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Existence of infinitely many homoclinic solutions for a class of $p-$Laplacian Hamiltonian systems

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Abstract. In this paper, we introduce a general and weak sufficient condition that guarantees the existence of infinitely many homoclinic solutions for a class of $p-$Laplacian Hamiltonian systems. Our findings are established using a new symmetric mountain pass theorem developed by Kajikia. We extend and refine several recent results from the literature and provide examples to demonstrate our key theoretical contributions.

1. Introduction and main results

The aim of this paper is to establish the existence of infinitely many homoclinic solutions for the following p-Laplacian Hamiltonian systems in the entire space

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0, \tag{1}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^N$, $p > 1$, $a \in C(\mathbb{R}, \mathbb{R})$ and $\nabla W(t, u)$ denotes the gradient of $W(t, u)$ with respect to u. Here, as usual, we say that a solution u of the problem (1) is homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \longrightarrow 0$ and $\dot{u}(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$. Furthermore, if $u \not\equiv 0$ then u is called a nontrivial homoclinic solution.

Laplacian systems, characterized by their linearity, are commonly employed in applications such as heat conduction, fluid dynamics, image processing (e.g., image denoising), and geometric modeling. In contrast, p-Laplacian systems are nonlinear and particularly pertinent in contexts where nonlinear effects are significant. These include modeling the flow of non-Newtonian fluids, phase transitions in materials, and the analysis of complex biological systems.

When $p = 2$, the p-Laplacian Hamiltonian system (1) takes the simple form:

$$\ddot{u} - a(t)u + \nabla W(t, u) = 0,$$

which is a special form of the extensively studied Hamiltonian systems

$$\ddot{u} - L(t)u + \nabla W(t, u) = 0. \tag{2}$$

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where $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$.

Many papers (see, e.g., [1]-[3], [14], [16]-[18], [26]-[29]) treated the periodic (including autonomous) case where $L(t)$ and $W(t,u)$ are either independent of t or periodic in t. By contrast, the problem is quite different in nature for the nonperiodic case due to the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [29], there are also many papers (see, e.g., [12], [13], [15], [17], [21]-[25], [30]-[43]) concerning the nonperiodic case. For this case, the function $L(t)$ plays an important role. As far as we know, almost all these mentioned papers assumed that $L(t)$ is either coercive or uniformly positively definite and $W(t,x)$ is subquadratic or superquadratic at infinity.

Recently, systems (2) was explored by a new symmetric mountain pass theorem due Kajikiya [19], when the matrix $L(t)$ is uniformly positively definite and the potential $W \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, even in u and satisfies the following main assumption or its equivalent:

(W) There exist $t_0 \in I$ and a constant $r_0 > 0$ such that $(t_0 - r_0, t_0 + r_0) \subset I$ and

$$\liminf_{|u|\longrightarrow 0} \inf_{t\in[r_0-\tau, r_0+\tau]} \frac{W(t,u)}{|u|^2} > -\infty$$

and

$$\limsup_{|u|\longrightarrow 0} \inf_{t\in[r_0-\tau, r_0+\tau]} \frac{W(t,u)}{|u|^2} = +\infty,$$

where I is a bounded domain in $\mathbb{R}$.

Remark 1.1. *The assumptions (W) is somewhat restrictive and eliminates many functions. For example, if we take:*

$$W(t,u) = \frac{a}{s}|u|^s - \frac{\gamma(t)}{r}|u|^r, \tag{3}$$

where s, r, a *are constants satisfying* $1 < r < 2,\ r < s < \frac{2}{3}(r+1),\ a > 0$ *and*

$$\gamma(t) = \begin{cases} t \text{ if } t \in [0, \frac{1}{2}], \\ 1 - t \text{ if } t \in [\frac{1}{2}, 1], \\ 0 \text{ if } t \in \mathbb{R} \setminus [0,1], \end{cases}$$

then, we have

$$\inf_{t\in[r_0-\tau, r_0+\tau]} \frac{W(t,u)}{|u|^2} = \frac{a}{s}|u|^{-(2-s)} - \frac{\overline{\gamma}}{r}|u|^{-(2-r)} \to -\infty \text{ as } u \to 0,$$

for any $[r_0 - \tau, r_0 + \tau] \subset (0,1)$*, where* $\overline{\gamma} := \max_{|t-t_0|\leq\tau} \gamma(t) > 0$*. Which implies that the assumption (W) is not satisfied. For more details, see Section 4.*

The purpose of this article is to weaken the assumption (W) and to find a general sufficient condition on $W(t,u)$ which covers more examples as in the previous remark. More precisely, we make the following assumptions:

(A_0) There exists a constant $a_0 > 0$ such that

$$a(t) + a_0 \geq 1, \quad \forall t \in \mathbb{R},$$

$$\int_{\mathbb{R}} (a(t) + a_0)^{-\frac{1}{p-1}} dt < \infty,$$

and $\{t \in \mathbb{R}/a(t) \equiv 0\} \supset (0,1)$.

(W_1) $W \in C^1(\mathbb{R}\times B(0,\varepsilon_0), \mathbb{R})$, it is even with respect to u and $W(t,0) \equiv 0$, where $B(0,\varepsilon_0)$ is the ball with center zero and radius ε_0 for some $\varepsilon_0 > 0$.

(W_2) There exists a constant $c_1 > 0$ such that

$$|\nabla W(t,u)| \leq c_1, \quad \forall (t,u) \in \mathbb{R} \times B(0,\varepsilon_0).$$

For $\rho > 0$, $t \in [0,1]$ satisfying $[t-\rho, t+\rho] \subset (0,1)$ and for $0 \neq u \in B(0,\varepsilon_0)$, we define

$$\overline{W}(t,u,\rho) := \inf\left\{ \frac{W(l,u)}{|u|^p}\rho^p : l \in [t-\rho, t+\rho] \right\}, \tag{4}$$

$$\underline{W}(t,u,\rho) := \inf\left\{ \frac{W(l,mu)}{|u|^p}\rho^p : l \in [t-\rho, t+\rho], 0 \leq m \leq 1 \right\}. \tag{5}$$

Substituting $m = 0$ into $\frac{W(l,mu)}{|u|^p}\rho^p$, we see that $\underline{W}(t,u,\rho) \leq 0$. We assume:

(W_3) There exists a positive integer k_0 satisfying the following condition:
For each $k \geq k_0$, there exist $0 \neq \mu_k \in B(0, \frac{\varepsilon_0}{2})$, $t_{k,i} \in (0,1)$, with $1 \leq i \leq 2k$ and $\rho_k > 0$ such that $[t_{k,i} - \rho_k, t_{k,i} + \rho_k] \subset (0,1)$, $[t_{k,i} - \rho_k, t_{k,i} + \rho_k] \cap [t_{k,j} - \rho_k, t_{k,j} + \rho_k] = \emptyset$ for $i \neq j$ and

$$\min_{1\leq i\leq 2k} \overline{W}(t_{k,i}, \mu_k, \rho_k) + 3 \min_{1\leq i\leq 2k} \underline{W}(t_{k,i}, \mu_k, \rho_k) > \frac{2^{p+2}}{p}. \tag{6}$$

Remark 1.2. • *We insist on the fact that in the hypothesis $(W_1) - (W_3)$, the conditions on the nonlinearity $W(t,u)$ are supposed only near $u = 0$ and there are no conditions for large $|u|$. This is essential and important. Indeed, this assumptions allows us to study equations having singularity or supercritical terms as $|u| \to \infty$. For example, let us consider the following p-Laplacian Hamiltonian systems:*

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) + \frac{|u|^{s-2}.u}{|\sin u|} = 0,$$

where $1 < s < 2$ and $u \in \mathbb{R}$. The function $u \longmapsto \frac{|u|^{s-2}.u}{|\sin u|}$ has singularities at $n\pi$ with $n \in \mathbb{Z}\setminus\{0\}$, but continuous at $u = 0$.

- *Under $(W_1) - (W_3)$, $W(t,u)$ can be subquadratic, superquadratic or asymptotically quadraticat infinity.*
- *To the best of our knowledge, there is no result concerning the existence and multiplicity of homoclinic orbits for the system with the conditions.*

Our main results reads as follows.

Theorem 1.3. *Suppose that (A_0) and $(W_1) - (W_3)$ are satisfied. Then, system (1) possesses a sequence of homoclinic solutions $\{u_k\}$ such that $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ as $k \to \infty$.*

Corollary 1.4. *Suppose that (A_0) and $(W_1) - (W_2)$ are satisfied and $\varepsilon_0 > 0$ be as in (W_1). We assume that there exist sequences $M_n \to \infty$ as $n \to \infty$, $0 \neq u_n \in B(0, \frac{\varepsilon_0}{2})$ and $\rho_n > 0$, $v_n \in (0,1)$ such that $[v_n - \rho_n, v_n + \rho_n] \subset (0,1)$ and a constant $c \geq 0$, satisfy*

$$W(t,u_n)\rho_n^p \geq M_n|u_n|^p,\ W(t,lu_n)\rho_n^p \geq -c|u_n|^p \text{ for } t \in [v_n - \rho_n, v_n + \rho_n],\ 0 \leq l \leq 1. \tag{7}$$

Then, system (1) possesses a sequence of homoclinic solutions $\{u_k\}$ such that $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ as $k \to \infty$.

Corollary 1.5. *Suppose that (A_0) and $(W_1) - (W_2)$ are satisfied and $\varepsilon_0 > 0$ be as in (W_1). We assume that there exist sequences $0 \neq u_n \in B(0, \frac{\varepsilon_0}{2})$, $\rho_n > 0$ and $v_n \in [0,1]$ such that $[v_n - \rho_n, v_n + \rho_n] \subset (0,1)$, and they satisfy*

$$\lim_{n\to\infty} \overline{W}(v_n, u_n, \rho_n) = \infty, \tag{8}$$

$$\liminf_{n\to\infty} \underline{W}(v_n, u_n, \rho_n) > -\infty. \tag{9}$$

Then, system (1) possesses a sequence of homoclinic solutions $\{u_k\}$ such that $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ as $k \to \infty$.

Corollary 1.6. *Suppose that* (A_0) *and* $(W), (W_1) - (W_2)$ *are satisfied. Then, system* (1) *possesses a sequence of homoclinic solutions* $\{u_k\}$ *such that* $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ *as* $k \to \infty$.

Corollary 1.7. *Suppose that* $(A_0), (W_1) - (W_2)$ *and*

$$\inf_{t\in[t_0-\rho_0,t_0+\rho_0]} \frac{W(t,u)}{|u|^p} \to \infty \text{ as } u \to 0, \tag{10}$$

are satisfied. Then, system (1) *possesses a sequence of homoclinic solutions* $\{u_k\}$ *such that* $\max_{t\in\mathbb{R}} |u_k(t)| \to 0$ *as* $k \to \infty$.

Following the idea of [38], we will first modify $W(t,u)$ for u outside a neighborhood of the origin 0 to get $\widetilde{W}(t,u)$ and introduce a modified Hamiltonian system $(\widetilde{pHS})$, where $(\widetilde{pHS})$ and $\widetilde{W}(t,u)$ will be specified in the following section (Section 2.). Then we show by variational methods that Hamiltonian system $(\widetilde{pHS})$ possesses a sequence of homoclinic solutions, which converges to zero in L^∞ norm. Consequently, we obtain infinitely many homoclinic solutions for the original Hamiltonian system (pHS).

Here and in the following $u.v$ denotes the inner product of $u, v \in \mathbb{R}^N$ and $|.|$ denotes the associated norm. Throughout the paper we denote by c, c_i the various positive constants which may vary from line to line and are not essential to the problem.

The remaining part of this paper is organized as follows: Some preliminary results are presented in Section 2. Section 3 is devoted to the proofs of our results and Section 4 is reserved for an example.

2. Preliminary results and variational setting

In order to prove our main result via the critical point theory, we need to establish the variational setting for (1). Before this, we have the following remark:

Remark 2.1. *Let* $\overline{a_0}(t) = a(t) + a_0$ *and* $W_0(t,u) = W(t,u) + \frac{a_0}{p}|u|^p$. *Consider the following p-Laplacian Hamiltonian system*

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a_0(t)|u(t)|^{p-2}u(t) + \nabla W_0(t,u(t)) = 0, \quad \forall t \in \mathbb{R}. \tag{11}$$

Then, system (11) *is equivalent to system* (1). *It is easy to check that the hypotheses* $(W1) - (W3)$ *still hold for* W_0 *provided that those hold for W. Moreover,* $\overline{a_0}(t) \geq 1$ *for all* $t \in \mathbb{R}$ *and* $\int_{\mathbb{R}} (\overline{a_0}(t))^{-\frac{1}{p-1}} dt < \infty$. *Hence, without loss of generality, we can assume in* (A_0) *that* $a(t) \geq 1$ *for all* $t \in \mathbb{R}$ *and* $\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}} dt < \infty$.

In view of Remark 2.1, we consider the space $E := \{u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N) | \int_{\mathbb{R}} |\dot{u}(t)|^p + a(t)|u(t)|^p dt < \infty\}$. Then E is a reflexive, separable Banach space with norm

$$||u|| = \left(\int_{\mathbb{R}} (|\dot{u}|^p + a(t)|u|^p) dt\right)^{\frac{1}{p}}.$$

In what follows, E becomes our working spaceMoreover, we write E^* for the topological dual of E, and $\langle \cdot, \cdot \rangle$: $E^* \times E \to \mathbb{R}$ for the dual pairing. Evidently, E is continuously embedded into $W^{1,p}(\mathbb{R}, \mathbb{R}^N)$. Using the Sobolev embedding theorem, we immediately get the following lemma.

Lemma 2.2. *If a satisfies* (A_0), *then E is continuously embedded in* L^1.

Proof. By (A_0) and Hölder inequality, we have for all $u \in E$

$$\begin{aligned}\int_{\mathbb{R}} |u|\, dt &= \int_{\mathbb{R}} \left|(a(t))^{\frac{-1}{p}} \, (a(t))^{\frac{1}{p}} \, u\right| dt \\ &\le \int_{\mathbb{R}} (a(t))^{\frac{-1}{p}} \left|(a(t))^{\frac{1}{p}} \, u\right| dt \\ &\le \left(\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}} \, dt\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}} a(t)|u|^p dt\right)^{\frac{1}{p}} \\ &\le \left(\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}} \, dt\right)^{\frac{p-1}{p}} \|u\| .\end{aligned} \tag{12}$$

□

Lemma 2.3. *If a satisfies (A_0) then E is compactly embedded into L^1.*

Proof. Let $(u_n) \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E. We will show that $u_n \longrightarrow u$ in L^1. By Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} |u_n - u|\, dt &= \int_{|t|\le R} |u_n - u|\, dt + \int_{|t|>R} |u_n - u|\, dt \\ &\le (2R)^{\frac{1}{p}} \left(\int_{|t|\le R} |u_n - u|^p \, dt\right)^{\frac{1}{p}} + \int_{|t|>R} \left|(a(t))^{\frac{-1}{p}} \, (a(t))^{\frac{1}{p}} \, (u_n - u)\right| dt \\ &\le (2R)^{\frac{1}{p}} \left(\int_{|t|\le R} |u_n - u|^p \, dt\right)^{\frac{1}{p}} + \int_{|t|>R} (a(t))^{-\frac{1}{p}} \left|(a(t))^{\frac{1}{p}} \, (u_n - u)\right| dt \\ &\le (2R)^{\frac{1}{p}} \left(\int_{|t|\le R} |u_n - u|^p \, dt\right)^{\frac{1}{p}} + \left(\int_{|t|>R} (a(t))^{-\frac{1}{p-1}} \, dt\right)^{\frac{p-1}{p}} \left(\int_{|t|>R} a(t)|u_n - u|^p dt\right)^{\frac{1}{p}} \\ &\le (2R)^{\frac{1}{p}} \left(\int_{|t|\le R} |u_n - u|^p \, dt\right)^{\frac{1}{p}} + \left(\int_{|t|>R} (a(t))^{-\frac{1}{p-1}} \, dt\right)^{\frac{p-1}{p}} \|u_n - u\| ,\end{aligned} \tag{13}$$

where $R > 0$. Then by (A_0) and the Sobolev embedding Theorem, for any $\varepsilon > 0$ there exits $R_0 > 0$ such that for $R > R_0$, we have

$$\int_{\mathbb{R}} |u_n - u|\, dt \le \varepsilon.$$

□

In order to define the corresponding variational functional on our working space E, we need modify $W(t, u)$ for u outside a neighborhood of the origin to get a globally defined $\widetilde{W}(t, u)$ as follows: Choose a constant $b \in (0, \frac{\varepsilon_0}{2})$ and define a cut-off function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\chi(t) = \begin{cases} 1 & \text{if } 0 \le t \le b \\ 0 & \text{if } t \ge 2b \end{cases} \quad \text{and, } -\frac{2}{b} \le \chi'(t) < 0 \text{ for } b < t < 2b. \tag{14}$$

Let $\widetilde{W}(t, u) = \chi(|x|)W(t, u)$, for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. By (14) and (W_2) we have, for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$,

$$\left|\widetilde{W}(t, u)\right| \le c_1 |u| \text{ and } \left|\nabla \widetilde{W}(t, u)\right| \le c_2, \tag{15}$$

for some $c_2 > 0$, where c_1 is the constant given in (W_2).

Now, we consider the following modified p-Laplacian Hamiltonian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a(t)|u(t)|^{p-2}u(t) + \nabla\widetilde{W}(t,u) = 0, \quad \forall t \in \mathbb{R}, \qquad (\widetilde{pHS})$$

and define the variational functional Φ associated with system $(\widetilde{pHS})$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\int_{\mathbb{R}}(|\dot{u}|^p + a(t)|u|^p)dt - \Psi(u)\\ &= \frac{1}{p}\|u\|^p - \Psi(u),\end{aligned}$$

where $\Psi(u) = \int_{\mathbb{R}} \widetilde{W}(t,u)dt$.

According to [34], we know that in order to find solutions of $(\widetilde{pHS})$ it suffices to obtain the critical points of Φ. For this purpose we recall the following definitions and results (see [15, 17, 18]).

Definition 2.4. *Let E be a real Banach space and $\phi \in C^1(E,\mathbb{R})$.*
• *ϕ is said to satisfy (PS) condition if any sequence $(u_k) \subset E$ for which $(\phi(u_k))$ is bounded and $\phi'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in E. Here $\phi'(u)$ denotes the Fréchet derivative of $\phi(u)$.*
• *Set $\Gamma := \{A \subset E\backslash\{0\} : A$ is closed and symmetric with respect to the origin$\}$. For $A \in \Gamma$, we say genus of A is n (denoted by $\sigma(A) = n$), if there is an odd mapping $\varphi \in C(A, \mathbb{R}^n\backslash\{0\})$, and n is the smallest integer with this property.*

Theorem 2.5 ([19, Theorem 1]). *Let ϕ be an even C^1 functional on E with $\phi(0) = 0$. Suppose that ϕ satisfies the (PS) condition and*

(1) ϕ is bounded from below.
(2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $sup_{u\in A_k}\phi(u) < 0$, where $\Gamma_k = \{A \in \Gamma : \sigma(A) \geq k\}$.

Then either (i) or (ii) below holds.

(i) There exists a critical point sequence (u_k) such that $\phi(u_k) < 0$ and $\lim_{k\to\infty} u_k = 0$.
(ii) There exist two critical point sequences (u_k) and (v_k) such that $\phi(u_k) = 0, u_k \neq 0, \lim_{k\to\infty} u_k = 0, \phi(v_k) < 0, \lim_{k\to\infty} \phi(v_k) = 0$, and (v_k) converges to a non-zero limit.

Lemma 2.6. *Let (A_0), (W_1) and (W_2) be satisfied. Then $\Psi \in C^1(E,\mathbb{R})$, and hence $\Phi \in C^1(E,\mathbb{R})$. Moreover,*

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}} \nabla\widetilde{W}(t,u)v\, dt, \qquad (16)$$

and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}} |\dot{u}|^{p-2}\dot{u}\dot{v} + a(t)|u|^{p-2}uv\, dt - \int_{\mathbb{R}} \nabla\widetilde{W}(t,u)v\, dt \qquad (17)$$

for all $u, v \in E$, and nontrivial critical points of Φ on E are homoclinic solutions of system $(\widetilde{pHS})$.

Proof. First, we show that Φ and Ψ are both well defined. For any $u \in E$, by (12) and (15), we have

$$\begin{aligned}\int_{\mathbb{R}} |\widetilde{W}(t,u)|dt &\leq c_1 \int_{\mathbb{R}} |u|dt\\ &\leq c_1 \left(\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}}\, dt\right)^{\frac{p-1}{p}} \|u\|.\end{aligned}$$

This implies that Φ and Ψ are both well defined.

Next, we prove $\Psi \in C^1(E, \mathbb{R})$. For any given $u \in E$, define an associated linear operator $J(u) : E \to \mathbb{R}$ by

$$\langle J(u), v\rangle = \int_{\mathbb{R}} \nabla\widetilde{W}(t,u)v\,dt, \quad \forall v \in E.$$

By (12) and (15), there holds

$$\begin{aligned}|\langle J(u), v\rangle| &= \int_{\mathbb{R}} \left|\nabla\widetilde{W}(t,u)\right||v|dt\\ &\leq c_2 \int_{\mathbb{R}} |v|dt\\ &\leq c_2 \left(\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}}\,dt\right)^{\frac{p-1}{p}} \|v\|.\end{aligned}$$

This implies that $J(u)$ is well defined and bounded. Observing (12) and (15), for any $u, v \in E$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{s\to 0} \frac{\Psi(u+sv) - \Psi(u)}{s} &= \lim_{s\to 0} \int_{\mathbb{R}} \nabla\widetilde{W}(t, u+\theta(t)sv)v\,dt\\ &= \int_{\mathbb{R}} \nabla\widetilde{W}(t,u)v\,dt\\ &= \langle J(u), v\rangle,\end{aligned} \tag{18}$$

where $\theta(t) \in [0,1]$ depends on u, v, s. This implies that Ψ is Gâteaux differentiable on E and the Gâteaux derivative of Ψ at $u \in E$ is $J(u)$. Now for any $u \in E$, suppose $u_n \to u$ in E, then $u_n \to u$ in L^∞. For any $\epsilon > 0$, by (L_0), there exists $T_\epsilon > 0$ such that

$$\left(\int_{|t|>T_\epsilon} (a(t))^{-\frac{1}{p-1}} dt\right)^{\frac{p-1}{p}} < \frac{\epsilon}{4c_2}. \tag{19}$$

For the T_ϵ given above, by virtue of the continuity of $\nabla\widetilde{W}$ and Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_{-T_\epsilon}^{T_\epsilon} \left|\nabla\widetilde{W}(t,u_n) - \nabla\widetilde{W}(t,u)\right|^p dt = 0.$$

Here we use the fact that $u_n \to u$ in L^∞. Consequently, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\left(\int_{-T_\epsilon}^{T_\epsilon} \left|\nabla\widetilde{W}(t,u_n) - \nabla\widetilde{W}(t,u)\right|^p dt\right) < \frac{\epsilon}{2}, \quad \forall n \geq N_\epsilon. \tag{20}$$

Combining (15), (19), (20) and the Hölder inequality, for each $n \geq N_\epsilon$, we have

$$
\begin{aligned}
\|J(u_n) - J(u)\|_{E^*} &= \sup_{\|v\|=1} |\langle J(u_n) - J(u), v\rangle| \\
&\leq \sup_{\|v\|=1} \left| \int_{\mathbb{R}} (\nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u)).vdt \right| \\
&\leq \sup_{\|v\|=1} \left| \int_{-T_\epsilon}^{T_\epsilon} (\nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u)).vdt \right| \\
&\quad + \sup_{\|v\|=1} \left| \int_{|t|>T_\epsilon} (\nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u)).vdt \right| \\
&\leq \sup_{\|v\|=1} \left(\int_{-T_\epsilon}^{T_\epsilon} \left| \nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u) \right|^p dt \right)^{\frac{1}{p}} \left(\int_{-T_\epsilon}^{T_\epsilon} |v|^p \, dt \right)^{\frac{1}{p}} \\
&\quad + 2c_2 \sup_{\|v\|=1} \left(\int_{|t|>T_\epsilon} (a(t))^{-\frac{1}{p-1}} dt \right)^{\frac{p-1}{p}} \left(\int_{|t|>T_\epsilon} a(t)|v|^p dt \right)^{\frac{1}{p}} \\
&\leq \frac{\epsilon}{2} + \frac{2c_2\epsilon}{4c_2} = \epsilon.
\end{aligned}
$$

This means that J is continuous in u. Thus $\Psi \in C^1(E,\mathbb{R})$ and (16) holds. Due to the form of ϕ, we know that $\Phi \in C^1(E,\mathbb{R})$ and (17) also holds.

Finally, a standard argument shows that nontrivial critical points of Φ on E are homoclinic solutions of $(\widetilde{pHS})$ (see, e.g., [29]). The proof is completed.

□

Lemma 2.7. *Let (A_0), (W_1) and (W_2) be satisfied. Then Φ is bounded from below and satisfies (PS) condition.*

Proof. We first prove that Φ is bounded from below. Combining $(W2)$, (12), (15) and the Hölder inequality, we have

$$
\begin{aligned}
\Phi(u) &\geq \frac{1}{2}\|u\|^2 - c_2 \int_{\mathbb{R}} |u| dt \\
&\geq \frac{1}{2}\|u\|^2 - c_2 \left(\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}} dt \right)^{\frac{p-1}{p}} \|u\|, \quad \forall u \in E,
\end{aligned}
\tag{21}
$$

where c_2 is the constant given in (15). Then it follows that Φ is bounded from below.

Next, we show that Φ satisfies (PS)-condition.

Let $\{u_n\} \subset E$ be a (PS)-sequence, i.e.,

$$
|\Phi(u_n)| \leq D_2 \quad \text{and} \quad \Phi'(u_n) \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \tag{22}
$$

for some $D_2 > 0$. By (21) and (22), we have

$$
D_2 \geq \frac{1}{2}\|u_n\|^2 - c_2 \left(\int_{\mathbb{R}} (a(t))^{-\frac{1}{p-1}} dt \right)^{\frac{p-1}{p}} \|u_n\|, \quad \forall n \in \mathbb{N}.
$$

This implies that $\{u_n\}$ is bounded in E. Thus, there exists a subsequence $\{u_{n_k}\}$ such that

$$
u_{n_k} \rightharpoonup u_0 \quad \text{as} \quad k \longrightarrow \infty \tag{23}
$$

for some $u_0 \in E$. By Lemma 2.3, it holds that

$$
u_{n_k} \longrightarrow u_0 \quad \text{in} \quad L^1 \quad \text{as} \quad k \longrightarrow \infty. \tag{24}
$$

This together with (15) yields

$$\left|\int_{\mathbb{R}}\left(\nabla\widetilde{W}(t,u_{n_k})-\nabla\widetilde{W}(t,u_0)\right)\cdot(u_{n_k}-u_0)dt\right|\leq 2c_2\int_{\mathbb{R}}|u_{n_k}-u_0|dt\longrightarrow 0\quad\text{as}\quad k\longrightarrow\infty. \tag{25}$$

Noting that $\{u_n\}$ is bounded in E, we infer from (22) and (23) that

$$\langle\Phi'(u_{n_k})-\Phi'(u_0),u_{n_k}-u_0\rangle\longrightarrow 0\quad\text{as}\quad k\longrightarrow\infty. \tag{26}$$

Combining (17), (25) and (26), we have

$$\begin{aligned}\|u_{n_k}-u_0\|^2&=\langle\Phi'(u_{n_k})-\Phi'(u_0),u_{n_k}-u_0\rangle\\&\quad+\int_{\mathbb{R}}\left(\nabla\widetilde{W}(t,u_{n_k})-\nabla\widetilde{W}(t,u_0)\right)\cdot(u_{n_k}-u_0)dt\longrightarrow 0\quad\text{as}\quad k\longrightarrow\infty.\end{aligned} \tag{27}$$

This means that $u_{n_k}\to u_0$ in E as $k\to\infty$. Thus Φ satisfies (PS)-condition. □

We introduce a closed symmetric set V_k as below:

$$V_k\equiv\{(l_1,l_2,...,l_{2k})\in\mathbb{R}^{2k};|l_i|\leq 1\text{ for all }i,\ \text{card}\{i:|l_i|=1\}\geq k\}.$$

Lemma 2.8 ([27, Lemma 4.5]). *V_k has the genus of $k+1$.*

Lemma 2.9. *Let (A_0), (W_1) and (W_3) be satisfied. Then for each $k\in\mathbb{N}$, there exists an $A_k\subseteq E$ with genus $\sigma(A_k)=k+1$ such that $\sup_{u\in A_k}\Phi(u)<0$.*

Proof. Let μ_k, $t_{k,i}$ and ρ_k with $k\geq k_0$ be given in assumption (W_3). Since $\Gamma_k\subset\Gamma_{k-1}$ by definition, it is enough to construct an $A_k\in\Gamma_k$ for $k\geq k_0$ such that $\sup_{u\in A_k}\Phi(u)<0$. Fix $k\geq k_0$. Instead of μ_k, $t_{k,i}$ and ρ_k we write μ, t_i and ρ for simplicity. Using $\overline{W}$ and $\underline{W}$ given by (4) and (5) respectively, we define

$$\overline{W}_i:=\overline{W}(t_i,\mu,\rho),\ \underline{W}_i:=\underline{W}(t_i,\mu,\rho),\ 1\leq i\leq 2k.$$

It follows from (4) and (5) and for $t\in[t_i-\rho,t_i+\rho]$, that

$$W(t,\mu)\geq\frac{1}{\rho^p}\overline{W}_i|\mu|^p, \tag{28}$$

$$W(t,l\mu)\geq\frac{1}{\rho^p}\underline{W}_i|\mu|^p,\ |l|\leq 1. \tag{29}$$

We define a function $\varphi(t)$ on $\mathbb{R}$ by $\varphi(t)=1$ for $t\leq\frac{1}{2}$, $\varphi(t)=2(1-|t|)$ for $\frac{1}{2}\leq|t|\leq 1$ and $\varphi(t)=0$ for $|t|\geq 1$. Put $\varphi_i(t):=\varphi(\frac{|t-t_i|}{\rho})$ for $t\in\mathbb{R}$. Define $D_i:=[t_i-\frac{\rho}{2},t_i+\frac{\rho}{2}]$. Then $0\leq\varphi_i(t)\leq 1$ in $\mathbb{R}$, $\varphi_i(t)=0$ for $t\in\mathbb{R}\backslash[t_i-\rho,t_i+\rho]$ and

$$\varphi_i(t)=1\ \text{ for }\ t\in D_i,|\dot{\varphi}_i(t)|\leq\frac{2}{\rho}\ \text{ for }\ t\in\mathbb{R}. \tag{30}$$

We define

$$A_k:=\left\{\sum_{i=1}^{2k}l_i\varphi_i(t)\mu:\ (l_1,...,l_{2k})\in V_k\right\}.$$

Since all the supports of φ_i are disjoint, they are linearly independent. Define $g(l_1,...,l_{2k}):=\sum_{i=1}^{2k}l_i\varphi_i(t)\mu$. Then g is a mapping from V_k onto A_k and it is an odd homeomorphism. By Lemma 2.8, the genus of V_k is $k+1$ and so is A_k. Thus $A_k\in\Gamma_k$.

We shall show that $\sup_{A_k}\Phi(u) < 0$. Fix $(l_1, ..., l_{2k}) \in V_k$ arbitrary. Then $u := \sum_{i=1}^{2k} l_i\varphi_i(t)\mu \in A_k$. Since the support of φ_i is $[t_i-\rho, t_i+\rho]$ and $[t_i-\rho, t_i+\rho]\cap[t_j-\rho, t_j+\rho] = \emptyset$ for $i \neq j$, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\int_{\mathbb{R}}(|\dot{u}|^p + a(t)|u|^p)dt - \int_{\mathbb{R}}\widetilde{W}(t,u)dt\\ &= \sum_{i=1}^{2k}\int_{t_i-\rho}^{t_i+\rho}\frac{1}{p}|\mu|^p|l_i|^p|\dot{\varphi}_i|^p dt + \frac{1}{p}\sum_{i=1}^{2k}\int_{t_i-\rho}^{t_i+\rho}\left(a(t)|l_i\varphi_i(t)\mu|^p\right)dt\\ &-\sum_{i=1}^{2k}\int_{t_i-\rho}^{t_i+\rho}W(t, l_i\varphi_i\mu)dt.\end{aligned}$$

By the assumption (L_0) and (30), we have

$$\Phi(u) \leq \frac{2^{p+2}k|\mu|^p\rho^{1-p}}{p} - \sum_{i=1}^{2k}\int_{t_i-\rho}^{t_i+\rho}W(t, l_i\varphi_i\mu)dt. \tag{31}$$

To estimate the second term, we define

$$\begin{aligned}\Lambda_1 &:= \{i \in \{1, ..., 2k\} : |l_i| = 1\}\\ \Lambda_2 &:= \{i \in \{1, ..., 2k\} : |l_i| < 1\}.\end{aligned}$$

By the definition of V_k, the cardinal number of Λ_1 greater than or equal to k. We compute the integral of W on $[t_i-\rho, t_i+\rho]$ for $i \in \Lambda_1$, and for $i \in \Lambda_2$, separately. Recall that $W(t,u)$ is even with respect to u and $\varphi_i(t) = 1$ on D_i. By (28) and (29) we obtain, for $i \in \Lambda_1$,

$$\begin{aligned}\int_{t_i-\rho}^{t_i+\rho}W(t, l_i\varphi_i\mu)dt &= \int_{D_i}W(t,\mu)dt + \int_{[t_i-\rho,t_i+\rho]\setminus D_i}W(t, l_i\varphi_i\mu)dt\\ &\geq \frac{|\mu|^p}{\rho}\overline{W}_i + \frac{|\mu|^p}{\rho}\underline{W}_i.\end{aligned} \tag{32}$$

We define

$$\alpha := \min_{1\leq i\leq 2k}\overline{W}_i, \quad \beta := \min_{1\leq i\leq 2k}\underline{W}_i.$$

As stated after (5), it holds that $\underline{W}_i \leq 0$, and hence $\beta \leq 0$. We rewrite (6) as

$$\alpha + 3\beta > \frac{2^{p+2}}{p}. \tag{33}$$

We reduce (32) to

$$\int_{t_i-\rho}^{t_i+\rho}W(t, l_i\varphi_i\mu)dt \geq [\alpha+\beta]\frac{|\mu|^p\rho^{1-p}}{p}.$$

The right hand side is positive because of (33) with $\beta \leq 0$. Recall that the cardinal number of Λ_1 is greather than ou equal to k. Summing up both sides of the inequality above over $i \in \Lambda_1$, we obtain

$$\sum_{i\in\Lambda_1}\int_{t_i-\rho}^{t_i+\rho}W(t, l_i\varphi_i\mu)dt \geq [\alpha+\beta]k\frac{|\mu|^p\rho^{1-p}}{p}. \tag{34}$$

Next, by (29), for $i \in \Lambda_2$, we have

$$\int_{t_i-\rho}^{t_i+\rho} W(t, l_i\varphi_i\mu)dt \geq 2|\mu|^p\rho^{1-p}\underline{W}_i \geq 2\beta|\mu|^p\rho^{1-p}. \tag{35}$$

Recall that the cardinal number of Λ_2 is less than or equal to k. Summing up both sides over $i \in \Lambda_2$ and using $\beta \leq 0$, we find

$$\sum_{i\in\Lambda_2}\int_{t_i-\rho}^{t_i+\rho} W(t, l_i\varphi_i\mu)dt \geq 2k\beta|\mu|^p\rho^{1-p}. \tag{36}$$

The set Λ_2 may be empty. In this case, we consider the left hand side to be zero. Then the inequality above is still valid because $\beta \leq 0$. Substituting (34) and (36) into (31) and using (33), we obtain

$$\Phi(u) \leq -[\alpha + 3\beta - \frac{2^{p+2}}{p}]k|\mu|^p\rho^{1-p} < 0,$$

which implies that $\sup_{u\in A_k}\Phi(u) < 0$. □

Now we are in the position to give the proofs of our main results.

3. Proofs of Theorem 1.3 and Corollary 1.4-1.7

The aim of this section is to establish the proofs of Theorem 1.3 and Corollary 1.4-1.7.

3.1. *Proof of Theorem 1.3*

Lemmas 2.7, 2.8 and 2.9 shows that the functional Φ satisfies conditions (1) and (2) in Theorem 2.5. Therefore, there exist a sequence of nontrivial critical points (u_k) of Φ such that $\Phi(u_k) \leq 0$ for all $k \in \mathbb{N}$ and $u_k \longrightarrow 0$ in E as $k \longrightarrow \infty$. By virtue of Lemma 2.6, $\{u_k\}$ is a sequence of homoclinic solutions of (pHS). Since E is continuously embedded into L^∞, then it follows that $\max_{t\in\mathbb{R}}|u_k(t)| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, there exists $k_0 \in \mathbb{N}$ such that u_k is a homoclinic solution of (1) for each $k \geq k_0$.

3.2. *Proof of Corollary 1.4 and 1.5*

It is enough to show that (8) and (9) ⇒(7) ⇒ (6). Impose (8) and (9). Then we shall construct $\mu_k, t_{k,i}$ and ρ_k satisfying (6). Fix k arbitrary. Let K be the smallest integer that satisfies $K \geq 2k$. We divide $[v_n - \rho_n, v_n + \rho_n]$ equally into K intervals. Denote them by $C_{n,i}$ with $i \leq K$. Then the edge of $C_{n,i}$ has the length of $\frac{2\rho_n}{K}$. Let $[t_{n,i} - r_n, t_{n,i} + r_n]$ the inscribed interval in $C_{n,i}$. Then $r_n = \frac{\rho_n}{K}$. Since $K \geq 2k$, $t_{n,i}$ is defined for all $1 \leq i \leq 2k$. We shall show that assumption (W_3) is fulfilled with $\mu_k, t_{k,i}$ and ρ_k replaced by $u_n, t_{n,i}$ and r_n, respectively, if n is large enough. It is clear that $[t_{n,i} - r_n, t_{n,i} + r_n] \subset (0, 1)$ and $[t_{n,i} - r_n, t_{n,i} + r_n] \cap [t_{n,j} - r_n, t_{n,j} + r_n] = \emptyset$ when $i \neq j$. Define $M_n := \overline{W}(v_n, u_n, \rho_n)$, which implies that

$$\frac{W(t, u_n)}{|u_n|^p}\rho_n^p \geq M_n \text{ for } t \in [v_n - \rho_n, v_n + \rho_n].$$

By (9), there exists a $c \geq 0$ such that

$$\frac{W(t, lu_n)}{|u_n|^p}\rho_n^p \geq -c \text{ for } t \in (v_n - \rho_n, v_n + \rho_n), 0 \leq l \leq 1.$$

Then we obtain (7). On the other hand, substituting $\rho_n = Kr_n$ in the two inequalities above we have,

$$\frac{W(t, u_n)}{|u_n|^p}K^p r_n^p \geq M_n, \ \frac{W(t, lu_n)}{|u_n|^p}K^p r_n^p \geq -c \text{ for } t \in (v_n - \rho_n, v_n + \rho_n), 0 \leq l \leq 1.$$

Taking the infimum on $(t_{n,i} - r_n, t_{n,i} + r_n)$, we have

$$\overline{W}(t_{n,i}, u_n, r_n) \geq \frac{M_n}{K^p}, \ \underline{W}(t_{n,i}, u_n, r_n) \geq -\frac{c}{K^p}.$$

Then we get

$$\min_{1\leq i\leq 2k} \overline{W}(t_{n,i}, u_n, r_n) + 3 \min_{1\leq i\leq 2k} \underline{W}(t_{n,i}, u_n, r_n) > \frac{M_n}{2K^p} - \frac{7c}{2K^p}.$$

Since $\lim_{n\to\infty} M_n = \infty$ by (8), the right hand side is larger than 8 for n large enough.

3.3. *Proof of Corollary 1.6*

To prove this corollary, it is enough to show that the assumption (W) implies (8) and (9). By (W) there exists a sequence u_n converging to zero such that

$$\inf_{t\in(t_0-\rho_0,t_0+\rho_0)} \frac{W(t, u_n)}{|u_n|^p} \to \infty \text{ as } n \to \infty.$$

Put $(v_n, \rho_n) := (t_0, \rho_0)$ for all n. Then the above inequality shows (8). Also, by (W), there exists a constant $c \geq 0$ such that

$$\inf_{t\in(t_0-\rho_0,t_0+\rho_0)} \frac{W(t, u)}{|u|^p} \geq -c \text{ for } 0 < |u| \leq 1.$$

Putting $u = lu_n$, we find

$$\inf_{t\in(t_0-\rho_0,t_0+\rho_0)} \frac{W(t, lu_n)}{|lu_n|^p} \geq -c \text{ for all large } n \text{ and } 0 < l \leq 1,$$

which leads to

$$\inf_{t\in(t_0-\rho_0,t_0+\rho_0)} \frac{W(t, lu_n)}{|u_n|^p} \geq -cl^p \geq -c.$$

Therefore (9) holds.

3.4. *Proof of Corollary 1.7*

We observe that (10) implies (W). Therefore, Corollary 1.6 yields Corollary 1.7.

4. Example

For the reader's convenience, we present one example to illustrate our main results.
Let

$$a(t) = \begin{cases} 0 \text{ if } 0 \leq t < q \\ (q^p + 1)^p (t - q), \text{ if } q \leq t < q + \frac{1}{q^p+1}, \\ (q^p + 1)^p, \text{ if } q + \frac{1}{q^p+1} \leq t < q + \frac{q^p}{q^p+1}, \\ (q^p + 1)^p (q + 1 - t), \text{ if } q + \frac{q^p}{q^p+1} \leq t < q + 1, \end{cases}$$

and

$$W(t, u) = \frac{a}{s}|u|^s - \frac{\alpha(t)}{r}|u|^r, \tag{37}$$

where $q \in \mathbb{N}^*$, and s, r, a are constants satisfying $1 < r < p,\ 1 < s < \frac{p}{p+1}(r+1),\ a > 0$ and

$$\alpha(t) = \begin{cases} t \text{ if } t \in [0, \frac{1}{2}], \\ 1 - t \text{ if } t \in [\frac{1}{2}, 1], \\ 0 \text{ if } t \in \mathbb{R} \setminus [0, 1]. \end{cases}$$

Then L and W match Theorem 1.3, but not satisfying the corresponding results on the above papers. Indeed, it is clear that $L(t)$ and $W(t, u)$ satisfy (A_0) and $(W_1) - (W_2)$ respectively. It remains to check that $W(t, u)$ satisfies (W_3). For this purpose we take k_0 be a positive integer such that $4k_0 e^{-k_0} < \frac{1}{2}$ and $k \geq k_0$. We define

$$\rho_k := e^{-k},\ t_i := (2i - 1)e^{-k} \text{ with } 1 \leq i \leq 2k. \tag{38}$$

Then $(t_i - \rho_k, t_i + \rho_k)$ becomes the interval $(2(i - 1)e^{-k}, 2ie^{-k})$. Theses intervals are disjoint. We have the estimate $\alpha(t) = t \leq 4ke^{-k} < \frac{1}{2}$ for $t \in (t_i - \rho_k, t_i + \rho_k)$ with $1 \leq i \leq 2k$. The function $W(t, u)$ is computed as

$$W(t, u) \geq \frac{a}{s}|u|^s - \frac{1}{r}4ke^{-k}|u|^r.$$

Define θ as follows

$$\frac{p}{p - s} < \theta < \frac{s}{p(s - r)} + 1 \text{ when } s > r. \tag{39}$$

$$\frac{p}{p - s} < \theta \text{ when } s \leq r. \tag{40}$$

It follows from (39) and (40) and $1 < s < \frac{p(r+1)}{p+1}$ that

$$-(p - s)\theta + p < 0,\ -(p - s)\theta + p < -(p - r)\theta + p + 1. \tag{41}$$

Put $\mu_k := \rho_k^\theta = e^{-k\theta}$. Then we have

$$\overline{W}(t_i, \mu_k, \rho_k) \geq \frac{a}{s}e^{[(p-s)\theta - p]k} - \frac{4}{r}ke^{[(p-r)\theta - p - 1]k} \longrightarrow \infty,$$

as $k \to \infty$ by (41). Furthermore, we have

$$\frac{W(t, m\mu_k)\rho_k^p}{|\mu|_k^p} \geq \frac{a}{s}m^s e^{[(p-s)\theta - p]k} - \frac{4}{r}m^r ke^{[(p-r)\theta - p - 1]k} \quad \text{for } 0 \leq m \leq 1. \tag{42}$$

Let

$$\xi_k(m) := \frac{a}{s}m^s e^{[(p-s)\theta - p]k} - \frac{4}{r}m^r ke^{[(p-r)\theta - p - 1]k} \text{ for } 0 \leq m \leq 1.$$

We shall show that $\xi_k(m)$ is bounded from below by a constant independent of k and $m \in [0, 1]$. By (42), $\xi_k(1) > 0$ for $k \geq k_0$ with a large k_0. We divide the proof into two cases.

- $s > r$. Then $\xi_k(m)$ achieves a negative minimum in $[0, 1]$, which is computed as

$$\min_{0 \leq m \leq 1} \xi_k(m) = \frac{s - r}{sr}\left(ae^{[(p-s)\theta - p]k}\right)^{\frac{-r}{s-r}}\left(4ke^{[(p-r)\theta - p - 1]k}\right)^{\frac{s}{s-r}}.$$

  By (39), the minimum of ξ_k converges to zero as $k \longrightarrow \infty$.

- $s \leq r$. Since $m^s \geq m^r$, we have $\xi_k(m) \geq 0$ for $k \geq k_0$ and $m \in [0, 1]$.

By Cases 1 and 2, we have the inequality $\underline{W}(t_i, \mu_k, \rho_k) \geq -c$ for all i and $k \geq k_0$ with $c \geq 0$.

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