{\int_0^t(t-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}$$

$$+C_1(t)\Bigg[\lambda_3\frac{\int_0^1(1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}$$

$$+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+C_2(t)\Bigg[\frac{\lambda_1\int_0^1(1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}$$

$$+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$

$$+C_3(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}$$

$$-\frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}+\frac{\sum_{j=1}^{m}\delta_j\int_0^{s_i}(s_i-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+C_4(t)\Bigg[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+C_5(t)\Bigg[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$

$$+C_6(t)\Bigg[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg].$$

For computational convenience, we set

$$\Theta_1=\frac{1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+A_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\mu_1^*+\frac{A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\mu_2^*$$

$$+\frac{(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+A_6^*c_1^{\alpha_3+\beta_3})}{\Gamma(\alpha_3+\beta_3+1)}\mu_3^*,$$

$$r_{11}=max\Bigg\{\Theta_1+\frac{|\lambda_1|(1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}+A_4^*a_1^{\alpha_1})}{\Gamma(\alpha_1+1)},\Theta_1$$

$$+\frac{|\lambda_2|(A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}+A_5^*b_1^{\alpha_2})}{\Gamma(\alpha_2+1)},\Theta_1+\frac{|\lambda_3|(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}+A_6^*c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}\Bigg\}$$

$$\Theta_2=\frac{1+B_1^*+B_3^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\mu_2^*+\frac{B_2^*+B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+B_5^*c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}\mu_3^*$$

$$+\frac{(B_3^*+B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+B_6^*a_1^{\alpha_1})}{\Gamma(\alpha_1+1)}\mu_1^*,$$

$$r_{12} = max\left\{\Theta_2 + \frac{|\lambda_2|(1 + B_1^* + B_3^*\sum_{k=1}^{n}\gamma_i s_i^{\alpha_2} + B_4^* b_1^{\alpha_2})}{\Gamma(\alpha_2+1)}, \Theta_2 + \frac{|\lambda_3|(B_2^* + B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3} + B_5^* c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}, \Theta_2 + \frac{|\lambda_1|(B_3^* + B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} + B_6^* a_1^{\alpha_1})}{\Gamma(\alpha_1+1)}\right\}.$$

$$\Theta_3 = \frac{1 + C_1^* + C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3} + C_4^* c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}\mu_3^* + \frac{C_2^* + C_1^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1} + C_5^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\mu_1^* + \frac{(C_3^* + C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2} + C_6^* b_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)}\mu_2^*,$$

$$r_{13} = max\left\{\Theta_3 + \frac{|\lambda_3|(1 + C_1^* + C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3} + C_4^* c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}, \Theta_3 + \frac{|\lambda_1|(C_2^* + C_1^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} + C_5^* a_1^{\alpha_1})}{\Gamma(\alpha_1+1)}, \Theta_3 + \frac{|\lambda_2|(C_3^* + C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} + C_6^* b_1^{\alpha_2})}{\Gamma(\alpha_2+1)}\right\},$$

where
$A_i = sup\{A_i(t), t \in [0,1]\}$, $B_i^* = sup\{B_i(t), t \in [0,1]\}$, $C_i^* = sup\{C_i(t), t \in [0,1]\}$, $\mu_j^* = sup\{\mu_j(t), t \in [0,1]\}$, for $i = 1;2;3;4;5;6$ and $j = 1;2;3$.
Before introducing the main results, we impose some assumptions :

(H_1) - $f, g, k : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

(H_2) - There exist non negative functions $\mu_1, \mu_2, \mu_3 \in C\Big([0,1],[0,+\infty)\Big)$ such that for all $t \in [0,1]$ and $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, we have

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \le \mu_1(t)\Big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\Big)$$

$$|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)| \le \mu_2(t)\Big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\Big),$$

$$|k(t,x_1,y_1,z_1) - k(t,x_2,y_2,z_2)| \le \mu_3(t)\Big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\Big),$$

(H_3) - $|f(t,x,y,z)| \le m_1(t)$; $|g(t,x,y,z)| \le m_2(t)$; $|k(t,x,y,z)| \le m_3(t)$, $\forall(t,x,y,z) \in [0,1]\times\mathbb{R}^3$ with $m_1, m_2, m_3 \in C([0,1];\mathbb{R}^+)$.

Theorem 3.1. *Let* $\Delta \neq 0$.
Suppose that $(H_1) - (H_2)$ *are satisfied.*
Then there exist a unique solution for the system (1) and (2) provided that $r_{11} + r_{12} + r_{13} < 1$.

Proof. Define $\sup_{0\le t\le 1}|f(t,0,0,0)| = D_1$, $\sup_{0\le t\le 1}|g(t,0,0,0)| = D_2$, $\sup_{0\le t\le 1}|k(t,0,0,0)| = D_3$
Let $B_r = \{(x,y,z) \in X\times X\times X : \|(x,y,z)\| \le r\}$ with,
$r \ge \dfrac{r_{21}+r_{22}}{1-(r_{11}+r_{12})}$,

where

$$r_{21} = \frac{1 + A_1^* + A_3^* \sum_{k=1}^{p} \sigma_k v_k^{\alpha_1+\beta_1} + A_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} D_1 + \frac{A_2^* + A_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + A_5^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} D_2$$
$$+ \frac{(A_3^* + A_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_3+\beta_3} + A_6^* c_1^{\alpha_3+\beta_3})}{\Gamma(\alpha_3+\beta_3+1)} D_3,$$
$$r_{22} = \frac{1 + B_1^* + B_3^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} D_2 + \frac{B_2^* + B_1^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_3+\beta_3} + B_5^* c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} D_3$$
$$+ \frac{(B_3^* + B_2^* \sum_{k=1}^{p} \sigma_k v_k^{\alpha_1+\beta_1} + B_6^* a_1^{\alpha_1})}{\Gamma(\alpha_1+1)} D_1$$
$$r_{23} = \frac{1 + C_1^* + C_3^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_3+\beta_3} + C_4^* c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} D_3 + \frac{C_2^* + C_1^* \sum_{k=1}^{p} \sigma_k v_k^{\alpha_1+\beta_1} + C_5^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} D_1$$
$$+ \frac{(C_3^* + C_2^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + C_6^* b_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)} D_2.$$

We show that $TB_r \subseteq B_r$.
For $(x, y, z) \in B_r$, $t \in [0, 1]$, we have:

$$\begin{aligned}
|f(t, x(t), y(t), z(t))| &\leq |f(t, x(t), y(t), z(t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\
&\leq \mu_1(t)\Big(|x| + |y| + |z(t)|\Big) + D_1 \\
&\leq \mu_1^*\Big(\|x\| + \|y\| + \|y\|\Big) + D_1, \\
|g(t, x(t), y(t), z(t))| &\leq |g(t, x(t), y(t), z(t)) - g(t, 0, 0, 0)| + |g(t, 0, 0, 0)| \\
&\leq \mu_2(t)\Big(|x| + |y| + |z(t)|\Big) + D_2 \\
&\leq \mu_2^*\Big(\|y\| + \|x\| + \|z\|\Big) + D_2, \\
|k(t, x(t), y(t), z(t))| &\leq |k(t, x(t), y(t), z(t)) - k(t, 0, 0, 0)| + |k(t, 0, 0, 0)| \\
&\leq \mu_3(t)\Big(|x| + |y| + |z(t)|\Big) + D_3 \\
&\leq \mu_3^*\Big(\|y\| + \|x\| + \|z\|\Big) + D_3.
\end{aligned}$$

Then,

$$|U_1(x(t), y(t))| \leq \Big[\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_1^*}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_3^* \sum_{k=1}^{p} \sigma_k v_k^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}$$
$$+ \frac{A_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Big]\Big[\mu_1^*\Big(\|x\| + \|y\| + \|z\|\Big) + D_1\Big] + \Big[\frac{A_2^*}{\Gamma(\alpha_2+\beta_2+1)}$$

$$
\begin{aligned}
&+\frac{A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}+\frac{A_5^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\Big]\Big[\mu_2^*\Big(\|x\|+\|y\|+\|z\|\Big)+D_2\Big]+\Big[\frac{A_3^*}{\Gamma(\alpha_3+\beta_3+1)}\\
&+\frac{A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}+\frac{A_6^*c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}\Big]\Big[\mu_3^*\Big(\|x\|+\|y\|+\|z\|\Big)+D_3\Big]\\
&+\Big(\frac{|\lambda_1|A_1^*}{\Gamma(\alpha_1+1)}+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}+\frac{|\lambda_1|A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{|\lambda_1|A_4^*a_1^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\|x\|\\
&+\Big(\frac{|\lambda_2|A_2^*}{\Gamma(\alpha_2+1)}+\frac{|\lambda_2|A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{|\lambda_2|A_5^*b_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big)\|y\|\\
&+\Big(\frac{|\lambda_3|A_3^*}{\Gamma(\alpha_3+1)}+\frac{|\lambda_3|A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}}{\Gamma(\alpha_3+1)}+\frac{|\lambda_3|A_6^*c_1^{\alpha_3}}{\Gamma(\alpha_3+1)}\Big)\|z\|\\
&\le\Big[\frac{1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+A_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\mu_1^*+\frac{A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\mu_2^*\\
&+\frac{(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}+A_6^*c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}\mu_3^*+\frac{|\lambda_1|(1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}+A_4^*a_1^{\alpha_1})}{\Gamma(\alpha_1+1)}\Big]\|x\|\\
&+\Big[\frac{1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+A_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\mu_1^*+\frac{A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\mu_2^*\\
&+\frac{(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}+A_6^*c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}\mu_3^*+\frac{|\lambda_2|(A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}+A_5^*b_1^{\alpha_2})}{\Gamma(\alpha_2+1)}\Big]\|y\|\\
&+\Big[\frac{1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+A_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\mu_1^*+\frac{A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\mu_2^*\\
&+\frac{(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}+A_6^*c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}\mu_3^*+\frac{|\lambda_3|(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}+A_6^*c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}\Big]\|z\|\\
&+\frac{1+A_1^*+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+A_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}D_1+\frac{A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}D_2\\
&+\frac{(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}+A_6^*c_1^{\alpha_3})}{\Gamma(\alpha_3+1)}D_3.
\end{aligned}
$$

Consequently,

$$\|U_1(x(t),y(t),z(t))\|\le r_{11}r+r_{21}.$$

Similarly, we can obtain that
$\|U_2(x(t), y(t), z(t))\| \le r_{12}r + r_{22}$,
$\|U_3(x(t), y(t), z(t))\| \le r_{13}r + r_{32}$,
therfore, we get
$\|U(x(t), y(t))\| = \|U_1(x, y, z)\| + \|U_2(x, y, z)\| + \|U_3(x, y, z)\| \le (r_{11} + r_{12} + r_{13})r + r_{21} + r_{22} + r_{32} \le r.$
Now, for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X \times X \times X$ and for $t \in [0; 1]$, we have

$$|U_1(x_1, y_1, z_1)(t) - U_1(x_2, y_2, z_2)(t)| \le \Big[\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_1^*}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}$$
$$+\frac{A_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Big]\Big[\mu_1^*\Big(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|\Big)\Big] + \Big[\frac{A_2^*}{\Gamma(\alpha_2+\beta_2+1)}$$
$$+\frac{A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{A_5^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\Big]\Big[\mu_2^*\Big(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|\Big)\Big] + \Big[\frac{A_3^*}{\Gamma(\alpha_3+\beta_3+1)}$$
$$+\frac{A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} + \frac{A_6^* c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}\Big]\Big[\mu_3^*\Big(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|\Big)\Big]$$
$$+\Big(\frac{|\lambda_1|A_1^*}{\Gamma(\alpha_1+1)} + \frac{|\lambda_1|}{\Gamma(\alpha_1+1)} + \frac{|\lambda_1|A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{|\lambda_1|A_4^* a_1^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big)\|x_1-x_2\| + \Big(\frac{|\lambda_2|A_2^*}{\Gamma(\alpha_2+1)}$$
$$+\frac{|\lambda_2|A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{|\lambda_2|A_5^* b_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big)\|y_1-y_2\| + \Big(\frac{|\lambda_3|A_3^*}{\Gamma(\alpha_3+1)} + \frac{|\lambda_3|A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}}{\Gamma(\alpha_3+1)} + \frac{|\lambda_3|A_6^* c_1^{\alpha_3}}{\Gamma(\alpha_3+1)}\Big)\|z_1-z_2\|$$
$$\le r_{11}(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|).$$

In similar manner, we can also have
$|U_2(x_1, y_1, z_1)(t) - U_2(x_2, y_2, z_2)(t)| \le r_{12}(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|)$,
$|U_3(x_1, y_1, z_1)(t) - U_3(x_2, y_2, z_2)(t)| \le r_{13}(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|)$,
which leads to,

$$\|U(x_1, y_1, z1) - U(x_2, y_2, z_2)\| \le (r_{11} + r_{12} + r_{13})(\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|).$$

As $r_{11} + r_{12} + r_{13} < 1$, we deduce that the operator U is a contraction mapping. Then, the system (1) and (2) has a unique solution. □

Theorem 3.2. *Let* $\Delta \ne 0$.
Assume that $(H_1), (H_3)$ *hold.*
Then, the system (1) and (2) has at least one solution on $[0, 1]$ *if* $R < 1$, *where*

$$R = max\Big\{\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\Big(1 + A_1^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}A_3^* + A_4^* a_1^{\alpha_1} + B_3^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}B_2^* + B_6^* a_1^{\alpha_1}$$
$$+C_2^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}C_1^* + C_5^* a_1^{\alpha_1}\Big), \frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\Big(A_2^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}A_1^* + A_5^* b_1^{\alpha_2} + 1 + B_1^*$$
$$+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_3^* + B_4^* b_1^{\alpha_2} + C_3^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^* + C_6^* b_1^{\alpha_2}\Big), \frac{|\lambda_3|}{\Gamma(\alpha_3+1)}\Big(A_3^* + \sum_{j=1}^{m}\delta_j u_j^{\alpha_3}A_2^*$$

$$+A_6^*c_1^{\alpha_3}+B_2^*+\sum_{j=1}^{n}\delta_j u_j^{\alpha_3}B_1^*+B_5^*c_1^{\alpha_3}+1+C_1^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}C_3^*+C_4^*c_1^{\alpha_3}\Big)\Big\}.$$

Proof. We define a bounded closed and convex ball $B_{r'}=\{(x,y,z)\in X\times X\times X:\|(x,y,z)\|\le r'\}$ with $r'\ge\dfrac{r_2'}{1-R}$, where,

$$r_2'=\frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\Big((1+A_1^*+A_4^*a_1^{\alpha_1+\beta_1}+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+B_3^*+B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}$$
$$+B_6^*a_1^{\alpha_1+\beta_1}+C_3^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^*+C_6^*b_1^{\alpha_2}\Big)+\frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\Big(A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}$$
$$+A_5^*b_1^{\alpha_2+\beta_2}+1+B_1^*+B_4^*b_1^{\alpha_2+\beta_2}+B_3^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}$$
$$+C_6^*b_1^{\alpha_2+\beta_2}\Big)+\frac{\|m_3\|}{\Gamma(\alpha_3+\beta_3+1)}\Big(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+A_6^*c_1^{\alpha_3+\beta_3}+B_2^*$$
$$+B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+B_5^*c_1^{\alpha_3+\beta_3}+1+C_1^*+C_4^*c_1^{\alpha_3+\beta_3}+C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}\Big).$$

Let us introduce the decomposition $U(x,y,z)(t)=W_1(x,y,z)(t)+W_2(x,y,z)(t)$,
where
$W_1(x,y,z)(t)=(T_1(x,y,z),R_1(x,y,z),P_1(x,y,z))(t)$,
$W_2(x,y,z)(t)=(T_2(x,y,z),R_2(x,y,z),P_2(x,y,z))(t)$,
with

$$T_1(x,y,z)(t)=\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t(t-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds+A_1(t)\Big[\sum_{i=1}^{n}\gamma_i$$
$$\frac{\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}-\tfrac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 f(s,x(s),y(s),z(s))$$
$$\times(1-s)^{\alpha_1+\beta_1-1}ds\Big]+A_2(t)\Big[\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}$$
$$-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Big]+A_3(t)\Big[\tfrac{-1}{\Gamma(\alpha_3+\beta_3)}\int_0^1(1-s)^{\alpha_3+\beta_3-1}$$
$$\times k(s,x(s),y(s),z(s))ds+\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$
$$+A_4(t)\Big[-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$
$$+A_5(t)\Big[-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Big]$$

$$+A_6(t)\left[-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\right]$$

$$T_2(x,y)(t)=-\frac{\int_0^{t}(t-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}+A_1(t)\left[\frac{\int_0^{1}(1-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}\right.$$

$$\left.-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}\right]+A_2(t)\left[\int_0^{1}\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\lambda_2 y(s)ds\right.$$

$$\left.-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)}\right]+A_3(t)\left[\int_0^{1}\frac{(1-s)^{\alpha_3-1}}{\Gamma(\alpha_3)}\lambda_3 z(s)ds\right.$$

$$\left.-\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}\right]+A_4(t)\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}$$

$$+\frac{A_5(t)\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}+\frac{A_6(t)\int_0^{c_1}(c_1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)},$$

$$R_1(x,y,z)(t)=\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{t}(t-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds+B_1(t)\left[\sum_{j=1}^{m}\delta_j\right.$$

$$\frac{\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{1}g(s,x(s),y(s),z(s))$$

$$\left.\times(1-s)^{\alpha_2+\beta_2-1}ds\right]+B_2(t)\left[\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\right.$$

$$\left.-\frac{\int_0^{1}(1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\right]+B_3(t)\left[\frac{-1}{\Gamma(\alpha_1+\beta_1)}\int_0^{1}(1-s)^{\alpha_1+\beta_1-1}\right.$$

$$\left.\times f(s,x(s),y(s),z(s))ds+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

$$+B_4(t)\left[-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

$$+B_5(t)\left[-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\right]$$

$$+B_6(t)\left[-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\right]$$

$$R_2(x,y,z)(t) = -\frac{\int_0^t (t-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)} + B_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_1)}$$

$$-\frac{\sum_{j=1}^{m}\delta_j \int_0^{u_j}(u_j-s)^{\alpha_3-1}\lambda_3 y(s)ds}{\Gamma(\alpha_3)}\Bigg] + B_2(t)\Bigg[\int_0^1 \frac{(1-s)^{\alpha_3-1}}{\Gamma(\alpha_3)}\lambda_3 z(s)ds$$

$$-\frac{\sum_{k=1}^{p}\sigma_k \int_0^{v_k}(v_k-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}\Bigg] + B_3(t)\Bigg[\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\lambda_1 x(s)ds$$

$$-\frac{\sum_{i=1}^{n}\gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}\Bigg] + B_4(t)\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}$$

$$+\frac{B_5(t)\int_0^{c_1}(c_1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)} + \frac{B_6(t)\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)},$$

$$P_1(x,y,z)(t) = \frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^t (t-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds + C_1(t)\Bigg[\sum_{k=1}^{p}\sigma_k$$

$$\frac{\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)} - \tfrac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^1 k(s,x(s),y(s),z(s))$$

$$\times(1-s)^{\alpha_3+\beta_3-1}ds\Bigg] + C_2(t)\Bigg[\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}$$

$$-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] + C_3(t)\Bigg[\tfrac{-1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}$$

$$\times g(s,x(s),y(s),z(s))ds + \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+C_4(t)\Bigg[-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$+C_5(t)\Bigg[-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$

$$+C_6(t)\Bigg[-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$P_2(x,y,z)(t) = -\frac{\int_0^t (t-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)} + C_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)}$$

$$-\frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)}\Bigg] + C_2(t)\Bigg[\int_0^1\frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\lambda_1 x(s)ds$$

$$-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}\Bigg] + C_3(t)\Bigg[\int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\lambda_2 y(s)ds$$

$$-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)}\Bigg] + C_4(t)\frac{\int_0^{c_1}(c_1-s)^{\alpha_3-1}\lambda_3 z(s)ds}{\Gamma(\alpha_3)}$$

$$+\frac{C_5(t)\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1 x(s)ds}{\Gamma(\alpha_1)} + \frac{C_6(t)\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2 y(s)ds}{\Gamma(\alpha_2)}.$$

For $(x,y,z)\in B_{r'}$, we have

$$|T_1(x,y,z)(t)+T_2(x,y,z)(t)| \le \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_1^*\|m_2\|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{A_1^*\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}$$

$$+\frac{A_2^*\|m_3\|\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} + \frac{A_2^*\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)} +$$

$$+\frac{A_3^*\|m_1\|\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_3^*\|m_3\|}{\Gamma(\alpha_3+\beta_3+1)} + \frac{A_4^* a_1^{\alpha_1+\beta_1}\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}$$

$$+\frac{A_5^* b_1^{\alpha_2+\beta_2}\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)} + \frac{A_6^* c_1^{\alpha_3+\beta_3}\|m_3\|}{\Gamma(\alpha_3+\beta_3+1)}$$

$$+\frac{|\lambda_1|\|x\|}{\Gamma(\alpha_1+1)} + \frac{A_1^*|\lambda_1|\|x\|}{\Gamma(\alpha_1+1)} + \frac{|\lambda_2|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}\|y\|A_1^*}{\Gamma(\alpha_2+1)} + \frac{A_2^*|\lambda_2|\|y\|}{\Gamma(\alpha_2+1)} + \frac{|\lambda_3|\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}\|z\|A_2^*}{\Gamma(\alpha_3+1)}$$

$$+\frac{A_3^*|\lambda_3|\|z\|}{\Gamma(\alpha_3+1)} + \frac{|\lambda_1|\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}\|x\|A_3^*}{\Gamma(\alpha_1+1)} + \frac{A_4^*|\lambda_1|a_1^{\alpha_1}\|x\|}{\Gamma(\alpha_1+1)} + \frac{A_5^*|\lambda_2|b_1^{\alpha_2}\|y\|}{\Gamma(\alpha_2+1)} + \frac{A_6^*|\lambda_3|c_1^{\alpha_3}\|z\|}{\Gamma(\alpha_3+1)}$$

$$\le \frac{|\lambda_1|\left(1+A_1^*+\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1}A_3^*+A_4^*a_1^{\alpha_1}\right)\|x\|}{\Gamma(\alpha_1+1)} + \frac{|\lambda_2|\left(A_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}A_1^*+A_5^*b_1^{\alpha_2}\right)\|y\|}{\Gamma(\alpha_2+1)}$$

$$+\frac{|\lambda_3|\left(A_3^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_3}A_2^*+A_6^*c_1^{\alpha_3}\right)\|z\|}{\Gamma(\alpha_3+1)} + \frac{\|m_1\|(1+A_1^*+A_4^*a_1^{\alpha_1+\beta_1}+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1})}{\Gamma(\alpha_1+\beta_1+1)}$$

$$+\frac{\|m_2\|(A_2^* + A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2} + A_5^* b_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\|m_3\|(A_3^* + A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3} + A_6^* c_1^{\alpha_3+\beta_3})}{\Gamma(\alpha_3+\beta_3+1)}.$$

By similar computation, we find

$$|R_1(x,y,z)(t) + R_2(x,y,z)(t)| \le \frac{|\lambda_2|\Big(1 + B_1^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} B_3^* + B_4^* b_1^{\alpha_2}\Big)\|y\|}{\Gamma(\alpha_2+1)}$$
$$+\frac{|\lambda_3|\Big(B_2^* + \sum_{j=1}^{n}\delta_j u_j^{\alpha_3} B_1^* + B_5^* c_1^{\alpha_3}\Big)\|z\|}{\Gamma(\alpha_3+1)}$$
$$+\frac{|\lambda_1|\Big(B_3^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} B_2^* + B_6^* a_1^{\alpha_1}\Big)\|x\|}{\Gamma(\alpha_1+1)} + \frac{\|m_2\|(1 + B_1^* + B_4^* b_1^{\alpha_2+\beta_2} + B_3^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)}$$
$$+\frac{\|m_3\|(B_2^* + B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3} + B_5^* c_1^{\alpha_3+\beta_3})}{\Gamma(\alpha_3+\beta_3+1)} + \frac{\|m_1\|(B_3^* + B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1} + B_6^* a_1^{\alpha_1+\beta_1})}{\Gamma(\alpha_1+\beta_1+1)},$$

$$|P_1(x,y,z)(t) + P_2(x,y,z)(t)| \le \frac{|\lambda_3|\Big(1 + C_1^* + \sum_{j=1}^{m}\delta_j u_j^{\alpha_3} C_3^* + C_4^* c_1^{\alpha_3}\Big)\|z\|}{\Gamma(\alpha_3+1)}$$
$$+\frac{|\lambda_1|\Big(C_2^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} C_1^* + C_5^* a_1^{\alpha_1}\Big)\|x\|}{\Gamma(\alpha_1+1)}$$
$$+\frac{|\lambda_2|\Big(C_3^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} C_2^* + C_6^* b_1^{\alpha_2}\Big)\|y\|}{\Gamma(\alpha_2+1)} + \frac{\|m_3\|(1 + C_1^* + C_4^* c_1^{\alpha_3+\beta_3} + C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3})}{\Gamma(\alpha_3+\beta_3+1)}$$
$$+\frac{\|m_1\|(C_2^* + C_1^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1} + C_5^* a_1^{\alpha_1+\beta_1})}{\Gamma(\alpha_1+\beta_1+1)} + \frac{\|m_2\|(C_3^* + C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2} + C_6^* b_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_1+2)}.$$

From above, we obtain

$$\|W_1(x,y,z)(t) + W_2(x,y,z)\| \le Rr' + r_2' \le r'.$$

Thus, $W_1(x,y,z)(t) + W_2(x,y,z)(t) \in B_{r'}$.
For $(x_1,y_1,z_1),(x_2,y_2,z_2) \in B_{r'}$ and $t \in [0,1]$, we have

$$|T_2(x_1,y_1,z_1)(t) - T_2(x_2,y_2,z_2)(t)| \le \frac{|\lambda_1|\Big(1 + A_1^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} A_3^* + A_4^* a_1^{\alpha_1}\Big)\|x_1 - x_2\|}{\Gamma(\alpha_1+1)}$$
$$+\frac{|\lambda_2|\Big(A_2^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} A_1^* + A_5^* b_1^{\alpha_2}\Big)\|y_1 - y_2\|}{\Gamma(\alpha_2+1)}$$

$$+\frac{|\lambda_3|\left(A_3^* + \sum_{j=1}^{m}\delta_j u_j^{\alpha_3} A_2^* + A_6^* c_1^{\alpha_3}\right)\|z_1 - z_2\|}{\Gamma(\alpha_3+1)}$$

$$\begin{aligned}|R_2(x_1,y_1,z_1)(t) - R_2(x_2,y_2,z_2)(t)| \leq\ & \frac{|\lambda_2|\left(1 + B_1^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} B_3^* + B_4^* b_1^{\alpha_2}\right)\|y_1 - y_2\|}{\Gamma(\alpha_2+1)} \\ & + \frac{|\lambda_3|\left(B_2^* + \sum_{j=1}^{n}\delta_j u_j^{\alpha_3} B_1^* + B_5^* c_1^{\alpha_3}\right)\|z_1 - z_2\|}{\Gamma(\alpha_3+1)} \\ & + \frac{|\lambda_1|\left(B_3^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} B_2^* + B_6^* a_1^{\alpha_1}\right)\|x_1 - x_2\|}{\Gamma(\alpha_1+1)},\end{aligned}$$

$$\begin{aligned}|P_2(x_1,y_1,z_1)(t) - P_2(x_2,y_2,z_2)(t)| \leq\ & \frac{|\lambda_3|\left(1 + C_1^* + \sum_{j=1}^{m}\delta_j u_j^{\alpha_3} C_3^* + C_4^* c_1^{\alpha_3}\right)\|z_1 - z_2\|}{\Gamma(\alpha_3+1)} \\ & + \frac{|\lambda_1|\left(C_2^* + \sum_{k=1}^{p}\sigma_k v_k^{\alpha_1} C_1^* + C_5^* a_1^{\alpha_1}\right)\|x_1 - x_2\|}{\Gamma(\alpha_1+1)} \\ & + \frac{|\lambda_2|\left(C_3^* + \sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} C_2^* + C_6^* b_1^{\alpha_2}\right)\|y_1 - y_2\|}{\Gamma(\alpha_2+1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|W_2(x_1,y_1,z_1) - W_2(x_2,y_2,z_2)\| &\leq R\|x_1 - x_2\| + R\|y_1 - y_2\| + R\|z_1 - z_2\| \\ &\leq R\|(x_1 - x_2, y_1 - y_2, z_1 - z_2)\|.\end{aligned}$$

As $R < 1$, we find that W_2 is a contraction.
Next, we prove that W_1 is compact and continuous. The continuity of f, g, k implies that the operator W_1 is continuous. Moreover, W_1 is uniformly bounded on $B_{r'}$.
Suppose that $0 \leq t_1 < t_2 \leq 1$. We have

$$\begin{aligned}|T_1(x,y,z)(t_2) - T_1(x,y,z)(t_1)| \leq\ & \frac{\Big|\int_0^{t_2}(t_2 - s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)} \\ & - \int_0^{t_1}(t_1 - s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),\Phi y(s))ds\Big| + |A_1(t_2) - A_1(t_1)|\Bigg[\sum_{i=1}^{n}\gamma_i \\ & \frac{\int_0^{s_i}(s_i - s)^{\alpha_2+\beta_2-1} g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)} - \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 f(s,x(s),y(s),z(s)) \\ & \times(1-s)^{\alpha_1+\beta_1-1}ds\Bigg] + |A_2(t_2) - A_2(t_1)|\Bigg[\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j - s)^{\alpha_3+\beta_3-1} k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\end{aligned}$$

$$-\frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1} g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] + |A_3(t_2)-A_3(t_1)|\Bigg[\frac{-1}{\Gamma(\alpha_3+\beta_3)}\int_0^1 (1-s)^{\alpha_3+\beta_3-1}$$

$$\times k(s,x(s),y(s),z(s))ds + \frac{\sum_{k=1}^{p}\sigma_k \int_0^{v_k} (v_k-s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$

$$+|A_4(t_2)-A_4(t_1)|\Bigg[-\frac{\int_0^{a_1} (a_1-s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$

$$+|A_5(t_2)-A_5(t_1)|\Bigg[-\frac{\int_0^{b_1} (b_1-s)^{\alpha_2+\beta_2-1} g(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$+|A_6(t_2)-A_6(t_1)|\Bigg[-\frac{\int_0^{c_1} (c_1-s)^{\alpha_3+\beta_3-1} k(s,x(s),y(s),z(s))ds}{\Gamma(\alpha_3+\beta_3)}\Bigg]$$

$$\leq \frac{\|m_1\|(t_2^{\alpha_1+\beta_1}-t_1^{\alpha_1+\beta_1})}{\Gamma(\alpha_1+\beta_1+1)} + |A_1(t_2)-A_1(t_1)|\Bigg[\frac{\|m_2\|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\Bigg]$$

$$+|A_2(t_2)-A_2(t_1)|\Bigg[\frac{\|m_3\|\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} + \frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\Bigg] + |A_3(t_2)-A_3(t_1)|$$

$$\times\Bigg[\frac{\|m_1\|\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{\|m_3\|}{\Gamma(\alpha_3+\beta_3+1)}\Bigg] + \frac{|A_4(t_2)-A_4(t_1)|\|m_1\|a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}$$

$$+\frac{|A_5(t_2)-A_5(t_1)|\|m_2\|b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|A_6(t_2)-A_6(t_1)|\|m_3\|c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}.$$

Similarly, we find that

$$|R_1(x,y,z)(t_2)-R_1(x,y,z)(t_1)| \leq \frac{\|m_2\|(t_2^{\alpha_2+\beta_2}-t_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)} + |B_1(t_2)-B_1(t_1)|$$

$$\times\Bigg[\frac{\|m_3\|\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} + \frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\Bigg] + |B_2(t_2)-B_2(t_1)|\Bigg[\frac{\|m_1\|\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}$$

$$+\frac{\|m_3\|}{\Gamma(\alpha_3+\beta_3+1)}\Bigg] + |B_3(t_2)-B_3(t_1)|\Bigg[\frac{\|m_2\|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\Bigg]$$

$$+\frac{|B_4(t_2)-B_4(t_1)|\|m_2\|b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|B_5(t_2)-B_5(t_1)|\|m_3\|c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)} + \frac{|B_6(t_2)-B_6(t_1)|\|m_1\|a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)},$$

$$|P_1(x,y,z)(t_2)-P_1(x,y,z)(t_1)| \leq \frac{\|m_3\|(t_2^{\alpha_3+\beta_3}-t_1^{\alpha_3+\beta_3})}{\Gamma(\alpha_3+\beta_3+1)} + |C_1(t_2)-C_1(t_1)|$$

$$\times\Bigg[\frac{\|m_1\|\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{\|m_3\|}{\Gamma(\alpha_3+\beta_3+1)}\Bigg]+|C_2(t_2)-C_2(t_1)|\Bigg[\frac{\|m_2\|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}$$
$$+\frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\Bigg]+|C_3(t_2)-C_3(t_1)|\Bigg[\frac{\|m_3\|\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}+\frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\Bigg]$$
$$+\frac{|C_4(t_2)-C_4(t_1)|\|m_3\|c_1^{\alpha_3+\beta_3}}{\Gamma(\alpha_3+\beta_3+1)}+\frac{|C_5(t_2)-C_5(t_1)|\|m_1\|a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|C_6(t_2)-C_6(t_1)|\|m_2\|b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}.$$

Thus, the operator W_1 is equicontinuous. Then, W_1 is relatively compact on $B_{r'}$. So, by Arzela Ascoli theorem, the operator W_1 is compact on $B_{r'}$. In conclusion, all terms of Krasnoselskii's theorm are satisfied. Hence, (1) and
(2) has at least one solution on $B_{r'}$. □

4. Ulam-Hyers Stability

Definition 4.1. *For some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we consider the system of inequalities*

$$\begin{cases}\left|{}^cD^{\beta_1}({}^cD^{\alpha_1}+\lambda_1)x^*(t)-f(t,x^*(t),y^*(t),z^*(t))\right|<\varepsilon_1, & t\in[0,1],\\ \left|{}^cD^{\beta_2}({}^cD^{\alpha_2}+\lambda_2)y^*(t)-g(t,x^*(t),y^*(t),z^*(t))\right|<\varepsilon_2, & t\in[0,1]\\ \left|{}^cD^{\beta_3}({}^cD^{\alpha_3}+\lambda_3)z^*(t)-k(t,x^*(t),y^*(t),z^*(t))\right|<\varepsilon_3, & t\in[0,1].\end{cases} \tag{5}$$

Then the system (1)- (2) is Ulam-Hyers stable if there exist $C_1, C_2, C_3 > 0$, such that there is a unique solution (x, y, z) of the problem (1)- (2) with

$$\|(x^*, y^*, z^*) - (x, y, z)\| \le C_1\varepsilon_1 + C_2\varepsilon_2 + C_3\varepsilon_3.$$

Remark: $(x^*, y^*, z*)$ is a solution of the system of inequalities (5) if we can find $\rho_1, \rho_2, \rho_3 \in \Big(C[0,1];\mathbb{R}\Big)$ such that $|\rho_1(t)| \le \varepsilon_1$, $|\rho_2(t)| \le \varepsilon_2$, $|\rho_3(t)| \le \varepsilon_3$, $\quad t \in [0,1]$ and

$$\begin{cases}{}^cD^{\beta_1}({}^cD^{\alpha_1}+\lambda_1)x^*(t)=f(t,x^*(t),y^*(t),z^*(t))+\rho_1(t), & t\in[0,1],\\ {}^cD^{\beta_2}({}^cD^{\alpha_2}+\lambda_2)y^*(t)=g(t,x^*(t),y^*(t),z^*(t))+\rho_2(t), & t\in[0,1],\\ {}^cD^{\beta_3}({}^cD^{\alpha_3}+\lambda_3)z^*(t)=k(t,x^*(t),y^*(t),z^*(t))+\rho_3(t), & t\in[0,1].\end{cases} \tag{6}$$

Theorem 4.2. *If $(H_1), (H_2)$ and $r_{11} + r_{12} + r_{13} < 1$ are satisfied, then the problem (1)- (2) is Ulam-Hyers stable.*

Proof. Let (x, y, z) be unique solution of the system (1)- (2) and (x^*, y^*, z^*) be a solution of (5)- (2). Then we have $\rho_1, \rho_2, \rho_3 \in \Big(C[0,1];\mathbb{R}\Big)$ such that

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1}+\lambda_1)x^*(t) = f(t, x^*(t), y^*(t), z^*(t)) + \rho_1(t), \quad t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2}+\lambda_2)y^*(t) = g(t, x^*(t), y^*(t), z^*(t)) + \rho_2(t), \quad t \in [0,1], \\ {}^cD^{\beta_3}({}^cD^{\alpha_3}+\lambda_3)z^*(t) = k(t, x^*(t), y^*(t), z^*(t)) + \rho_3(t), \quad t \in [0,1], \\ x(0) = 0; \quad x(a_1) = 0; \quad x(1) = \sum_{i=1}^{n} \gamma_i y(s_i), \\ y(0) = 0; \quad y(b_1) = 0; \quad y(1) = \sum_{j=1}^{m} \delta_i z(u_j), \\ z(0) = 0; \quad z(c_1) = 0; \quad z(1) = \sum_{k=1}^{p} \sigma_k x(v_k), \\ 0 < a_1 < b_1 < c_1 < s_1 < s_2 < ... < s_n < u_1 < u_2 < ... < u_m < v_1 < v_2 < ... < v_p < 1. \end{cases} \tag{7}$$

By Lemma 2.7, we get

$$\begin{aligned} x^*(t) &= \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}(f(s, x^*(s), y^*(s), z^*(s)) + \rho_1(s))ds - \lambda_1 \frac{\int_0^t (t-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)} \\ &+A_1(t)\Bigg[\lambda_1 \frac{\int_0^1 (1-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^{n}\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)} \\ &+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}(g(s,s,x^*(s),y^*(s),z^*(s)) + \rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)} \\ &-\frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1}(f(s,s,x^*(s),y^*(s),z^*(s)) + \rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \\ &+A_2(t)\Bigg[\frac{\lambda_2\int_0^1 (1-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)} \\ &+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}(k(s,s,x^*(s),y^*(s),z^*(s)) + \rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)} \\ &-\frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1}(g(s,s,x^*(s),y^*(s),z^*(s)) + \rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\ &+A_3(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_3-1}\lambda_3 z^*(s)ds}{\Gamma(\alpha_3)} - \frac{\int_0^1 (1-s)^{\alpha_3+\beta_3-1}(k(s,s,x^*(s),y^*(s),z^*(s)) + \rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)} \\ &-\frac{\sum_{k=1}^{p}\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)} + \frac{\sum_{k=1}^{p}\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}(f(s,s,x^*(s),y^*(s),z^*(s)) + \rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \end{aligned}$$

$$+A_4(t)\left[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\right]$$

$$+A_5(t)\left[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

$$+A_6(t)\left[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)}\right],$$

$$y^*(t)=\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds-\lambda_2\frac{\int_0^t(t-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}$$

$$+B_1(t)\left[\lambda_2\frac{\int_0^1(1-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}-\frac{\sum_{j=1}^m\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}\right.$$

$$+\frac{\sum_{j=1}^m\delta_j\int_0^{u_j}(u_j-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)}$$

$$\left.-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

$$+B_2(t)\left[\frac{\lambda_3\int_0^1(1-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}-\frac{\sum_{k=1}^p\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)}\right.$$

$$+\frac{\sum_{k=1}^p\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}$$

$$\left.-\frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)}\right]$$

$$+B_3(t)\left[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1x^*(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\right.$$

$$\left.-\frac{\sum_{i=1}^n\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}+\frac{\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

$$+B_4(t)\left[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

$$+B_5(t)\left[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)}\right]$$

$$+B_6(t)\Big[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\Big].$$

$$z^*(t)=\frac{1}{\Gamma(\alpha_3+\beta_3)}\int_0^t(t-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds-\lambda_3\frac{\int_0^t(t-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}$$

$$+C_1(t)\Big[\lambda_3\frac{\int_0^1(1-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}-\frac{\sum_{k=1}^p\sigma_k\lambda_1\int_0^{v_k}(v_k-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)}$$

$$+\frac{\sum_{k=1}^p\sigma_k\int_0^{v_k}(v_k-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}$$

$$-\frac{\int_0^1(1-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_2(t)\Big[\frac{\lambda_1\int_0^1(1-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^n\gamma_i\lambda_2\int_0^{s_i}(s_i-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}$$

$$+\frac{\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}$$

$$-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$+C_3(t)\Big[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2y^*(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}$$

$$-\frac{\sum_{j=1}^m\delta_j\lambda_3\int_0^{u_j}(u_j-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}+\frac{\sum_{j=1}^m\delta_j\int_0^{s_i}(s_i-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))ds}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_4(t)\Big[\frac{\lambda_3\int_0^{c_1}(c_1-s)^{\alpha_3-1}z^*(s)ds}{\Gamma(\alpha_3)}-\frac{\int_0^{c_1}(c_1-s)^{\alpha_3+\beta_3-1}(k(s,x^*(s),y^*(s),z^*(s))+\rho_3(s))}{\Gamma(\alpha_3+\beta_3)}\Big]$$

$$+C_5(t)\Big[\frac{\lambda_1\int_0^{a_1}(a_1-s)^{\alpha_1-1}x^*(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}(f(s,x^*(s),y^*(s),z^*(s))+\rho_1(s))ds}{\Gamma(\alpha_1+\beta_1)}\Big]$$

$$+C_6(t)\Big[\frac{\lambda_2\int_0^{b_1}(b_1-s)^{\alpha_2-1}y^*(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}(g(s,x^*(s),y^*(s),z^*(s))+\rho_2(s))ds}{\Gamma(\alpha_2+\beta_2)}\Big].$$

By, $|\rho_1(t)|\le\varepsilon_1$, $|\rho_2(t)|\le\varepsilon_2$ and $|\rho_3(t)|\le\varepsilon_3$, $\quad t\in[0,1]$, we obtain

$$\left|x^*(t)-U_1(x^*,y^*,z^*)\right| \le \frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Big(1+A_1^*+A_4^*a_1^{\alpha_1+\beta_1}+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}\Big)$$
$$+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Big(A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}\Big)$$
$$+\frac{\varepsilon_3}{\Gamma(\alpha_3+\beta_3+1)}\Big(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+A_6^*c_1^{\alpha_3+\beta_3}\Big),$$
$$\left|y^*(t)-U_2(x^*,y^*,z^*)\right| \le +\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Big(B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+B_3^*+B_6^*a_1^{\alpha_1+\beta_1}\Big)$$
$$+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Big(1+B_1^*+B_3^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\Big)$$
$$+\frac{\varepsilon_3}{\Gamma(\alpha_3+\beta_3+1)}\Big(B_2^*+B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+B_5^*c_1^{\alpha_3+\beta_3}\Big)$$
$$\left|z^*(t)-U_3(x^*,y^*,z^*)\right| \le \frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Big(C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*+C_6^*b_1^{\alpha_2+\beta_2}\Big)$$
$$+\frac{\varepsilon_3}{\Gamma(\alpha_3+\beta_3+1)}\Big(1+C_1^*+C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+C_4^*c_1^{\alpha_3+\beta_3}\Big)$$
$$+\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Big(C_2^*+C_1^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+C_5^*a_1^{\alpha_1+\beta_1}\Big).$$

Thus,

$$\|U(x^*,y^*,z^*)-(x^*,y^*,z^*)\| = \|U_1(x^*,y^*,z^*)-x^*\|+\|U_2(x^*,y^*,z^*)-y^*\|+\|U_3(x^*,y^*,z^*)-z^*\|$$
$$\le \frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Big(1+A_1^*+A_4^*a_1^{\alpha_1+\beta_1}+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+B_3^*+B_6^*a_1^{\alpha_1+\beta_1}+C_2^*$$
$$+C_1^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+C_5^*a_1^{\alpha_1+\beta_1}\Big)+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Big(A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^*b_1^{\alpha_2+\beta_2}$$
$$+1+B_1^*+B_3^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*+C_6^*b_1^{\alpha_2+\beta_2}\Big)$$
$$+\frac{\varepsilon_3}{\Gamma(\alpha_3+\beta_3+1)}\Big(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+A_6^*c_1^{\alpha_3+\beta_3}+B_2^*+B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+B_5^*c_1^{\alpha_3+\beta_3}+1+C_1^*$$
$$+C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+C_4^*c_1^{\alpha_3+\beta_3}\Big).$$

We also know that

$$\left\|U(x,y,z)-U(x^*,y^*,z^*)\right\| = \left\|(x,y,z)-U(x^*,y^*,z^*)\right\| \le (r_{11}+r_{12}+r_{13})\left\|(x,y,z)-(x^*,y^*,z^*)\right\|,$$

this implies,

$$\left\|(x,y,z)-(x^*,y^*,z^*)\right\| \le \frac{\left\|U(x^*,y^*,z^*)-(x^*,y^*,z^*)\right\|}{1-(r_{11}+r_{12}+r_{13})},$$

then,

$$\left\|(x,y,z)-(x^*,y^*,z^*)\right\| \le \frac{1}{1-(r_{11}+r_{12}+r_{13})}\Big[\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Big(1+A_1^*+A_4^*a_1^{\alpha_1+\beta_1}+A_3^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}$$
$$+B_2^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+B_3^*+B_6^*a_1^{\alpha_1+\beta_1}+C_2^*+C_1^*\sum_{k=1}^{p}\sigma_k v_k^{\alpha_1+\beta_1}+C_5^*a_1^{\alpha_1+\beta_1}\Big)$$

$$+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Big(A_2^*+A_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+A_5^* b_1^{\alpha_2+\beta_2}+1+B_1^*+B_3^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^* b_1^{\alpha_2+\beta_2}$$
$$+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*+C_6^* b_1^{\alpha_2+\beta_2}\Big)+\frac{\varepsilon_3}{\Gamma(\alpha_3+\beta_3+1)}\Big(A_3^*+A_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+A_6^* c_1^{\alpha_3+\beta_3}+B_2^*$$
$$+B_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+B_5^* c_1^{\alpha_3+\beta_3}+1+C_1^*+C_3^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_3+\beta_3}+C_4^* c_1^{\alpha_3+\beta_3}\Big)\Big].$$

Hence, the system (1)- (2) is Ulam-Hyers stable. □

5. Examples

Example 5.1. *Consider the following problem:*

$$\begin{cases} {}^cD^{\frac{12}{8}}\Big({}^cD^{\frac{6}{8}}+\frac{1}{10^7}\Big)x(t)=\frac{t^2}{4\times10^4}\Big(x(t)+y(t)+cos(z(t))\Big), & t\in[0,1],\\ {}^cD^{\frac{13}{8}}\Big({}^cD^{\frac{7}{8}}+\frac{1}{10^7}\Big)y(t)=\frac{\Big(sin(x(t))+y(t)+z(t)\Big)}{4\times10^4+t^2}, & t\in[0,1],\\ {}^cD^{\frac{14}{8}}\Big({}^cD^{\frac{5}{8}}+\frac{1}{10^7}\Big)z(t)=t^2\frac{\Big(x(t)+sin(y(t))+z(t)\Big)}{4\times10^4}, & t\in[0,1],\\ x(0)=0;\quad x\Big(\frac{1}{10000}\Big)=0;\quad x(1)=\frac{1}{3000}\Big(y(\frac{1}{90})+y(\frac{1}{80})+y(\frac{1}{70})\Big),\\ y(0)=0;\quad y\Big(\frac{1}{1000}\Big)=0;\quad y(1)=\frac{1}{4000}\Big(z(\frac{1}{60})+z(\frac{1}{50})+z(\frac{1}{40})\Big)\\ z(0)=0;\quad z\Big(\frac{1}{100}\Big)=0;\quad z(1)=\frac{1}{4000}\Big(x(\frac{1}{30})+x(\frac{1}{20})+x(\frac{1}{10})\Big). \end{cases} \tag{8}$$

Where $\beta_1=\frac{12}{8}$, $\alpha_1=\frac{6}{8}$, $\beta_2=\frac{13}{8}$, $\alpha_2=\frac{7}{8}$, $\beta_3=\frac{14}{8}$, $\alpha_3=\frac{5}{8}$, $\lambda_1=\lambda_2=\lambda_3=\frac{1}{10^7}$, *and*
$f(t,x,y,z)=\frac{t^2\big(x(t)+y(t)+cos(z(t))\big)}{40000}$, $g(t,x,y,z)=\frac{\big(sin(x(t))+y(t)+z(t)\big)}{40000+t^2}$, $k(t,x,y,z)=\frac{1}{40000+t^2}\big(x(t))+sin(y(t))+z(t)\big)$, $a_1=\frac{1}{10000}$, $b_1=\frac{1}{1000}$,
$c_1=\frac{1}{100}$, $\gamma_1=\gamma_2=\gamma_3=\frac{1}{3000}$, $s_1=\frac{1}{90}$, $s_2=\frac{1}{80}$, $s_3=\frac{1}{70}$, $\delta_1=\delta_2=\delta_3=\sigma_1=\sigma_2=\sigma_3=\frac{1}{4000}$, $u_1=\frac{1}{60}$, $u_2=\frac{1}{50}$, $u_3=\frac{1}{40}$, $v_1=\frac{1}{30}$, $v_2=\frac{1}{20}$, $v_3=\frac{1}{10}$.
Clearly, $\mu_1^*=\mu_2^*=\mu_3^*=\frac{1}{40000}$,
then, we have $r_{11}+r_{12}+r_{13}\approx 0.02015<1$.
So, by Theorem 3.1, the system (8) has a unique solution.

Example 5.2. *Consider the following system of fractional Langevin equations:*

$$\begin{cases} {}^cD^{\frac{15}{8}}\Big({}^cD^{\frac{5}{8}}+\frac{1}{2\times10^4}\Big)x(t)=\frac{t^2\Big(sin(x(t))+cos(y(t))+cos(z(t))\Big)}{4\times10^4}, & t\in[0,1],\\ {}^cD^{\frac{14}{8}}\Big({}^cD^{\frac{6}{8}}+\frac{1}{2\times10^4}\Big)y(t)=\frac{t^2\Big(cos(x(t))+sin(y(t))+sin(z(t)\Big)}{4\times10^4} & t\in[0,1],\\ {}^cD^{\frac{13}{8}}\Big({}^cD^{\frac{7}{8}}+\frac{1}{2\times10^4}\Big)z(t)=\frac{t^2\Big(sin(x(t))+cos(y(t))+sin(z(t))\Big)}{4\times10^4} & t\in[0,1],\\ x(0)=0;\quad x\Big(\frac{1}{500}\Big)=0;\quad x(1)=\frac{1}{6000}\Big(y(\frac{1}{190})+y(\frac{1}{170})+y(\frac{1}{160})\Big),\\ y(0)=0;\quad y\Big(\frac{1}{300}\Big)=0;\quad y(1)=\frac{1}{5000}\Big(z(\frac{1}{150})+z(\frac{1}{140})+z(\frac{1}{130})\Big),\\ z(0)=0;\quad z\Big(\frac{1}{200}\Big)=0;\quad z(1)=\frac{1}{5000}\Big(x(\frac{1}{120})+x(\frac{1}{115})+x(\frac{1}{110})\Big). \end{cases} \tag{9}$$

Where $\beta_1 = \dfrac{15}{8}, \alpha_1 = \dfrac{5}{8}, \beta_2 = \dfrac{14}{8}, \alpha_2 = \dfrac{6}{8}, \beta_3 = \dfrac{13}{8}, \alpha_3 = \dfrac{7}{8}, \lambda_1 = \lambda_2 = \lambda_3 = \dfrac{1}{20000}$, *and*

$$f(t,x,y,z) = \frac{t^2\Big(sin(x(t)) + cos(y(t)) + cos(z(t))\Big)}{4\times 10^4},$$

$$g(t,x,y,z) = \frac{t^2\Big(cos(x(t)) + sin(y(t)) + sin(z(t)\Big)}{4\times 10^4},$$

$$k(t,x,y,z) = \frac{t^2\Big(sin(x(t)) + cos(y(t)) + sin(z(t))\Big)}{4\times 10^4},$$

$a_1 = \frac{1}{500}, b_1 = \frac{1}{300}, c_1 = \frac{1}{200}, \gamma_1 = \gamma_2 = \gamma_3 = \dfrac{1}{6000}$,

$s_1 = \frac{1}{190}, \quad s_2 = \frac{1}{170}, \quad s_3 = \frac{1}{160}, \delta_1 = \delta_2 = \delta_3 = \sigma_1 = \sigma_2 = \sigma_3 = \dfrac{1}{5000}, \quad u_1 = \frac{1}{150}, \quad u_2 = \frac{1}{140}, \quad u_3 = \frac{1}{130}, \quad v_1 = \frac{1}{120}, \quad v_2 = \frac{1}{115}, \quad v_3 = \frac{1}{110}$.

Then, we get $R \approx 0.0423 < 1$.

Thus, by theorem 3.2 the problem (9) has a least one solution.

6. Conclusion

In this research, we studied the existence and uniqueness results for a tripled system of nonlinear fractional Langevin equations supplemented with multipoint boundary conditions by the application of the Banach contraction principle and Krasnoselskii's fixed point theorem. In addition, we have improved Ulam stability to the solution of mentioned system. Finally, we have presented two examples to demonstrate our results.

Data Availability

No data were used to support this study.

Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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