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Existence and multiplicity results for a class of anisotropic Robin elliptic problems

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Abstract. The aim of this paper is to study a class of anisotropic Robin elliptic equations with variable exponents where the nonlinearity may depend on the gradient of the solution. First, we demonstrate the existence of at least one weak solution using the surjectivity result of pseudomonotone operators. Moreover, under additional conditions on the data, we show that the solution is unique. Furthermore, we prove the existence of at least three weak solutions using the direct Ricceri variational principle when the nonlinearity does not depend on the gradient.

1. Introduction

Through recent years, Neumann and Robin elliptic problems have sparked immense interest for their ability to unravel the complexities of physical phenomena that vary in intensity across different areas within a domain. These intriguing problems, characterized by variable exponents, often emerge from the non-linear dynamics inherent in physical processes or the diverse nature of the domain itself. Understanding these nuances is crucial, as it allows us to create more accurate models of real-world scenarios, including materials with spatially varying properties, porous media flow, and non-Newtonian fluid flows. Henceforth, investigating Robin problems, especially those that may involve convection terms, is pivotal as they play starring roles in fields such as fluid dynamics, combustion, heat transfer, and the intricate world of environmental modeling.

In this paper, we deal with a class of anisotropic elliptic equations with variable exponents, convection terms, and Robin nonlinear boundary conditions, that is the following problem

$$\begin{cases} \mathcal{A}(u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x,u,\nabla u) + h(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) = \mu g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain with smooth boundary, $\mathcal{A}(u) := -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is an anisotropic elliptic operator defined by the functions a_i subjected to the conditions (A_i) below and associated

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to the continuous variable exponents p_i with $\inf_{x\in\bar{\Omega}} p_i(x) > 1$, and where $p_M(\cdot) := \max_{1\le i\le N} p_i(\cdot)$, while $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $h \in L^{p'_M(x)}(\Omega)$ with $p'_M(x)$ such that $\dfrac{1}{p_M(x)} + \dfrac{1}{p'_M(x)} = 1$. Finally, $b(x) \in L^\infty(\Omega)$ with $ess \inf_{x\in\Omega} b(x) = b_0 > 0$, $\lambda > 0$ and $\mu \ge 0$ are real parameters, and ν_i constitute the elements comprising the outer normal unit vector for all $i \in \{1, \dots, N\}$.

This paper addresses a problem featuring the forementioned class of operators, associated with the operaor

$$\mathcal{A}(u) := -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u), \tag{2}$$

where, for all $i \in \{1, \dots, N\}$, $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are applications of Carathéodory type verifying

(A_1) $A_i(x, 0) = 0$, for all $x \in \Omega$ and $t \in \mathbb{R}$, where A_i is such that

$$a_i(x, t) = \frac{\partial A_i}{\partial t}(x, t),$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

(A_2) $a_i(x, \cdot)$ is strictly monotone in $\mathbb{R}$, that is

$$(a_i(x, s) - a_i(x, t))\,(s - t) > 0,$$

for all $(s, t) \in \mathbb{R}^2$ with $s \ne t$ and $x \in \Omega$.

(A_3) There exist positive constants σ_i and ρ_i with $\sigma_i \le \rho_i$ such that

$$\sigma_i |t|^{p_i(x)} \le a_i(x, t)t \quad \text{and} \quad |a_i(x, t)| \le \rho_i \left(1 + |t|^{p_i(x)-1}\right),$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Numerous studies have addressed questions regarding the dependence of the right-hand side on the gradient of the solution in the context of anisotropic elliptic equations. For instance, Benboubker et al. [3] (see the references cited in this article) mentioned this category of problems in the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$. Our focus here is twofold: First, to tackle gradient-dependent nonlinearity in variable-exponent Sobolev space under anisotropic conditions in the context of parametric elliptic equations. This class of problems presents many challenges, including the limitation of variational methods owing to the dependence of the right-hand side on the gradient. The complexity is further aggravated by the nonhomogeneous nature of the $p(\cdot)$-Laplacian and $\vec{p}(\cdot)$-Laplacian operators, which is distinct from the homogeneous p-Laplacian operator.

Second, our focus extends to establishing the existence of multiple solutions for nonlinearities that do not depend on the gradient. Notably, previous papers employing variational methods have addressed the existence of solutions concerning quasilinear anisotropic elliptic equations with constant or variable exponents, specifically, when the right-hand side term remains independent of the gradient. Furthermore, our problem adds another layer of difficulty when dealing with nonlinear Robin conditions, as well as an additional term $h(x)$.

Now, we situate our work in relation to some publications already published on this subject. Colasuonno et al. [10] studied the following isotropic Robin boundary type problem

$$\begin{cases} -div(\vec{a}(x, \nabla u)) = \lambda \left(b(x)|u|^{p(x)-2}u + f(x, u)\right) & \text{in } \Omega, \\ \vec{a}(x, \nabla u) \cdot \nu = -a(x)|u|^{p(x)-2}u + \lambda\mu g(x, u) & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $p(x) = p$ and the operator $-div(\vec{a}(x, \nabla u))$ is driven by the p-Laplacian. In the case where $b(x) \equiv 0$, they showed the existence of either two nontrivial solutions or only trivial solutions under specific growth conditions verified by functions f and g.

In parallel, in the case where $\lambda = \mu = 1$ and $h(x) \equiv 0$, the nonparametric isotropic problem

$$\begin{cases} \mathcal{A}(u) + b(x)|u|^{p_M(x)-2}u = f(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{4}$$

was treated by Boureanu and Rădulescu [7]. The authors proved the existence of at least a nonnegative solution using the standard minimization result for a weakly lower semicontinuous and coercive functional.

On the other hand, the authors in [6, 11] dealt with problem (3) in the case where $a(x) \equiv 0$ and obtained the existence of infinitely many solutions. Interesting multiplicity results of solutions for some problems related to problem (1) have been obtained by several authors, especially Ahmed and Elemine Vall [1], Aydin and Unal [2], Khademloo et al. [19], Kim and Park [20], and Marano and Motreanu [23]. Furthermore, Ourraoui and Ragusa [25] studied a slightly different problem than (1) when $\mu = 0$ and obtained results regarding the existence of solutions without requiring Ambrosetti-Rabinowitz-type conditions. Ellahyani and El Hachimi [14] also studied a similar problem to (1) with a Robin boundary condition and established existence and multiplicity results. All these authors established the existence of infinitely many solutions using different variational methods (see also Chems Eddine and Ragusa [9], Jleli et al. [18], Papageorgiou and Scapellato [26] and the references therein).

Finally, let us point out that the paper by Kim and Park [20] seems to be closer to our present work. These authors obtained the existence of at least three solutions for problem (3) in the isotropic case and functions f that satisfy the following hypotheses

(KP_1) $|f(x,t)| \leq k_1(x) + \sigma_1(x)|t|^{r_1(x)-1}$, with $r_1^+ < p^-$, for all $(x,t) \in \Omega \times \mathbb{R}$,

(KP_2) $\displaystyle\limsup_{s\to 0} \left(ess \sup_{x\in\Omega} \frac{\left|\int_0^s f(x,t)dt\right|}{|s|^{q(x)}} \right) < +\infty$, with $q \in C_+(\bar{\Omega})$, $p^+ < q_- \leq q < p^*$, where $C_+(\bar{\Omega})$ and the parameters p^-, p^+ and p^* are defined in the next section.

Note that our hypotheses (H_1), (H_2) and (H_3) or (H_3') in Section 4 differ from these assumptions. For example, the function given in Example 4.6 in the isotropic case, does not verify the conditions (KP_1) and (KP_2) for $\beta(x) < p^+$, for a.e. $x \in \Omega$. However, it satisfies our hypotheses (H) and the existence of three solutions is assured.

The remainder of this paper is organized as follows. In Section 2, we provide definitions and propositions for generalized Lebesgue-Sobolev spaces alongside generalized anisotropic Sobolev spaces. In Section 3, using the surjectivity result for pseudomonotone operators, we demonstrate the existence of a solution concerning problem (1). Moreover, under suitable hypotheses, we show that the solution is unique. In Section 4, we demonstrate the existence of a minimum of three weak solutions by applying a recent Ricceri variational principle in the case where there is no dependence of nonlinearity f on the gradient.

2. Functional framework

We now introduce the variable exponent Lebesgue-Sobolev setting. For further information on the properties of the variable exponent Lebesgue-Sobolev spaces, we refer to Edmunds et al. [12], Edmunds and Rákosník [13], Kovacik and Rákosník [22], Samko and Vakulov [28]. For $s \in C_+(\bar{\Omega})$, define the Lebesgue space with variable exponents as

$$L^{s(x)}(\Omega) = \left\{ u, u \text{ is a measurable real-valued function and } \rho_{s(\cdot)}(u) = \int_\Omega |u(x)|^{s(x)}dx < \infty \right\},$$

characterized by the Luxemburg norm

$$\|u\|_{L^{s(x)}(\Omega)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{s(x)} dx \le 1\right\},$$

where

$$C_+(\bar{\Omega}) = \left\{s \in C(\bar{\Omega};\mathbb{R}) : \inf_{x\in\Omega} s(x) > 1\right\}.$$

For $s \in C_+(\bar{\Omega})$, denote

$$s^+ = \sup_{x\in\Omega} s(x), \quad s^- = \inf_{x\in\Omega} s(x).$$

Note that $L^{s(x)}(\Omega)$ is a separable and reflexive Banach space and recall the following embedding result.

Proposition 2.1. *(see Kovacik and Rákosník [22]) Assume that Ω is bounded and that $s_1, s_2 \in C_+(\bar{\Omega})$ such that $s_1 \le s_2$ in Ω. Then, the embedding $L^{s_2(x)}(\Omega) \hookrightarrow L^{s_1(x)}(\Omega)$ is continuous.*

Denote $L^{s'(x)}(\Omega)$ the conjugate space $\left(L^{s(x)}(\Omega)\right)'$ of $L^{s(x)}(\Omega)$, where $\dfrac{1}{s(x)} + \dfrac{1}{s'(x)} = 1$. Then, for any $u \in L^{s(x)}(\Omega)$ and $v \in L^{s'(x)}(\Omega)$, the following Hölder-type inequality

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{s^-} + \frac{1}{s'^-}\right)\|u\|_{L^{s(x)}(\Omega)}\|v\|_{L^{s'(x)}(\Omega)} \le 2\|u\|_{L^{s(x)}(\Omega)}\|v\|_{L^{s'(x)}(\Omega)},$$

holds. The next proposition sheds light on the relation between the norm $\|u\|_{s(.)}$ and the convex modular $\rho_{s(\cdot)}$.

Proposition 2.2. *(see Fan and Zhao [17]) Let $s \in C_+(\bar{\Omega})$,. Then, one has*

(a) $\|u\|_{L^{s(x)}(\Omega)} > 1 \Longrightarrow \|u\|^{s^-}_{L^{s(x)}(\Omega)} \le \rho_{s(\cdot)}(u) \le \|u\|^{s^+}_{L^{s(x)}(\Omega)}$.

(b) $\|u\|_{L^{s(x)}(\Omega)} < 1 \Longrightarrow \|u\|^{s^+}_{L^{s(x)}(\Omega)} \le \rho_{s(\cdot)}(u) \le \|u\|^{s^-}_{L^{s(x)}(\Omega)}$.

(c) $\|u\|_{L^{s(x)}(\Omega)} = a > 0 \Longleftrightarrow \rho_{s(\cdot)}\left(\dfrac{u}{a}\right) = 1$.

(d) $\|u\|_{L^{s(x)}(\Omega)} < 1(= 1; > 1) \Longleftrightarrow \rho_{s(\cdot)}(u) < 1(= 1; > 1)$.

Remark 2.3. *From Proposition 2.2, it follows that, for all $u \in L^{s(x)}(\Omega)$, we have*

(e) $\|u\|_{L^{s(x)}(\Omega)} < \rho_{s(\cdot)}(u) + 1$.

(f) $\rho_{s(\cdot)}(u) < \|u\|^{s^+}_{L^{s(x)}(\Omega)} + 1$.

Proposition 2.4. *(see Zhao et al. [16]) Let $s \in C_+(\bar{\Omega})$. If $u, u_n \in L^{s(x)}(\Omega)$, then the following statements are equivalent*

(i) $\lim_{k\to\infty} \|u_k - u\|_{L^{s(x)}(\Omega)} = 0$.

(ii) $\lim_{k\to\infty} \rho_{s(\cdot)}(u_k - u) = 0$.

(iii) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to\infty} \rho_{s(\cdot)}(u_k) = \rho_{s(\cdot)}(u)$.

Let $s \in C_+(\bar{\Omega})$ and denote $W^{1,s(x)}(\Omega)$ the variable exponent Sobolev space defined by

$$W^{1,s(x)}(\Omega) = \left\{u \in L^{s(x)}(\Omega) : \partial_{x_i} u \in L^{s(x)}(\Omega), \forall i \in \{1, \dots, N\}\right\},$$

endowed with the norm

$$\|u\|_{W^{1,s(x)}(\Omega)} = \|u\|_{L^{s(x)}(\Omega)} + \sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{s(x)}(\Omega)}.$$

The space $\left(W^{1,s(x)}(\Omega), \|\cdot\|_{W^{1,s(x)}(\Omega)}\right)$ is a separable and reflexive Banach space. In general, the smooth functions are not dense in $W^{1,s(x)}(\Omega)$, but if the variable exponent $s \in C_+(\bar{\Omega})$ is logarithmic Hölder continuous, that is

$$|s(x) - s(y)| \leq -\frac{M}{\log(|x-y|)} \text{ for all } x, y \in \Omega, \text{ such that } |x-y| \leq \frac{1}{2},$$

then the smooth functions are dense in $W^{1,s(x)}(\Omega)$. We now define $\vec{p}$ as the vector function $\vec{p} : \bar{\Omega} \to \mathbb{R}^N$ such that $\vec{p}(x) = (p_1(x), \ldots, p_N(x))$, where $p_i \in C_+(\bar{\Omega})$, for all $i \in \{1, \ldots, N\}$ and denote

$$p_M(x) = \max\{p_1(x), \ldots, p_N(x)\}, \quad p_m(x) = \min\{p_1(x), \ldots, p_N(x)\}.$$

The anisotropic variable exponent Sobolev space is given by

$$X = W^{1,\vec{p}(x)}(\Omega) = \left\{v \in L^{p_M(x)}(\Omega) : \partial_{x_i} v \in L^{p_i(x)}(\Omega), \forall i \in \{1, \ldots, N\}\right\},$$

and it is endowed with the norm

$$\|v\| = \|v\|_{W^{1,\vec{p}(x)}(\Omega)} = \|v\|_{L^{p_M(x)}(\Omega)} + \sum_{i=1}^{N} \left\|\partial_{x_i} v\right\|_{L^{p_i(x)}(\Omega)}.$$

We emphasize that $\left(W^{1,\vec{p}(x)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(x)}(\Omega)}\right)$ is a reflexive Banach space. Over the course of this paper, we suppose that

(A_0) $1 < \displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} < N+1.$

Next, we introduce the following notations

$$(\bar{p})^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \quad \text{and} \quad p_i^{\partial}(x) := \begin{cases} \dfrac{(N-1)p_i(x)}{Np_i(x)}, & \text{if } p_i(x) < N, \\ +\infty, & \text{if } p_i(x) \geq N. \end{cases}$$

For any $q \in C_+(\bar{\Omega})$ with $1 < q(x) \leq \max\left\{(\bar{p})^*, p_M^-\right\}$, for a.e $x \in \bar{\Omega}$, we denote $S_{q,\Omega}$ the best constant in the continuous embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, that is,

$$S_{q(x),\Omega} = \inf_{v \in W^{1,\vec{p}(x)}(\Omega) \setminus \{0\}} \frac{\|v\|_{W^{1,\vec{p}(x)}(\Omega)}}{\|v\|_{L^{q(x)}(\Omega)}}.$$

Similarly, for any $q \in C_+(\bar{\Omega})$ verifying $1 \leq q(x) \leq \min\limits_{x \in \partial\Omega}\{p_1^{\partial}(x), \ldots, p_N^{\partial}(x)\}$, where

$$p_i^{\partial}(x) = \begin{cases} (N-1)p_i(x)/[N - p_i(x)] & \text{if } p_i(x) < N, \\ \infty & \text{if } p_i(x) \geq N, \end{cases}$$

for all $x \in \bar{\Omega}$ and for $i \in \{1, \ldots, N\}$, we denote $S_{q(x),\partial\Omega}$ the best constant in the embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.

Proposition 2.5. *(see Fan [15]) If $q \in C_+(\bar{\Omega})$ satisfies $1 < q(x) < \max\{(\bar{p})^*, p_M(x)\}$ for all $x \in \bar{\Omega}$, then the embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Proposition 2.6. *(see Boureanu and Rădulescu [7]) Suppose that* $\vec{p} \in (C_+(\bar{\Omega}))^N$ *and* $q \in C_+(\bar{\Omega})$ *satisfying* $1 < q(x) < \min_{x\in\partial\Omega}\left\{p_1^{\partial}(x), \dots, p_N^{\partial}(x)\right\}$*, for all* $x \in \partial\Omega$*. Then, the embedding* $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\partial\Omega)$ *is compact.*

Definition 2.7. *An element* $u \in W^{1,\vec{p}(x)}(\Omega)$ *is termed a weak solution of problem* (1) *if*

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \int_\Omega b(x)|u|^{p_M(x)-2} uv dx = \lambda \int_\Omega f(x, u, \nabla u) v dx + \mu \int_{\partial\Omega} g(x, u) v d\sigma + \langle h, v\rangle, \tag{5}$$

for all $v \in W^{1,\vec{p}(x)}(\Omega)$*, where* $\langle h, v\rangle$ *is the duality pairing between* $W^{1,\vec{p}(x)}(\Omega)$ *and its dual space.*

Definition 2.8. *Let X be a reflexive Banach space, X^* its dual space and denote by* $\langle\cdot,\cdot\rangle$ *the duality pairing. Consider an application* $\mathcal{J} : X \to X^*$*. Then,* $\mathcal{J}$ *is called*

(*a*) *to verify the* (S^+)*-property if*

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{J} u_n, u_n - u\rangle \le 0 \text{ imply } u_n \to u \text{ in } X.$$

(*b*) *pseudomonotone if*

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{J} u_n, u_n - u\rangle \le 0 \text{ imply } \mathcal{J} u_n \rightharpoonup \mathcal{J} u \text{ and } \langle \mathcal{J} u_n, u_n\rangle \to \langle \mathcal{J} u, u\rangle.$$

(*c*) *coercive if*

$$\lim_{\|u\|_X\to\infty} \frac{\langle \mathcal{J} u, u\rangle}{\|u\|_X} = \infty.$$

3. Existence and uniqueness of the solution

Now, in order to state the main existence result for problem (1), we assume the following assumptions on the functions f and g.

(F_1) There exists $r_1, r_2 \in C_+(\bar{\Omega})$ with $1 < r_1(x) < \max\{(\bar{p})^*, p_M(x)\}$ for all $x \in \Omega$, and $1 < r_2(x) < \min_{x\in\partial\Omega}\left\{p_1^{\partial}(x), \dots, p_N^{\partial}(x)\right\}$ for all $x \in \partial\Omega$, $k_1 \in L^{r_1'(x)}(\Omega)$, $k_2 \in L^{r_2'(x)}(\partial\Omega)$ and $c_1, c_2, c_1' > 0$ such that

$$|f(x, t, \xi)| \le k_1(x) + c_1|t|^{r_1(x)-1} + c_2 \sum_{i=1}^N |\xi_i|^{p_i(x)\frac{r_1(x)-1}{r_1(x)}}, \quad \text{for a.e. } x \in \Omega \text{ and all } (t, \xi) \in \mathbb{R}\times\mathbb{R}^N,$$

and

$$|g(x, t)| \le k_2(x) + c_1'|t|^{r_2(x)-1}, \text{ for a.e. } x \in \partial\Omega \text{ and all } (t, \xi) \in \mathbb{R}\times\mathbb{R}^N.$$

(F_2) There exists $\theta_1 \in L^1(\Omega)$, $\theta_2 \in L^1(\partial\Omega)$ and $c_3, c_4, c_2' \ge 0$ such that

$$f(x, t, \xi)t \le \theta_1(x) + c_3|t|^{r_1(x)} + c_4 \sum_{i=1}^N |\xi_i|^{p_i(x)},$$
$$g(x, t)t \le \theta_2(x) + c_2'|t|^{r_2(x)},$$

for a.e. $x \in \Omega$ and all $(t, \xi) \in \mathbb{R}\times\mathbb{R}^N$.

Proposition 3.1. *We define* $X = W^{1,\vec{p}(x)}(\Omega)$ *and*

$$\langle \mathcal{J}(u), v\rangle = \int_{\Omega} \Big(\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + b(x)|u|^{p_M(x)-2} uv \Big) dx,$$

for all $u, v \in X$. Then

i) *$\mathcal{J} : X \to X^{\star}$ is bounded and coercive.*

ii) *$\mathcal{J}$ is of type (S^+).*

Proof. *i)* The boundedeness of $\mathcal{J}$ is a consequence of hypothesis (A_3) and the coerciveness is obtained in the proof of Theorem 3.3 below.

ii) The proof is similar to that in Boureanu [5, Lemma 2] and is omitted here.
□

Now, we state the primary tool employed in this section that relies on the surjectivity result for pseudomonotone operators, as presented in Carl et al. [8].

Lemma 3.2. *(see Carl et al. [8]) Given a real reflexive Banach space X and a bounded, pseudomonotone, and coercive operator $\mathcal{J} : X \to X^*$, there exists a solution to the equation $\mathcal{J}(u) = h$, for any $h \in X^*$.*

Theorem 3.3. *Assume that hypotheses $(A_0) - (A_3)$, (F_1) and (F_2) are satisfied. Furthermore, suppose that $1 < r_i^- \leq r_i^+ < p_m^-$, for all $i \in \{1, 2\}$. Then, for any $\lambda \in \left] -\frac{\sigma_0}{c_4}, \frac{\sigma_0}{c_4} \right[$ and any $h \in \left(W^{1,\vec{p}(x)}(\Omega)\right)^{\star}$, problem (1) admits a weak solution in $W^{1,\vec{p}(x)}(\Omega)$.*

Proof. We define the Nemytskii operators $\bar{N}_f : X \subseteq L^{r_1(x)}(\Omega) \to L^{r_1'(x)}(\Omega)$ and $\bar{N}_g : L^{r_2(x)}(\partial\Omega) \to L^{r_2'(x)}(\partial\Omega)$ by $\left(\bar{N}_f u\right)(x) = f(x, u(x), \nabla u(x))$ and $\left(\bar{N}_g u\right)(x) = g(x, u(x))$ respectively. Furthermore, denote $i^* : L^{r_1'(x)}(\Omega) \to X^*$ the adjoint operator for the embedding $i : X \to L^{r_1(x)}(\Omega)$ and $j^* : L^{r_2'(x)}(\partial\Omega) \to X^*$ the adjoint operator for the embedding $j : X \to L^{r_2(x)}(\partial\Omega)$. Subsequently, define $N_f = i^* \circ \bar{N}_f : X \to X^*$ and $N_g = j^* \circ \bar{N}_g \circ j : X \to X^*$, which are well-defined, bounded and continuous operators by assumption (F_1).
Now, define the operator $\mathbb{A} : X \to X^*$ as follows

$$\mathbb{A}(u) = \mathcal{J}(u) - \lambda N_f(u) - \mu N_g(u) - h.$$

Our aim is to apply Lemma 3.2. Hence, we need to show that $\mathbb{A}$ is bounded, pseudomonotone and coercive.
• $\mathbb{A}$ ***is bounded:***
Thanks to growth conditions on functions f and g stated in (F_1) and the boundedness of $\mathcal{J}$, we obtain the boundedness of $\mathbb{A}$.
• $\mathbb{A}$ ***is pseudomonotone:***
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ \limsup_{n\to+\infty} \langle \mathbb{A}\left(u_n\right), u_n - u\rangle \leq 0. \end{cases}$$

Thanks to the compact embeddings $X \hookrightarrow\hookrightarrow L^{r_1(x)}(\Omega)$ for $1 < r_1(x) < \max\{\bar{p}^*(x), p_M(x)\}$ for all $x \in \bar{\Omega}$, and $X \hookrightarrow\hookrightarrow L^{r_2(x)}(\partial\Omega)$ for $1 \leq r_2(x) < \min_{x\in\partial\Omega} \left\{p_1^{\partial}(x), \ldots, p_N^{\partial}(x)\right\}$ for all $x \in \partial\Omega$, we obtain

$$u_n \to u \text{ in } L^{r_1(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{r_2(x)}(\partial\Omega) \tag{6}$$

Thus, using (F_1) alongside Hölder's inequality and the boundedness of $(u_n)_{n\in\mathbb{N}}$, we obtain

$$\begin{aligned}\left|\int_\Omega f(x,u_n,\nabla u_n)\cdot(u_n-u)dx\right| &\le \Bigg[\int_\Omega k_1(x)|u_n-u|dx + c_1\int_\Omega |u_n|^{r_1(x)-1}|u_n-u|dx \\ &+ c_2\int_\Omega \sum_{i=1}^N \left|\partial_{x_i}u_n\right|^{p_i(x)\frac{r_1(x)-1}{r_1(x)}} |u_n-u|dx\Bigg] \\ &\le \Bigg[2\|k_1\|_{L^{r_1'(x)}(\Omega)}\|u_n-u\|_{L^{r_1(x)}(\Omega)} + 2c_1\left\||u_n|^{r_1(x)-1}\right\|_{L^{r_1'(x)}(\Omega)}\|u_n-u\|_{L^{r_1(x)}(\Omega)} \\ &+ 2c_2\sum_{i=1}^N \left\|\left|\partial_{x_i}u_n\right|^{p_i(x)\frac{r_1(x)-1}{r_1(x)}}\right\|_{L^{r_1'(x)}(\Omega)}\|u_n-u\|_{L^{r_1(x)}(\Omega)}\Bigg],\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\partial\Omega} g(x,u_n)\cdot(u_n-u)dx\right| &\le \int_{\partial\Omega} k_2(x)|u_n-u|dx + c_1'\int_{\partial\Omega}|u_n|^{r_2(x)-1}|u_n-u|dx \\ &\le 2\|k_2\|_{L^{r_2'(x)}(\partial\Omega)}\|u_n-u\|_{L^{r_2(x)}(\partial\Omega)} + 2c_1'\left\||u_n|^{r_2(x)-1}\right\|_{L^{r_2'(x)}(\partial\Omega)}\|u_n-u\|_{L^{r_2(x)}(\partial\Omega)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\left|\int_\Omega f(x,u_n,\nabla u_n)\cdot(u_n-u)dx\right| &\le \Bigg[2\|k_1\|_{L^{r_1'(x)}(\Omega)}\|u_n-u\|_{L^{r_1(x)}(\Omega)} \\ &+ 2c_1\left(\left\||u_n|^{r_1^+-1}\right\|_{L^{r_1(x)}(\Omega)} + \left\||u_n|^{r_1^--1}\right\|_{L^{r_1(x)}(\Omega)}\right)\|u_n-u\|_{L^{r_1(x)}(\Omega)} \\ &+ 2c_2\sum_{i=1}^N\left(\left\|\left|\partial_{x_i}u_n\right|^{p_i^+(\frac{r_1^+-1}{r_1^+})}\right\|_{L^{p_i(x)}(\Omega)} + \left\|\left|\partial_{x_i}u_n\right|^{p_i^-(\frac{r_1^--1}{r_1^-})}\right\|_{L^{p_i(x)}(\Omega)}\right)\|u_n-u\|_{L^{r_1(x)}(\Omega)}\Bigg],\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\partial\Omega} g(x,u_n)\cdot(u_n-u)dx\right| &\le 2\|k\|_{L^{r_2'(x)}(\partial\Omega)}\|u_n-u\|_{L^{r_2(x)}(\partial\Omega)} \\ &+ 2c_1\left(\left\||u_n|^{r_2^+-1}\right\|_{L^{r_2(x)}(\partial\Omega)} + \left\||u_n|^{r_2^--1}\right\|_{L^{r_2(x)}(\partial\Omega)}\right)\|u_n-u\|_{L^{r_2'(x)}(\partial\Omega)}.\end{aligned}$$

This, in combination with (6), leads to the conclusion that

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n,\nabla u_n)\cdot(u_n-u)dx = 0,$$

and

$$\lim_{n\to+\infty}\int_{\partial\Omega} g(x,u_n)\cdot(u_n-u)dx = 0.$$

Taking the limit in (5), and substituting u with u_n and v with u_n-u, yields

$$\limsup_{n\to+\infty}\langle\mathbb{A}(u),u_n-u\rangle = \limsup_{n\to+\infty}\langle\mathcal{J}(u),u_n-u\rangle \le 0.$$

Therefore, $u_n\to u$ follows from $\mathcal{J}$ being of type (S^+). Moreover, considering the continuity of $\mathbb{A}$, we deduce $\mathbb{A}(u_n)\to\mathbb{A}(u)$ in X^*, establishing the pseudomonotonicity of $\mathbb{A}$.

• $\mathbb{A}$ ***is coercive:***

We need to show that

$$\lim_{\|u\|\to+\infty}\frac{\langle\mathbb{A}(u),u\rangle}{\|u\|} = +\infty. \tag{7}$$

Initially, let us give the following notations

$$\mathcal{L}_1 = \left\{ i \in \{1, \dots, N\} : \left\| \partial_{x_i} u_n \right\|_{L^{p_i(x)}(\Omega)} \leq 1 \right\},$$
$$\mathcal{L}_2 = \left\{ i \in \{1, \dots, N\} : \left\| \partial_{x_i} u_n \right\|_{L^{p_i(x)}(\Omega)} > 1 \right\}.$$

Then, we get

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega \left| \partial_{x_i} u_n \right|^{p_i(x)} dx &= \sum_{i \in \mathcal{L}_1} \int_\Omega \left| \partial_{x_i} u_n \right|^{p_i(x)} dx + \sum_{i \in \mathcal{L}_2} \int_\Omega \left| \partial_{x_i} u_n \right|^{p_i(x)} dx \\
&\geq \sum_{i \in \mathcal{L}_1} \left\| \partial_{x_i} u_n \right\|^{p_M^+}_{L^{p_i(x)}(\Omega)} + \sum_{i \in \mathcal{L}_2} \left\| \partial_{x_i} u_n \right\|^{p_m^-}_{L^{p_i(x)}(\Omega)} \\
&\geq \sum_{i=1}^{N} \left\| \partial_{x_i} u_n \right\|^{p_m^-}_{L^{p_i(x)}(\Omega)} - \sum_{i \in \mathcal{L}_1} \left\| \partial_{x_i} u_n \right\|^{p_m^-}_{L^{p_i(x)}(\Omega)} \\
&\geq \sum_{i=1}^{N} \left\| \partial_{x_i} u_n \right\|^{p_m^-}_{L^{p_i(x)}(\Omega)} - N \\
&\geq \frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^{N} \left\| \partial_{x_i} u_n \right\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} - N.
\end{aligned}$$

Case 1: If $\|u\|_{L^{p_M(x)}(\Omega)} \geq 1$, we have

$$\begin{aligned}
\langle \mathcal{J}(u), u \rangle &\geq \min\{\sigma_0, b_0\} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^{N} \left\| \partial_{x_i} u \right\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} - N + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)} \right] \\
&\geq \min\{\sigma_0, b_0\} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^{N} \left\| \partial_{x_i} u \right\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)} \right] - N \min\{\sigma_0, b_0\} \\
&\geq \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}} \left[\sum_{i=1}^{N} \left\| \partial_{x_i} u \right\|_{L^{p_i(x)}(\Omega)} + \|u\|_{L^{p_M(x)}(\Omega)} \right]^{p_m^-} - N \min\{\sigma_0, b_0\} \\
&\geq \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}} \|u\|^{p_m^-} - N \min\{\sigma_0, b_0\}.
\end{aligned}$$

Case 2: If $\|u\|_{L^{p_M(x)}(\Omega)} < 1$, we have

$$\begin{aligned}
\langle \mathcal{J}(u), u \rangle &\geq \min\{\sigma_0, b_0\} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^{N} \left\| \partial_{x_i} u \right\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)} - 1 - N \right] \\
&\geq \min\{\sigma_0, b_0\} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^{N} \left\| \partial_{x_i} u \right\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)} \right] - (N+1) \min\{\sigma_0, b_0\} \\
&\geq \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}} \|u\|^{p_m^-} - (N+1) \min\{\sigma_0, b_0\}.
\end{aligned}$$

Therefore, in both cases, we obtain

$$\langle \mathcal{J}(u), u \rangle \geq d_0 \|u\|^{p_m^-} - d_1, \tag{8}$$

where

$$d_0 = \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}},$$

and

$$d_1 = \begin{cases} N\min\{\sigma_0, b_0\}, & \text{if } \|u\|_{L^{p_M(x)}(\Omega)} \geq 1, \\ (N+1)\min\{\sigma_0, b_0\}, & \text{if } \|u\|_{L^{p_M(x)}(\Omega)} < 1. \end{cases}$$

Now, denote the dual norm in X^* by $\|\cdot\|_*$. For any $u \in X$, using (F_2) and Proposition 2.2, we obtain

$$\begin{aligned}
\langle \mathbb{A}(u), u\rangle &\geq \sigma_0 \sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} u_n\right|^{p_i(x)} dx + b_0 \int_\Omega |u|^{p_M(x)} u dx - \lambda \int_\Omega f(x,u,\nabla u) u dx - \mu \int_{\partial\Omega} g(x,u) u d\sigma - \langle h, u\rangle \\
&\geq \sigma_0 \sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} u_n\right|^{p_i(x)} dx + b_0 \int_\Omega |u|^{p_M(x)} u dx - |\lambda| \left(\int_\Omega |\theta_1(x)| dx + c_3 \int_\Omega |u|^{r_1(x)} + c_4 \int_\Omega \sum_{i=1}^{N} \left|\partial_{x_i} u\right|^{p_i(x)} dx \right) \\
&- |\mu| \left(\int_{\partial\Omega} |\theta_2(x)| d\sigma + c_2' \int_{\partial\Omega} |u|^{r_2(x)} d\sigma \right) - \|u\|\|h\|_\star \\
&\geq \min\{\sigma_0 - |\lambda| c_4, b_0\} \left(\int_\Omega \sum_{i=1}^{N} \left|\partial_{x_i} u\right|^{p_i(x)} dx + \int_\Omega |u|^{p_M(x)} dx \right) - |\lambda| c_3 \max\left\{ \|u\|_{L^{r_1(x)}(\Omega)}^{r_1^+}, \|u\|_{L^{r_1(x)}(\Omega)}^{r_1^-} \right\} \\
&- |\mu| c_2' \max\left\{ \|u\|_{L^{r_2(x)}(\partial\Omega)}^{r_2^+}, \|u\|_{L^{r_2(x)}(\partial\Omega)}^{r_2^-} \right\} - |\lambda| \|\theta_1\|_{L^1(\Omega)} - |\mu| \|\theta_2\|_{L^1(\partial\Omega)} - \|u\|\|h\|_\star.
\end{aligned}$$

Now using either Case 1 or Case 2, it follows

$$\begin{aligned}
\langle \mathbb{A}(u), u\rangle \geq \min\{\sigma_0 - \lambda c_4, b_0\} \frac{1}{(2N)^{p_m^- - 1}} \|u\|^{p_m^-} - \lambda c_3 \max\left\{ \frac{\|u\|^{r_1^+}}{S_{r_1(x),\Omega}^{r_1^+}}, \frac{\|u\|^{r_1^-}}{S_{r_1(x),\Omega}^{r_1^-}} \right\} \\
- \mu c_2' \max\left\{ \frac{\|u\|^{r_2^+}}{S_{r_2(x),\partial\Omega}^{r_2^+}}, \frac{\|u\|^{r_2^-}}{S_{r_2(x),\partial\Omega}^{r_2^-}} \right\} - \lambda \|\theta_1\|_{L^1(\Omega)} - \mu \|\theta_2\|_{L^1(\partial\Omega)} - \|u\|\|h\|_\star.
\end{aligned}$$

Since $1 < r_i^- \leq r_i^+ < p_m^-$, for all $i \in \{1, 2\}$, then (7) is satisfied. Hence, $\mathbb{A}$ is coercive.

Thus, all the assumptions of Lemma 3.2 are satisfied. Therefore, there exists $u \in X$ such that $\mathbb{A}(u) = h$. This completes the proof. □

In what follows, we consider the uniqueness of the solution for problem (1) under the following additional hypotheses.

(F_3) For all $\xi \in \mathbb{R}^N$ and for a.e $x \in \Omega$, we have $s \to f(x, s, \xi)$ and $s \to g(x, s)$ are decreasing.

(F_4) For $s \in \mathbb{R}$, a.e $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, we have $f(x, s, \xi) = f(x, s, |\xi|)$ and $|\xi| \to f(x, s, |\xi|)$ is decreasing.

(F_5) $\lambda > 0$ and $p_M(x) \geq 2$, for a.e $x \in \Omega$.

Theorem 3.4. *Suppose that hypotheses of Theorem 3.3 alongside with* (F_3) *to* (F_5) *are verified. Then, problem* (1) *admits a unique solution.*

Proof. Consider u_1 and u_2 as two weak solutions to problem (1). By using the weak formulation of u_1 and u_2, and selecting $\phi = (u_1 - u_2)_+$ as a test function, we derive

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^{N} \left(a_i(x, \partial_{x_i} u_1) - a_i(x, \partial_{x_i} u_2)\right) \left(\partial_{x_i} u_1 - \partial_{x_i} u_2\right)_+ dx \\
&+ \int_\Omega b(x) \left(|u_1|^{p_M(x)-2} u_1 - |u_2|^{p_M(x)-2} u_2 \right) (u_1 - u_2)_+ dx \\
&= \lambda \int_\Omega \left(f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2)\right) (u_1 - u_2)_+ dx + \mu \int_{\partial\Omega} \left(g(x, u_1) - g(x, u_2)\right) (u_1 - u_2)_+ d\sigma.
\end{aligned}$$

By hypothesis (F_5), we obtain

$$\begin{aligned}0 \leq b_0 \int_{\Omega} \left|(u_1-u_2)_+\right|^{p_M(x)} dx &\leq \int_{\Omega} \sum_{i=1}^{N} \left(a_i\left(x, \partial_{x_i} u_1\right)-a_i\left(x, \partial_{x_i} u_2\right)\right)\left(\partial_{x_i} u_1-\partial_{x_i} u_2\right)_+ dx \\ &+ \int_{\Omega} b(x)\left(|u_1|^{p_M(x)-2} u_1-|u_2|^{p_M(x)-2} u_2\right)(u_1-u_2)_+ dx \\ &\leq \lambda \int_{\Omega}\left(f(x, u_1, |\nabla u_1|)-f(x, u_2, |\nabla u_2|)\right)(u_1-u_2)_+ dx \\ &+ \mu \int_{\partial\Omega}\left(g(x, u_1)-g(x, u_2)\right)(u_1-u_2)_+ d\sigma.\end{aligned}$$

On the other hand,

$$\lambda \int_{\Omega}\left(f(x, u_1, |\nabla u_1|)-f(x, u_2, |\nabla u_2|)\right)(u_1-u_2)_+ dx+\mu \int_{\partial\Omega}\left(g(x, u_1)-g(x, u_2)\right)(u_1-u_2)_+ d\sigma=\mathcal{I}_1+\mathcal{I}_2,$$

where

$$\begin{aligned}\mathcal{I}_1 &= \lambda \int_{\Omega}\left(f\left(x, u_1, |\nabla u_1|\right)-f\left(x, u_2, |\nabla u_1|\right)\right) \cdot (u_1-u_2)_+ dx+\mu \int_{\partial\Omega}\left(g(x, u_1)-g(x, u_2)\right)(u_1-u_2)_+ d\sigma, \\ \mathcal{I}_2 &= \lambda \int_{\Omega}\left(f\left(x, u_2, |\nabla u_1|\right)-f\left(x, u_2, |\nabla u_2|\right)\right) \cdot (u_1-u_2)_+ dx.\end{aligned}$$

According to hypothesis (F_1), we have $\mathcal{I}_1 \leq 0$. On the other hand, we get

$$\mathcal{I}_2=\lambda \int_{\Omega_2}\left(f\left(x, u_2, |\nabla u_1|\right)-f\left(x, u_2, |\nabla u_2|\right)\right) \cdot \left(|\nabla u_1|-|\nabla u_2|\right)_+ \frac{(u_1-u_2)_+}{\left(|\nabla u_1|-|\nabla u_2|\right)_+} dx,$$

where $\Omega_2=\{x, (|\nabla u_1|-|\nabla u_2|)_+(x) \neq 0\}$. Thus, according to hypothesis (F_2), we have $\mathcal{I}_2 \leq 0$. Therefore, we obtain

$$\lambda \int_{\Omega}\left(f(x, u_1, \nabla u_1)-f(x, u_2, \nabla u_2)\right) \cdot (u_1-u_2)_+ dx \leq 0.$$

which implies that

$$b_0 \int_{\Omega}\left|(u_1-u_2)_+\right|^{p_M(x)} dx=0.$$

From this, we find $u_1(x)=u_2(x)$ for a.e x on $\mathcal{U}=\{x \in \Omega: u_1(x)>u_2(x)\}$; while for $x \in \Omega \backslash \mathcal{U}$, we have $(u_1-u_2)_+(x)=0$. Consequently, we have $u_1 \leq u_2$ on Ω. Similarly, we find $u_1 \geq u_2$, in Ω. Hence, $u_1=u_2$ and the solution is unique. □

Example 3.5. *Let the functions f and g be defined by*

$$\begin{aligned}f(x, s, \xi) &= -a(x) s-\frac{2}{\pi}\left(\arctan s+\frac{s}{1+s^2}\right)\left(1+d(x) \frac{|\xi|^2}{1+|\xi|^2}\right), \\ g(x, s) &= -b(x) s-\frac{2}{\pi}\left(\arctan s+\frac{s}{1+s^2}\right),\end{aligned}$$

where a, b and d are positive functions in $L^{\infty}(\Omega)$. Then, f and g satisfy assumptions of Theorem 3.4.

4. Three weak solutions

In this section, we investigate and obtain the existence of three weak solutions for the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x,u) + h(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) = \mu g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain with smooth boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, $h \in L^{(p_M^+)'}(\Omega)$ with $\frac{1}{p_M^+} + \frac{1}{(p_M^+)'} = 1$, $1 < r_i^- \le r_i^+ < p_m^- \le p_M^+$, $b \in L^\infty(\Omega)$ verifying $ess \inf_{x\in\Omega} b(x) = b_0 > 0$, $\lambda > 0$ and $\mu \ge 0$ are real parameters, and ν_i constitute the elements comprising the outer normal unit vector for all $i \in \{1, \dots, N\}$.

Definition 4.1. *An element $u \in W^{1,\vec{p}(x)}(\Omega)$ is termed a weak solution of problem* (9) *if*

$$\int_\Omega \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \int_\Omega b(x)|u|^{p_M(x)-2}uv dx = \lambda \int_\Omega f(x,u)v dx + \mu \int_{\partial\Omega} g(x,u)v d\sigma + \langle h, v\rangle, \tag{10}$$

for all $v \in W^{1,\vec{p}(x)}(\Omega)$, where $\langle h, v\rangle$ is the duality pairing between $W^{1,\vec{p}(x)}(\Omega)$ and its dual space.

Define

$$F(x,u) := \int_0^u f(x,s)ds, \text{ for all } (x,u) \in \Omega\times\mathbb{R} \quad \text{and} \quad G(x,u) := \int_0^u g(x,s)ds, \text{ for all } (x,u) \in \partial\Omega\times\mathbb{R}.$$

First, we make the following assumptions regarding f and g to obtain the desired result.

(H_1) There exists $q, r_1, r_2 \in C_+(\bar{\Omega})$ with $1 < q(x) < p_m^-$, $1 < q, r_1(x) < \max\{(\bar{p})^*, p_M(x)\}$, for all $x \in \Omega$, $1 < r_2(x) < \min_{x\in\partial\Omega}\left\{p_1^\partial(x), \dots, p_N^\partial(x)\right\}$ for all $x \in \partial\Omega$, $k_1 \in L^{r_1'(x)}(\Omega)$, $k_2 \in L^{r_2'(x)}(\partial\Omega)$ and $c_6, c_7 > 0$ such that

$$|f(x,t)| \le k_1(x) + c_6|t|^{r_1(x)-1}, \text{ for all } (x,u) \in \Omega\times\mathbb{R},$$
$$|g(x,t)| \le k_2(x) + c_7|t|^{r_2(x)-1}, \text{ for all } (x,u) \in \partial\Omega\times\mathbb{R}.$$

(H_2) $\limsup_{|t|\to 0} \frac{F(x,t)}{|t|^{p_m^-}} = 0,$ uniformly with respect to $x \in \Omega$, and

(H_3) $\limsup_{|t|\to+\infty} q(x)\frac{F(x,t)}{|t|^{q(x)}} = A(x),$ uniformly with respect to $x \in \Omega$ with A such that $A_0 := \|A\|_\infty > 0$, or

(H_3') $\limsup_{|t|\to+\infty} p_M(x)\frac{F(x,t)}{|t|^{p_M(x)}} = 0,$ uniformly with respect to $x \in \Omega$.

Remark 4.2. *Note that, under hypotheses* (H_2) *and* (H_3) *(resp.* (H_3')*), for all $\epsilon > 0$, there exist $M_\epsilon \in L^1(\Omega)$ such that, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, we have*

$$F(x,t) \le \epsilon|t|^{p_m^-} + \frac{(A_0+\epsilon)}{q(x)}|t|^{q(x)} + M_\epsilon(x), \tag{11}$$

(resp.

$$F(x,t) \le \epsilon\left(|t|^{p_m^-} + \frac{1}{p_M(x)}|t|^{p_M(x)}\right) + M_\epsilon(x)). \tag{12}$$

Let $\epsilon > 0$. Using (H_3), there exists $\eta_\epsilon > 0$ such that

$$F(x,t) \leq \epsilon |t|^{q(x)}, \text{ for all } |t| \leq \eta_\epsilon, \text{ uniformly for a.e } x \in \Omega. \tag{13}$$

Moreover, using (H_2), there exists an ϵ-uniformly integrable function $L_\epsilon \in L^1(\Omega)$ and $\nu_\epsilon > 0$ such that

$$F(x,t) \leq \frac{(A(x)+\epsilon)}{q(x)}|t|^{q(x)} + L_\epsilon(x), \text{ for all } |t| \geq \nu_\epsilon, \text{ uniformly for a.e } x \in \Omega. \tag{14}$$

Now, using hypothesis (H_1), (13) and (14), there exists $k_0(\epsilon) > 0$ that depends on ϵ, q, k_1 and c_6 such that

$$F(x,t) \leq \epsilon |t|^{p_m^-} + \frac{(A_0+\epsilon)}{q(x)}|t|^{q(x)} + L_\epsilon(x) + k_0(\epsilon), \text{ for all } t \in \mathbb{R}, \text{ uniformly for a.e } x \in \Omega.$$

Then, it suffices to take $M_\epsilon(x) = L_\epsilon(x) + k_0(\epsilon)$ and $A_0 = \|A\|_\infty$. Note that the same remark stands for relation (12), starting from hypotheses (H_2) and (H_3').

For the last hypotheses (H), we suppose that there exists a real $e_0 > 0$ such that

- (H_4) $F(x, e_0) + e_0 h(x) \geq 0$, for a.e $x \in \mathcal{B}(x_0, D)$,
- (H_5) There exists $0 < \alpha_0 < 1$ such that $F(x, e_0) + e_0 h(x) > 0$, for a.e $x \in \mathcal{B}(x_0, \alpha_0 D)$,
- (H_6) There exists $\gamma_0 \in \mathbb{R}$ such that $G(x,y) := F(x,y) + yh(x) \geq \gamma_0$, $\forall (x,y) \in \mathcal{B}(x_0, D) \times]-e_0, e_0[$, where $x_0 \in \Omega$ and $D > 0$ are such that

$$\mathcal{B}(x_0, D) := \left\{ x \in \mathbb{R}^N : \|x - x_0\|_1 = \sum_{i=1}^{N} |x_i - x_{0i}| \leq D \right\} \subset \Omega.$$

Proposition 4.3. *(see Bonanno and Candito [4]) Let X be a reflexive real Banach space and I be a real interval. $\mathcal{H} : X \to \mathbb{R}$ is a continuously differentiable and sequentially weakly lower semi-continuous functional whose derivative admits a continuous inverse on X^*, $\mathcal{K} : X \to \mathbb{R}$ is a continuously differentiable functional whose derivative is compact. Assume that*

- (i) $\lim\limits_{\|u\| \to \infty} (\mathcal{H}(u) + \lambda \mathcal{K}(u)) = \infty$, *for all $\lambda \in I$.*
- (ii) *There exists $\gamma \in \mathbb{R}$ such that*

$$\sup_{\lambda \in I} \inf_{u \in X} (\mathcal{H}(u) + \lambda(\mathcal{K}(u) + \gamma)) < \inf_{u \in X} \sup_{\lambda \in I} (\mathcal{H}(u) + \lambda(\mathcal{K}(u) + \gamma)).$$

Then, there exist an open interval $\Lambda \subset I$ and a positive real number ρ such that, for each $\lambda \in \Lambda$ and for each $\mathcal{M} : X \to \mathbb{R}$ continuously differentiable, with compact derivative, there exists $\delta > 0$ such that for any $\mu \in [0, \delta]$, the equation

$$\mathcal{H}'(u) + \lambda \mathcal{K}'(u) + \mu \mathcal{M}'(u) = 0,$$

has at least three solutions in X whose norms are less than ρ.

Put $\theta_0 := \dfrac{b_0}{A_0}$. The main result of this section is the following.

Theorem 4.4. *Assume that hypotheses $(A_0) - (A_4)$, (H_1), (H_2), (H_3) (resp. (H_3')), and $(H_4) - (H_6)$ are verified and suppose that $h \in \left(L^{p_M^+}(\Omega)\right)'$. Additionally, suppose that $|\lambda| < \theta_0$ (resp. $\lambda \in]-\infty, +\infty[$). Then, there exist an open interval $\Lambda \subset]-\theta_0, \theta_0[$ (resp. $\subset]-\infty, +\infty[$), two positive constants ρ and δ such that for any $\lambda \in \Lambda$ and any $\mu \in [0, \delta]$, problem (9) has at least three weak solutions in $W^{1,\vec{p}(x)}(\Omega)$, whose norms are less than ρ.*

Proof. In order to apply Ricceri's result [27], we define the functionals $\mathcal{H},\mathcal{K},\mathcal{M} : W^{1,\vec{p}(x)}(\Omega) \to \mathbb{R}$ by

$$\begin{cases} \mathcal{H}(u) = \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)} dx \\ \mathcal{K}(u) = -\int_\Omega F(x,u)dx - \int_\Omega u(x)h(x)dx, \\ \mathcal{M}(u) = -\int_{\partial\Omega} G(x,u)d\sigma. \end{cases}$$

One can see that $\mathcal{H},\mathcal{K},\mathcal{M} \in C^1(W^{1,\vec{p}(x)}(\Omega),\mathbb{R})$ just by drawing on similar reasoning as demonstrated in the proof of [21, Lemma 3.4.], with their respective derivatives given by

$$\begin{cases} \langle \mathcal{H}'(u), v\rangle = \int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \int_\Omega b_i(x)|u|^{p_M(x)-2} uv dx \\ \langle \mathcal{K}'(u), v\rangle = -\int_\Omega f(x,u)v dx - \int_\Omega hv dx, \\ \langle \mathcal{M}'(u), v\rangle = -\int_{\partial\Omega} g(x,u)v d\sigma, \end{cases}$$

for any $u, v \in W^{1,\vec{p}(x)}(\Omega)$. Hence, if there exists a critical point u of the operator $\mathcal{H} + \lambda\mathcal{K} + \mu\mathcal{M}$, we conclude that $u \in W^{1,\vec{p}(x)}(\Omega)$ is a weak solution to equation (9). Then, we can apply Theorem 4.3 to look for weak solutions to problem (9). First, the fact that $\mathcal{M}'$ is compact can be shown easily by adapting to the case of $\partial\Omega$ (instead of Ω) the proof of Colasuonno et al. [10, Lemma 3.2.]. Next, we prove (*i*) in Proposition 4.3.

Case 1: Suppose hypothesis (H_3) is satisfied. We denote $\Phi(u) = \mathcal{H}(u) + \lambda\mathcal{K}(u)$. Let $\epsilon_0 > 0$ be fixed such that $|\lambda| < \frac{b_0}{\epsilon_0}$ and $u \in W^{1,\vec{p}(x)}(\Omega)$ with $\|u\| > \max\{1, \eta_{\epsilon_0}\}$ (where η_{ϵ_0} is given in (13)). By using (14), it follows that

$$\begin{aligned} \Phi(u) &= \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)} dx - \lambda \int_\Omega F(x,u)dx - \lambda \int_\Omega u(x)h(x)dx \\ &\geq \sigma_0 \int_\Omega \sum_{i=1}^N \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} dx + b_0 \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx - |\lambda| \int_\Omega \left(\frac{(A_0+\epsilon)}{q(x)} |u(x)|^{q(x)} + |M_{\epsilon_0}(x)| \right) dx - |\lambda| \|u\| \|h\|_*. \end{aligned}$$

Now, we have

$$\int_\Omega \frac{1}{q(x)} |u(x)|^{q(x)} dx \leq \frac{1}{q^-} \int_\Omega |u(x)|^{q(x)} dx \leq c\left(\|u\|^{q^-} + \|u\|^{q^+} \right).$$

Then, we get

$$\begin{aligned} \Phi(u) \geq{}& \min\{\sigma_0, b_0\} \left(\int_\Omega \sum_{i=1}^N \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx \right) - |\lambda|(A_0+\epsilon_0) \int_\Omega \frac{1}{q(x)} |u(x)|^{q(x)} dx \\ &- |\lambda| \|M_{\epsilon_0}\|_1 - \|u\| \|h\|_* \\ \geq{}& \min\{\sigma_0, b_0\} \left(\int_\Omega \sum_{i=1}^N \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx \right) - c_\lambda \left(\|u\|^{q^-} + \|u\|^{q^+} \right) - |\lambda| \|M_{\epsilon_0}\|_1 - \|u\| \|h\|_* \\ \geq{}& \frac{d_0}{p_M^+} \|u\|^{p_m^-} - c_\lambda \left(\|u\|^{q^-} + \|u\|^{q^+} \right) - |\lambda| \|M_{\epsilon_0}\|_1 - |\lambda| \|u\| \|h\|_* - \frac{d_1}{p_M^+}, \end{aligned}$$

where $c_\lambda := c|\lambda|(A_0 + \epsilon_0)$. For $1 < q^- < q^+ < p_m^-$, we deduce that $\Phi(u) \to \infty$ is $\|u\| \to \infty$, which means that Φ is coercive for any $\lambda \in \mathbb{R}$. Therefore, (i) in Proposition 4.3 is verified.

Case 2: Suppose hypothesis (H_3') is satisfied. Let $\epsilon_0 > 0$ be fixed such that $|\lambda| < \dfrac{b_0}{\epsilon_0}$ and let $u \in W^{1,\vec{p}(x)}(\Omega)$ with $\|u\| > \max\{1, \eta_{\epsilon_0}\}$ (where η_{ϵ_0} is mentioned in (13)). By using (14), it follows that

$$\begin{aligned}\Phi(u) &= \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)} dx - \lambda \int_\Omega F(x,u) dx - \lambda \int_\Omega u(x)h(x) dx \\ &\geq \sigma_0 \int_\Omega \sum_{i=1}^N \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} dx + b_0 \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx - |\lambda| \int_\Omega \left(\frac{\epsilon_0}{p_M(x)} |u(x)|^{p_M(x)} + |M_{\epsilon_0}(x)| \right) dx - |\lambda| \|u\| \|h\|_*.\end{aligned}$$

Then, by (8), we get

$$\begin{aligned}\Phi(u) &\geq \sigma_0 \int_\Omega \sum_{i=1}^N \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} dx + (b_0 - |\lambda|\epsilon_0) \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx - |\lambda| \|M_{\epsilon_0}\|_1 - |\lambda| \|u\| \|h\|_* \\ &\geq \min\{\sigma_0, b_0 - |\lambda|\epsilon_0\} \left(\int_\Omega \sum_{i=1}^N \frac{\left|\partial_{x_i} u\right|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx \right) - |\lambda| \|M_{\epsilon_0}\|_1 - |\lambda| \|u\| \|h\|_* \\ &\geq \frac{d_0 \min\{\sigma_0, b_0 - |\lambda|\epsilon_0\}}{p_M^+} \|u\|^{p_m^-} - |\lambda| \|M_{\epsilon_0}\|_1 - |\lambda| \|u\| \|h\|_* - \frac{d_1}{p_M^+}.\end{aligned}$$

Because $p_m^- > 1$, we deduce that $\Phi(u) \to \infty$ as $\|u\| \to \infty$, which means that Φ is coercive for any $\lambda \in \left]-\infty, +\infty\right[$. Therefore, (i) in Proposition 4.3 is verified.

Now, in order to establish (ii) of Proposition 4.3, by Bonanno and Candito [4, Proposition 1.3], it suffices to show that there exists $w \in X$ and $r > 0$ such that

(B_1) $\mathcal{H}(w) > \rho$

(B_2) $\displaystyle\sup_{\mathcal{H}(u) < r} \mathcal{K}(u) < \rho \frac{\mathcal{K}(w)}{\mathcal{H}(w)}$.

We now prove (B_1). Consider $x_0 \in \Omega$, e_0 and $D > 0$ as defined in hypothesis (H_6). For α with $0 < \alpha < 1$, define u_α such that

$$u_\alpha(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus \mathcal{B}(x_0, D), \\ \dfrac{e_0}{D(1-\alpha)}(D - \|x - x_0\|_1) & \text{if } x \in \mathcal{B}(x_0, D) \setminus \mathcal{B}(x_0, \alpha D), \\ e_0 & \text{if } x \in \mathcal{B}(x_0, \alpha D). \end{cases}$$

Straightforward calculations show that $u_\alpha \in X$. Moreover, for α less and close to 1, we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N \left|\partial_{x_i} u_\alpha\right|^{p_i(x)} dx + \int_\Omega |u_\alpha|^{p_M(x)} dx &> \int_{T_\alpha} \sum_{i=1}^N \left(\frac{e_0}{D(1-\alpha)} \right)^{p_i(x)} dx \\ &> N \left(\frac{e_0}{D(1-\alpha)} \right)^{p_m^-} meas(T_\alpha),\end{aligned}$$

where $T_\alpha = \mathcal{B}(x_0, D) \setminus \mathcal{B}(x_0, \alpha D)$. Therefore, we get

$$\begin{aligned}\mathcal{H}(u_\alpha) &\geq \frac{\sigma_0}{p_M^+}\int_\Omega \sum_{i=1}^N \left|\partial_{x_i} u_\alpha\right|^{p_i(x)} dx + \frac{b_0}{p_M^+}\int_\Omega |u_\alpha|^{p_M(x)} dx \\ &> \frac{\sigma_0 N}{p_M^+}\left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha),\end{aligned}$$

Denote

$$r_\alpha = \frac{\sigma_0 N}{p_M^+}\left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha),$$

and recall that

$$meas(\mathcal{B}(x_0, \tau D)) = \frac{(2\tau D)^N}{N!},$$

for any $\tau > 0$. Therefore, we have

$$\lim_{\alpha\to 1^-} r_\alpha = +\infty.$$

To complete the proof of (B_1), it suffices to consider $w = u_\alpha$ and $\rho = r_\alpha$, with $0 < \alpha < 1$. Next, we prove (B_2). Let $u \in X$ be such that $\mathcal{H}(u) < r_\alpha$.

Case 1: Suppose that hypothesis (H_3) is satisfied. We have

$$\begin{aligned}\mathcal{H}(u) &\geq \frac{\sigma_0}{p_M^+}\int_\Omega \sum_{i=1}^N \left|\partial_{x_i} u\right|^{p_i(x)} dx + \frac{b_0}{p_M^+}\int_\Omega |u|^{p_M(x)} dx \\ &\geq \frac{\min\{\sigma_0, b_0\}}{p_M^+}\left(\int_\Omega \sum_{i=1}^N \left|\partial_{x_i} u\right|^{p_i(x)} dx + \int_\Omega |u|^{p_M(x)} dx\right) \\ &\geq \frac{1}{p_M^+}\left(d_0\|u\|^{p_m^-} - d_1\right).\end{aligned}$$

Therefore, we obtain

$$\|u\| \leq \left(\frac{p_M^+}{d_0}\left(r_\alpha + \frac{d_1}{p_M^+}\right)\right)^{\frac{1}{p_m^-}}.$$

That is,

$$\|u\| \leq \Lambda r_\alpha^{\frac{1}{p_m^-}}. \tag{15}$$

for certain $\Lambda > 0$.

Case 2: Suppose that hypothesis (H_3') is satisfied. We have

$$\begin{aligned}\mathcal{H}(u) &\geq \frac{\sigma_0}{p_M^+}\int_\Omega \sum_{i=1}^N \left|\partial_{x_i} u\right|^{p_i(x)} dx + \frac{b_0}{p_M^+}\int_\Omega |u|^{p_M(x)} dx \\ &\geq \frac{b_0}{p_M^+}\int_\Omega |u|^{p_M(x)} dx \\ &= \frac{b_0}{p_M^+}\rho_{p_M(x)}(u).\end{aligned}$$

It gives

$$\rho_{p_M(x)}(u) \leq \frac{p_M^+}{b_0} r_\alpha. \tag{16}$$

To obtain (B_2), we prove the following lemma.

Proposition 4.5. *Suppose that hypotheses* (H_1), (H_2), (H_3) *(resp.*(H_3')*) and* $(H_4)-(H_6)$ *are satisfied. Then, we have*

$$\lim_{\alpha\to 1^-}\frac{\sup\{\mathcal{K}(u)/\mathcal{H}(u)<r_\alpha\}}{r_\alpha}=0. \tag{17}$$

Let us first point out that $\mathcal{K}(u_\alpha)$ is positive for α close to 1. Straightforward calculations give

$$\mathcal{K}(u_\alpha)=I_\alpha+J_\alpha,$$

where

$$I_\alpha=\int_{\mathcal{B}(x_0,\alpha D)}(F(x,e_0)+e_0h(x))\,dx,$$

and

$$J_\alpha=\int_{T_\alpha}\left(F(x,\frac{e_0}{(1-\alpha)D}(D-\|x-x_0\|_1)+\frac{e_0}{(1-\alpha)D}h(x)(D-\|x-x_0\|_1)\right)dx,$$

where $T_\alpha=\mathcal{B}(x_0,D)\setminus\mathcal{B}(x_0,\alpha D)$. Choosing $0<\alpha<1$ very close to 1, and using hypotheses (H_4) and (H_5), we have $I_\alpha>I_{\alpha_0}>0$, for any $\alpha_0<\alpha<1$. Similarly, based on hypothesis (H_6), J_α can be made sufficiently small such that we can obtain $\mathcal{K}(u_\alpha)>0$ (note that $\lim_{\alpha\to1^-} meas(T_\alpha)=0$). On the other hand, it is evident that $\mathcal{H}(u_\alpha)$ is positive. Now, if (17) is satisfied, then for $0<\epsilon<\dfrac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})}$, there exists $0<\eta<1$ such that, for any α satisfying $0<1-\eta<\alpha<1$, we have

$$\sup\{\mathcal{K}(u)/\mathcal{H}(u)<r_\alpha\}<\epsilon r_\alpha<r_\alpha\frac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})},$$

which yields (B_2).

Proof of Proposition 4.5.

Case 1: Suppose hypothesis (H_3) is satisfied. Let $\epsilon>0$. By using (11) we derive that, for some positive constant c_8 that may depend on ϵ, we have

$$\mathcal{K}(u)\le\epsilon\|u\|^{p_m^-}+c_8\left(\|u\|^{q^-}+\|u\|^{q^+}\right)+\|M_\epsilon\|_{L^1(\Omega)}+\|h\|_*\|u\|, \tag{18}$$

for all $u\in X$. Therefore, for $u\in X$ with $\mathcal{H}(u)<r_\alpha$, thanks to (15) and (18), there exist positive constants K_1, K_2 and K_3 (K_2 depending on ϵ) such that

$$\mathcal{K}(u)\le\epsilon\Lambda^{p_m^-}r_\alpha+K_1\left(r_\alpha^{\frac{q^-}{p_m^-}}+r_\alpha^{\frac{q^+}{p_m^-}}\right)+K_2+K_3r_\alpha^{\frac{1}{p_m^-}}.$$

Consequently, we obtain

$$\frac{\mathcal{K}(u)}{r_\alpha}\le\epsilon\Lambda^{p_m^-}+K_1\left(r_\alpha^{\frac{q^-}{p_m^-}-1}+r_\alpha^{\frac{q^+}{p_m^-}-1}\right)+\frac{K_2}{r_\alpha}+K_3r_\alpha^{\frac{1}{p_m^-}-1}. \tag{19}$$

Because $p_m^->\max\{1,q^-,q^+\}$, by letting α tend to 1 and ϵ tend to 0, we can make the second term in (19) as small as desired.

Case 2: Suppose that hypothesis (H_3') is satisfied. Let $\epsilon>0$. By using (12), we derive

$$\mathcal{K}(u)\le\epsilon\|u\|^{p_m^-}+\frac{\epsilon}{p_M^-}\rho_{p_M(x)}+\|M_\epsilon\|_{L^1(\Omega)}+\|h\|_*\|u\|, \tag{20}$$

for all $u\in X$. Then, for $u\in X$ with $\mathcal{H}(u)<r_\alpha$, thanks to (16) and (20), there exist positive constants K_1, K_2 and K_3 (K_2 depending on ϵ) such that

$$\mathcal{K}(u) \leq \epsilon \left(\Lambda^{p_m^-} + \frac{p_M^+}{b_0 p_M^-} \right) r_\alpha + K_2 + K_3 r_\alpha^{\frac{1}{p_m^-}}.$$

Consequently, we obtain

$$\frac{\mathcal{K}(u)}{r_\alpha} \leq \epsilon \left(\Lambda^{p_m^-} + \frac{p_M^+}{b_0 p_M^-} \right) + \frac{K_2}{r_\alpha} + K_3 r_\alpha^{\frac{1}{p_m^-}-1}. \tag{21}$$

Because $p_m^- > 1$, by letting α tend to 1 and ϵ to 0, we can make the second term in (21) as small as desired. Hence, the proof of Proposition 4.5 is complete.

Now, we conclude the proof of Theorem 4.3. Let $0 < \alpha_1 < 1$ such that $\mathcal{K}(u_{\alpha_1}) > 0$ and choose ϵ such that $0 < \epsilon < \epsilon_0 := \dfrac{1}{2} \dfrac{\mathcal{K}(u_{\alpha_1})}{\mathcal{H}(u_{\alpha_1})}$. Using Proposition 4.5, there exists η_0 such that, for any $1 - \eta_0 < \alpha < 1$, we have

$$\frac{\sup \{\mathcal{K}(u) / \mathcal{H}(u) < r_\alpha\}}{r_\alpha} < \epsilon_0. \tag{22}$$

By choosing $\alpha_1 > 1 - \eta_0$ and taking

$$\xi_0 := \frac{\sup \{\mathcal{K}(u) / \mathcal{H}(u) < r_{\alpha_1}\}}{r_{\alpha_1}} \quad \text{and} \quad \delta = \frac{b}{2\epsilon_0 - \xi_0},$$

with $b > 1$ and applying the result by Bonanno and Candito [4, Proposition 1.3], we deduce that

$$\sup_{\lambda \in \mathbb{R}} \inf_{u \in X} (\mathcal{H}(u) + \lambda(\beta - \mathcal{K}(u))) = \inf_{u \in X} \sup_{\lambda \in [0,\delta]} (\mathcal{H}(u) + \lambda(\beta - \mathcal{K}(u))),$$

for a suitable $\beta > 0$. Then, by using Ricceri [27, Theorem 1], there exist a non-empty set $U \subset]-\sigma_0, \sigma_0[$ (where $\sigma_0 := \theta_0$ if (H_3) is satisfied and $\sigma_0 := +\infty$ when (H_3') is supposed) and $\rho > 0$ such that for any $\lambda \in U$, there exists $\delta > 0$, such that the equation

$$\mathcal{H}'(u) + \lambda \mathcal{K}'(u) + \mu \mathcal{M}'(u) = 0,$$

has at least three solutions in X whose norms are less than ρ. This completes the proof of Theorem 4.4. □

Example 4.6. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $p_i \in C^+(\bar{\Omega})$ with $p_i(x) \geq 2$ for all $i \in \{1, \dots, N\}$ and $(r_1, r_2) \in (C^+(\bar{\Omega}))^2$ with $2 \leq r_1(x) < \max\{(\bar{p})^*, p_M(x)\}$, for all $x \in \Omega$ and $1 < r_2(x) < \min_{x \in \partial\Omega} \left\{ p_1^\partial(x), \dots, p_N^\partial(x) \right\}$. Consider β and q in $C^+(\bar{\Omega})$ such that $1 < q(x) < p_m^- < \beta(x)$ and $1 + q(x) \leq r_1(x) \leq \beta(x)$ for all $x \in \Omega$. Take $h \in L^{p_M'(\cdot)}(\Omega)$ and $A \in L^\infty(\Omega)$ with*

$$h(x) := \left(\frac{1}{1 + x^2} \right)^{\frac{1}{p_M'(x)}} \quad \text{and} \quad A(x) := \frac{\beta(x)}{q(x)} \left(\frac{\pi}{2} \right)^{\beta(x) - q(x)},$$

for all $x \in \Omega$. Note that $\|A\|_\infty = \dfrac{\beta^+}{q^-} \left(\dfrac{\pi}{2} \right)^{\beta^+ - q^-}$. Now, define

$$f(x,u) := \begin{cases} \beta(x) |u|^{\beta(x)-2} u & \text{if } |u| \leq \frac{\pi}{2}, \\ q(x) A(x) |u|^{q(x)-2} u \sin u + A(x) |u|^{q(x)} \cos u & \text{if } |u| \geq \frac{\pi}{2}, \end{cases}$$

for all $(x,u) \in \Omega \times \mathbb{R}$, and

$$g(x,u) = \frac{1}{1 + x^2} |u|^{\sigma(x)} \cos u, \quad \forall (x,u) \in \partial\Omega \times \mathbb{R},$$

with $\sigma \in C^+(\bar{\Omega})$ such that $1 < \sigma(x) < r_2(x) - 1$, $\forall x \in \partial\Omega$. Straightforward calculations give that

$$|f(x,u)| \leq c_6 |u|^{r_1(x)-1}, \quad \forall (x,u) \in \Omega \times \mathbb{R}, \quad (\textit{resp. } |g(x,u)| \leq |u|^{r_2(x)-1}, \quad \forall (x,u) \in \partial\Omega \times \mathbb{R}),$$

with $c_6 := \max\left\{\beta^+, \left(q^+ + \frac{2}{\pi}\right)\|A\|_\infty\right\}$ *and*

$$F(x,u) := \int_0^u f(x,s)ds = \begin{cases} |u|^{\beta(x)} & \text{if } |u| \leq \frac{\pi}{2}, \\ A(x)|u|^{q(x)} & \text{if } |u| \geq \frac{\pi}{2}, \end{cases}$$

for all $(x,u) \in \Omega \times \mathbb{R}$. *Moreover, we have*

$$F(x,y) + yh(x) > 0, \quad \forall (x,y) \in \Omega \times \mathcal{B}_R(O, e_0),$$

with $0 < e_0 < \frac{\pi}{2}$ *and*

$$\limsup_{|u|\to 0} \frac{F(x,u)}{|u|^{p_m^-}} = 0 \quad \text{and} \quad \limsup_{|u|\to +\infty} \frac{F(x,u)}{|u|^{q(x)}} = A(x),$$

uniformly for a.e $x \in \Omega$. *Then, all hypotheses of Theorem* 4.4 *are satisfied and we obtain the existence of at least three solutions for the following problem*

$$\begin{cases} -\sum_{i=1}^{N} |\partial_{x_i} u(x)|^{p_i(x)-2}\partial_{x_i} u + |u|^{p_M(x)-2}u = \lambda f(x,u) + h(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} |\partial_{x_i} u(x)|^{p_i(x)-2}\partial_{x_i} u(x)\nu_i(x) = \mu g(x,u) & \text{on } \partial\Omega. \end{cases}$$

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