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A generalized mean-variance model for portfolio optimization problem in the simultaneous presence of random and uncertain returns

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Abstract. In this paper, a composite mean-variance model for portfolio optimization problems in the simultaneous presence of random and uncertain returns has been revisited and generalized. The expressions for the mean and variance of the total uncertain random return have been obtained using chance distribution. The model is flexible, as it is capable of dealing with both types of stocks: those with sufficient past records and those that are newly introduced. A generalized uncertainty distribution is defined to represent the returns of newly introduced stocks. And, the return vector of the stocks with sufficient past records is assumed to follow a multivariate normal distribution. By varying the parameter(s) involved in the generalized uncertain return distribution, different representative problems can be obtained. Thus, the model provides the scope for incorporating subjective preferences. A comparative analysis of the solutions obtained for different problems has been conducted. The most suitable one may be selected by the analysis. The method of solution of the proposed model has been illustrated by constructing a numerical example involving 30 stocks randomly selected from Bombay Stock Exchange (BSE) India, out of which 20 stocks give a random return and the remaining 10 stocks give an uncertain return. The problem has been solved using the function "fmincon" in Matlab R2018a.

1. Introduction

In 1952, Markowitz [12] did fundamental research work for modern finance theory, and then analysis of the modern portfolio selection problem started. In his model, Markowitz respectively maximized and minimized the expected value and variance of the total return. The expected value and variance of the total return were calculated, considering the returns as random variables. Such estimations require enough historical data. But in the stock market, there are some stocks that have recently entered the market, and they naturally lack sufficient past records. In this scenario, a randomly selected portfolio will be a combination of stocks, some of which have sufficient past records and others that are newly introduced to the market.

To deal with the newly introduced stocks, which lack enough historical data, researchers resort to fuzzy set theory and uncertainty theory. In this regard a comprehensive review of the literature is placed bellow:

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1.1. Literature review

Here we put forward the recent works in the field of portfolio optimization using fuzzy and uncertainty theory.

1.1.1. Works using fuzzy set theory

Considering the returns of newly listed stocks as fuzzy variables, some researchers have developed fuzzy portfolio optimization theory. Few such works are given by: Bhattacharya er al. [1] proposed multi-objective fuzzy stock portfolio selection models that maximize mean and skewness as well as minimize portfolio variance and cross-entropy. While Wang et al. [21] for the first time considered fuzzy Sharpe ratio and then utilized it in defining value-at-Risk ratio for constructing a multi-objective portfolio selection problem. Liu and Zhang [11] constructed a multi-period model where they considered the behavior, of investors, which varies with time, in respect of loss and gains earned from the investment. Nath et al. [14] employed goal programming (GP) method for solving a fuzzy model. In the portfolio model proposed by Zhang et al. [23], an investor can immediately sell out the assets which gains a preassigned price. Considering the dealings of investors, Chen et al. [2] proposed a model of portfolio where the proportions of stocks could be adjusted suitably.

1.1.2. Works using uncertainty theory

To estimate inexact quantities using expert opinions, uncertainty theory introduced by Liu [6] is used as an alternative approach. Uncertainty theory can also be used to model portfolio selection problems and find their solutions. For application of the theory in portfolio selection problem, pioneering work has been done by Qin et al. [18]. Further, Huang and Ying [3] established risk index-based models for portfolio adjustment problems with uncertain returns subject to experts' evaluation. Qin et al. [19] defined an uncertain portfolio-adjusting model using the semi-absolute deviation. Li and Zhang [4] constructed portfolio optimization model giving liquidity and entropy some preassigned lower bound. Mehlawat et al. [13] studied portfolio selection problem using multiple objectives and higher order moments and compare their finding with that of Qin [16]. Zhai et al. [22] studied portfolio optimization problem under uncertain and random environment. It was observed that, to get higher return, one has to take higher risk and skewness of the total return. Multi-period portfolio optimization problem with bankruptcy control and liquidity was considered by Li et al. [5]. These are some of the works. Thus, it is evident that portfolio selection problems, considering the stock return as an uncertain variable, are a vibrant area of research.

From the literature review, we note the following research gap:

1.2. Research gap

The zigzag uncertainty distribution [6] has three parameters, namely, $a, b,$ and c respectively, $a < b < c$. The values of the distribution function at $a, b,$ and c are fixed at $0, 1/2,$ and 1, respectively. Many researchers used this particular uncertainty distribution to represent rate of newly listed stocks. But the return distributions of the newly listed stocks are unknown. Thus, it would be interesting to see the effect of variation in the values of the distribution function assigned at a and b, in practical applications.

1.3. Motivation

The literature survey and research gap analysis motivated us to take the uncertainty return distribution to be represented by a generalized uncertainty distribution having five parameters $a, b, c; h_1, h_2$, where h_1 and h_2 are respectively the values of the distribution function at a and b, with $0 \leq h_1 < h_2 < 1$. Also apply the same to the portfolio optimization problem and compare the results obtained with those of the zigzag distribution.

Thus, in this paper, we have considered a hybrid portfolio of stocks, some of which have sufficient historical data, and the rest are newly introduced. The returns from stocks with enough historical data are considered random variables, and those with a lack of historical data are taken as uncertain variables, which follow a generalized zigzag uncertainty distribution introduced in this paper. The expected value and the variance of the total uncertain random return are calculated using chance distribution [10], in a generalized way.

1.4. Contribution of the study

The main contributions of our study can be listed as follows:

- A new uncertainty distribution has been introduced as a generalization of the zigzag uncertainty distribution, and its properties have been studied. The results generalize earlier findings obtained by using the zigzag uncertainty distribution.

- For non-zero values of h_1, the expression of the variance of the total return contains a term that represents the combined influence of randomness and uncertainty, which is a noble observation. Such joint action is not present in the previous work with zigzag uncertainty distribution. Thus, the generalized zigzag uncertainty distribution introduced in this paper is capable of reflecting the coexistence of randomness and uncertainty more aptly than the zigzag uncertainty distribution.

- Four sets of representative values of h_1 and h_2 are taken for analyzing the effect of variation. From the comparative analysis and expected value vs. variance graphs, the most suitable one is identified.

- It is shown in the form of a theorem that, the proposed method yields efficient solutions.

1.5. The reason for using uncertainty theory together with statistical approach in portfolio selection

In a portfolio selection problem, the most important variable is the return rate of a stock. Now, the stocks that have recently been introduced to the market (risky stocks) lack sufficient past records. Hence, the determination of the probability distribution of the return of such stocks is not possible. Since the determination of the parameters of a distribution requires a reasonable amount of data, statistical methods in such a scenario cannot be applied. One has to resort to fuzzy set theory or uncertainty theory to find the membership or distribution function of the return of the risky stocks, using experts' opinions. Further, fuzzy methods have some limitations in this regard. A paradox will appear when fuzzy variables are used to describe the return of the stocks, as was observed by Huang and Ying [3] . Thus, under such situations, the uncertainty theory introduced by Liu [6] becomes an appropriate tool for working. Also, uncertainty theory is established on a solid theoretical background. Hence, in our present work, we have used uncertainty theory to find the return distributions of newly listed stocks.

Our portfolio is a composite one consisting of stocks, some of which have enough historical data, and the remaining have been recently introduced to the market. Separately, we can apply statistical theory to the first category of stocks and uncertainty theory to the second. But for the composite portfolio, the only way is to use uncertainty theory. This is because uncertainty theory also permits the coexistence of randomness and uncertainty. Using chance distribution introduced by Liu [10] , one can calculate the mean and variance of the total uncertain random return of the portfolio. The same has been applied to the paper.

1.6. Organization of the paper

To fulfill our aim, the paper is organized as follows: In Section 2, we put forward some preliminaries related to our study. Next, in Section 3, a generalized zigzag uncertainty distribution, is introduced, and its properties are studied. Then in Section 4, the problem of portfolio selection is stated. In Section 5, the expected value and variance of the total uncertain random return have been obtained. In Section 6, we illustrated the procedure for solving our proposed problem. This has been done by constructing a numerical example. Next in Section 7 a comparative analysis of the result obtained with existing result has been done. Managerial implications and the perspective of the proposed methodology has been discussed in Section 8. Section 9 contains the conclusions derived from the study, and the references relevant to our study have been listed.

2. Preliminaries

Definition 2.1. [6] Let Γ be a non-empty set and $\mathcal{L}$ be a σ algebra over Γ. Each element $A \in \mathcal{L}$ is called an event. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

1. $\mathcal{M}(\Gamma) = 1$.
1. $\mathcal{M}(A_1) \leq \mathcal{M}(A_2)$ whenever $A_1 \subset A_2$.
2. $\mathcal{M}(A) + \mathcal{M}(A^c) = 1$, for any event A, $A^c = \Gamma - A$.
3. For every countable sequence of events $\{A_i\}$, we have

$$\mathcal{M}\left\{ \bigcup_{i=1}^{\infty} A_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2.2. [6] An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, that is, for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

Definition 2.3. [6] For any $x \in \mathbb{R}$, the uncertainty distribution function Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\gamma \in \Gamma \mid \xi(\gamma) \leq x\}$. Here, $\mathbb{R}$ is the set of all real numbers.

Definition 2.4. [8] An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution function given by

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \frac{x-a}{2(b-a)}, & \text{if } a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)}, & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c, \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$, where a, b, c are real numbers with $a < b < c$.

Definition 2.5. [7] The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{ \bigcap_{i=1}^{n} \{\xi_i \in B_i\} \right\} = \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\},$$

Theorem 2.1. [8] Let $\xi_1, \xi_2, ..., \xi_n$ be independent uncertain variables with regular uncertainty distribution functions $\Phi_1, \Phi_2, ..., \Phi_n$ respectively. If f is strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, ..., \xi_n)$$

has an inverse uncertainty distribution function

$$\Psi^{-1}(\delta) = f(\Phi_1^{-1}(\delta), \Phi_2^{-1}(\delta), ..., \Phi_n^{-1}(\delta)).$$

Definition 2.6. [6] The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\, dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\, dx,$$

provided that at least one of the two integrals exits finitely.

Theorem 2.2. [8] Let ξ_1 and ξ_2 be two independent uncertain variables with finite expected values $E[\xi_1]$ and $[\xi_2]$ respectively. Then for any real numbers a and b,

$$E[a\xi_1 + b\xi_2] = aE[\xi_1] + bE[\xi_2].$$

Definition 2.7. [10] An uncertain random variable is a measurable function $\tilde{\xi}$ from the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P},)$ to the set of real numbers, that is, for any Borel set B of set of real numbers R, $\{\tilde{\xi} \in B\} = \{(\gamma, \omega) : \tilde{\xi}(\gamma, \omega) \in B\} \in \mathcal{L} \times \mathcal{A}$, that is an event in $\mathcal{L} \times \mathcal{A}$, where $\gamma \in \Gamma, \omega \in \Omega$. Its chance distribution is defined by $\tilde{\Phi}(x) = ch\{\tilde{\xi} \leq x\}$, for any $x \in \mathbb{R}$,
where $\{\tilde{\xi} \leq x\} = \{(\gamma, \omega) \ \in \Theta \mid \gamma \in \Gamma, \omega \in \Omega \mid \tilde{\xi}(\gamma, \omega) \leq x\} \in \mathcal{L} \times \mathcal{A}$.

Definition 2.8. [10] Let $\tilde{\xi}$ be an uncertain random variable. Then its expected value is defined by

$$E[\tilde{\xi}] = \int_0^{+\infty} ch\{\tilde{\xi} \geq x\}\, dx \ - \int_{-\infty}^0 ch\{\tilde{\xi} \leq x\}\, dx,$$

provided at least one of the two integrals exits finitely.

Result 2.1. [10] If η is a random variable and ξ is an uncertain variable, then $\eta + \xi$ is an uncertain random variable.

Result 2.2. [9] If η is a random variable and ξ is an uncertain variable, then the expected value of the uncertain random variable $\eta + \xi$ is given by

$$E[\eta + \xi] = E[\eta] + E[\xi].$$

Definition 2.9. [10] Let $\tilde{\xi}$ be an uncertain random variable with finite expected value e. Then its variance is defined by

$$V[\tilde{\xi}] = E[(\tilde{\xi} - e)^2].$$

Theorem 2.3. [20] Let $\tilde{\xi}$ be an uncertain random variable with chance distribution $\tilde{\Phi}$ and finite expected value e. Then

$$V[\tilde{\xi}] = \int_{-\infty}^{+\infty} (x - e)^2 d\tilde{\Phi}(x).$$

3. A new uncertainty distribution and its properties

In this section, we propose a new uncertainty distribution and study its properties.

Definition 3.1. An uncertain variable ξ is called generalized zigzag if its uncertainty distribution is given by

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ h_1 + \frac{(h_2-h_1)(x-a)}{(b-a)}, & \text{if } a \leq x \leq b \\ h_2 + \frac{(1-h_2)(x-b)}{(c-b)}, & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c, \end{cases} \tag{1}$$

where a, b, c, h_1, h_2 are real numbers with $a < b < c$ and $0 \leq h_1 < h_2 < 1$ and we write $\xi \sim \mathcal{GZ}(a, b, c; h_1, h_2)$. The only point of discontinuity of Φ is at $x = a$, for $h_1 \neq 0$.

For a given pair of values of h_1 and h_2, the values of a, b, c are estimated using expert's opinion on the basis of available information.

Clearly, the newly defined uncertainty distribution is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$. So, by Theorem 3.1 of [15] and definition 1.16 of [8], the generalized zigzag uncertainty distribution is a regular uncertainty distribution. Hence, the Operational law (theorem 2.1) holds for a finite collection of independent generalized zigzag uncertain variables. The discontinuity of $\mathcal{GZ}(a, b, c; h_1, h_2)$ at $x = a$, (when $h_1 \neq 0$) does not have any effect on the regularity property or the operational law to hold.

Also a generalized zigzag uncertain variable becomes zigzag uncertain variable when $h_1 = 0$ and $h_2 = 1/2$.

The distribution function $\xi \sim \mathcal{GZ}(a, b, c; h_1, h_2)$ can be expressed in the form

$$\Phi(x) = \Big(h_1 + \frac{(h_2 - h_1)(x - a)}{(b - a)}\Big)I_{\{x:\ a\leq x\leq b\}} + \Big(h_2 + \frac{(1 - h_2)(x - b)}{(c - b)}\Big)I_{\{x:\ b\leq x\leq c\}} + I_{\{x:\ x\geq c\}}, \tag{2}$$

where $I_{\{.\}}$ is the indicator function of the set $\{.\}$.

Next we find the inverse uncertainty distribution function for $\mathcal{GZ}(a, b, c; h_1, h_2)$, at an uncertainty level δ, $0 < \delta < 1$ and put in the form of a lemma. The following lemma can easily be proved.

Lemma 3.1. The inverse uncertainty distribution of the generalized zigzag uncertain variable $\mathcal{GZ}(a, b, c; h_1, h_2)$ is given by

$$\Phi^{-1}(\delta) = \begin{cases} \frac{(h_2-\delta)a+(\delta-h_1)b}{h_2-h_1}, & \text{if } 0 < h_1 \leq \delta < h_2 \\ \frac{(1-\delta)b+(\delta-h_2)c}{1-h_2}, & \text{if } h_2 \leq \delta < 1. \end{cases}$$

Now we have the following results regarding the sum and scalar multiplication operation of generalized zigzag uncertain variables. Using the lemma 3.1, one can prove that:

Theorem 3.1. Let $\xi_1 \sim \mathcal{GZ}(a_1, b_1, c_1; h_1, h_2)$ and $\xi_2 \sim \mathcal{GZ}(a_2, b_2, c_2; h_1, h_2)$ are independent then $\xi_1 + \xi_2 \sim \mathcal{GZ}(a_1 + a_2, b_1 + b_2, c_1 + c_2; h_1, h_2)$. And if $\xi \sim \mathcal{GZ}(a, b, c; h_1, h_2)$ then for a scalar $w > 0$, $w\xi \sim \mathcal{GZ}(wa, wb, wc; h_1, h_2)$.

Corollary 3.1. If $\beta_j \sim \mathcal{GZ}(a_j, b_j, c_j; h_1, h_2)$, for $j = 1, 2, ..., n$, are independent uncertain variables, then for the scalars $d_j \geq 0$, with $\sum_{j=1}^{n} d_j > 0$, $\Sigma d_j\beta_j \sim \mathcal{GZ}(\Sigma d_j a_j, \Sigma d_j b_j, \Sigma d_j c_j; h_1, h_2)$.

The corollary is a direct consequence of theorem 3.1.

4. Problem statement

Here we find an optimal portfolio in a scenario where some of the stocks give random returns and for the rest it is uncertain. For the purpose of stating our proposed problem of portfolio selection, we adopt the following notations and symbols for convenience:

X =collection of stocks having enough historical data.
Y =collection of stocks that have recently entered the market and are lacking sufficient historical data.
$m =| X |$, the cardinality of the set X.
$n =| Y |$, the cardinality of the set Y.
α_i denotes the random return of one unit of the i^{th} stock belonging to the set X, for $i = 1, 2, ...m$.

$\kappa_i = E[\alpha_i]$, that is, the expected value of random variable α_i, for $i = 1, 2, ...m$.
σ_{ij} is the covariance of random variables α_i and α_j, for $i, j = 1, 2, ..., m$.
$\sigma_{ii}^2 \equiv \sigma_i^2$ is the variance of random variables α_i, for $i = 1, 2, ..., m$.
$\mathbf{\Sigma} = (\sigma_{ij})_{m\times m}$ = the covariance matrix for the random variables $\alpha_1, \alpha_2, ..., \alpha_m$.
Ψ_i is the probability distribution of the random variable α_i, for $i = 1, 2, ..., m$.
β_j denotes the uncertain return of one unit of the j^{th} stock belonging to the set Y, for $j = 1, 2, ..., n$.
$\rho_j = E[\beta_j]$, that is, the expected value of the uncertain variable β_j, for $j = 1, 2, ..., n$.
τ_j^2 is the variance of the uncertain variable β_j, for $j = 1, 2, ..., n$.
ζ_j denotes the uncertainty distribution of the uncertain variable β_j, for $j = 1, 2, ..., n$.
x_i denotes the proportion of the total budget invested in the i^{th} stock belonging to set X, for $i = 1, 2, ...m$.
y_j denotes the proportion of the total budget invested in the j^{th} stock belonging to set Y, for $j = 1, 2, ..., n$.
$\tilde{\Phi}$ is the chance distribution of the uncertain random variable $\sum_{i=1}^{m} x_i\alpha_i + \sum_{j=1}^{n} y_j\beta_j$ (result 2.1).
The following notations for the frequently accruing matrices will be used throughout with out any specific reference

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}, \boldsymbol{\kappa} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_m \end{pmatrix}, \boldsymbol{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{pmatrix},$$

$$\boldsymbol{\sigma}^2 = \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_m^2 \end{pmatrix}, \boldsymbol{\tau}^2 = \begin{pmatrix} \tau_1^2 \\ \tau_2^2 \\ \vdots \\ \tau_n^2 \end{pmatrix}.$$

4.1. Construction of the objectives and constraints

Here, we form the objectives and constraints for our proposed portfolio optimization problem.

4.1.1. Formation of objectives

The, the total return earned from the investment is $x_1\alpha_1 + x_2\alpha_2 + ... + x_m\alpha_m + y_1\beta_1 + y_2\beta_2 + ... + y_n\beta_n = \boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta} = r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})$. Here, $\boldsymbol{x}'\boldsymbol{\alpha} = x_1\alpha_1 + x_2\alpha_2 + ... + x_m\alpha_m$ and $\boldsymbol{y}'\boldsymbol{\beta} = y_1\beta_1 + y_2\beta_2 + ... + y_n\beta_n$ are respectively random and uncertain variables.

It follows from result 2.1 and 2.2, that $r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an uncertain random variable and $E[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = E[\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}] = E[\boldsymbol{x}'\boldsymbol{\alpha}] + E[\boldsymbol{y}'\boldsymbol{\beta}]$. Also the variance of the total return is given by, $V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = V[\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}]$.

In any portfolio optimization problem, the primary aim of an investor is to maximize the total expected return and minimize the total variance, representing the risk in the investment. Thus, we have set the following two objectives in this paper.

- Maximization of expected value of total return: Our first objective is to maximize the total return $E[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})]$ earned from the investment.

- Minimization of variance of total return: Also we wish to minimize the investment risk measured by the total variance $V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = V[\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}]$.

4.1.2. Formation of constraint

- Total budget constraint: According to this constraint the sum of the proportions of wealth invested in different stocks must be equal to 1. Thus one constraint is $x_1 + x_2 + ... + x_m + y_1 + y_2 + ...y_n = 1$, where $x_i, y_j \geq 0$, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

- Upper and lower bounds constraint: For the sake of reducing the risk of investment in the portfolio, we put restrictions on the upper and lower bounds of the variables x_i, y_j. Let $l_i \leq x_i \leq u_i$ and $l_{m+j} \leq y_j \leq u_{m+j}$, to reduce the risk. The values of these bounds are decided by the decision-maker.

Thus, the portfolio selection problem under the simultaneous presence of random and uncertain returns, is expressed as

$$\begin{cases} \text{Max} \quad E[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})] = E[\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}] \\ \text{Min} \quad V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})] = V[\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}] \\ \text{subject to, } \boldsymbol{I}_m'\boldsymbol{x} + \boldsymbol{I}_n'\boldsymbol{y} = 1 \\ \qquad \boldsymbol{L_1} \leq \boldsymbol{x} \leq \boldsymbol{U_1} \\ \qquad \boldsymbol{L_2} \leq \boldsymbol{y} \leq \boldsymbol{U_2} \\ \qquad \boldsymbol{x},\boldsymbol{y} \geq 0, \end{cases} \tag{3}$$

where

$$\boldsymbol{I_m} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \ \boldsymbol{I_n} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \ \boldsymbol{L_1} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix}, \ \boldsymbol{L_2} = \begin{pmatrix} l_{m+1} \\ l_{m+2} \\ \vdots \\ l_{m+n} \end{pmatrix}, \ \boldsymbol{U_1} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \ \boldsymbol{U_2} = \begin{pmatrix} u_{m+1} \\ u_{m+2} \\ \vdots \\ u_{m+n} \end{pmatrix}$$

are column matrices. Here, $\boldsymbol{I_m}$ and $\boldsymbol{I_n}$ are of orders $m \times 1$ and $n \times 1$ respectively.

4.2. *Distribution of random and uncertain returns*

In problem (3), the distribution of the random and uncertain variables involved has not been specified. Now we assume that $\beta_j \sim \mathcal{GZ}(a_j, b_j, c_j; h_1, h_2), \quad j = 1, 2, ..., n$ are independent. We further assume that the random vector $\boldsymbol{\alpha}$ follows multivariate normal distribution having density

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{z}) = \frac{1}{\sqrt{(2\pi)^m \mid \boldsymbol{\Sigma} \mid}} \exp\Big(-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{\kappa})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{z}-\boldsymbol{\kappa})\Big), \ \forall \boldsymbol{z} \in \mathbb{R}^m, \ \mid \boldsymbol{\Sigma} \mid \neq 0.$$

Under our assumptions $\boldsymbol{x}'\boldsymbol{\alpha}$ follows normal distribution with mean $x_1\kappa_1 + x_2\kappa_2 + ... + x_m\kappa_m = \boldsymbol{x}'\boldsymbol{\kappa}$ and variance $\Sigma_{i=1}^m \Sigma_{j=1}^m x_i x_j \sigma_{ij} = \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}$. The density function of the normal variable $\boldsymbol{x}'\boldsymbol{\alpha}$ is given by

$$\psi(s) = \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \exp\Big(-\frac{(s - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\Big), \tag{4}$$

where $\sigma(\boldsymbol{x}) = \sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}$ and $\sigma^2(\boldsymbol{x}) = (\sigma(\boldsymbol{x}))^2$.

Also it follows from theorem 3.1,

$$\boldsymbol{y}'\boldsymbol{\beta} = \sum_{j=1}^n y_j\beta_j \sim \mathcal{GZ}(\sum_{j=1}^n y_j a_j, \sum_{j=1}^n y_j b_j, \sum_{j=1}^n y_j c_j; \ h_1, h_2). \tag{5}$$

4.2.1. *Formulae for expected value and variance of the total uncertain random return*

Here we put forward the formulae for calculating the expected value and variance of the total return $r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})$.

Using the independence of β_j, for $j = 1, 2, ...n$, we have $E[\boldsymbol{y}^{'}\boldsymbol{\beta}] = E[\sum_{j=1} y_j\beta_j] = \sum_{j=1}^{n} y_j E[\beta_j] = \sum_{j=1}^{n} y_j\rho_j = \boldsymbol{y}^{'}\boldsymbol{\rho}$ (theorem 2.2). Therefore, from result 2.2, we have

$$E[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})] = E[\boldsymbol{x}^{'}\boldsymbol{\alpha} + \boldsymbol{y}^{'}\boldsymbol{\beta}] = E[\boldsymbol{x}^{'}\boldsymbol{\alpha}] + E[\boldsymbol{y}^{'}\boldsymbol{\beta}] = \boldsymbol{x}^{'}\boldsymbol{\kappa} + \boldsymbol{y}^{'}\boldsymbol{\rho} \tag{6}$$

Thus, the variance of uncertain random variable $r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})$ is given by, Sheng and Yao [20]

$$V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})] = \int_{-\infty}^{+\infty} [t - (\boldsymbol{x}^{'}\boldsymbol{\kappa} + \boldsymbol{y}^{'}\boldsymbol{\rho})]^2 d\tilde{\Phi}(t), \tag{7}$$

where the chance distribution $\tilde{\Phi}(t)$ of $r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})$ is given by

$$\tilde{\Phi}(t) = \int_{-\infty}^{+\infty} \zeta(t-s)d\Psi(s), \tag{8}$$

where $d\Psi(s) = \psi(s)ds$.
In the above expression $\zeta(.)$ and $\Psi(.)$ denotes respectively the uncertainty distribution and probability distribution of the variables $\boldsymbol{y}^{'}\boldsymbol{\beta}$ and $\boldsymbol{x}^{'}\boldsymbol{\alpha}$.

5. Calculation of total expected value and variance of the uncertain random return

Here our aim is to find $E[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})]$ and $V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta})]$ in terms of the parameters of the distributions and holding proportions. For this first we calculate the expected value of uncertain variable $\beta \sim \mathcal{GZ}(a,b,c;h_1,h_2)$, which is placed in the form of the following theorem.

Theorem 5.1. The Generalized zigzag uncertain variable $\beta \sim \mathcal{GZ}(a,b,c;h_1,h_2)$ has expected value

$$E[\beta] = \frac{1}{2}\Big[(h_1+h_2)a + (1-h_1)b + (1-h_2)c\Big]. \tag{9}$$

Proof: We know that, the uncertainty distribution ζ of an uncertain variable β having generalized zigzag uncertainty distribution is given by

$$\zeta(x) = \begin{cases} 0, & \text{if } x < a, \\ h_1 + \frac{(h_2-h_1)(x-a)}{(b-a)}, & \text{if } a \le x \le b, \\ h_2 + \frac{(1-h_2)(x-b)}{(c-b)}, & \text{if } b \le x \le c, \\ 1, & \text{if } x \ge c. \end{cases}$$

Hence, by the definition, the expected value of an uncertain variable β is given by,

$$\begin{aligned} E[\beta] &= \int_0^{+\infty} \mathcal{M}\{\beta \ge x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\beta \le x\}dx \\ &= \int_0^{+\infty} (1-\zeta(x))dx - \int_{-\infty}^{0} \zeta(x)dx, \\ &= \int_0^{a} 0\,dx + \int_a^b \{1 - (h_1 + \frac{(x-a)(h_2-h_1)}{(b-a)})\}dx \\ &+ \int_b^c \{1-(h_2 + \frac{(1-h_2)(x-b)}{(c-b)})\}dx + \int_c^{+\infty} (1-1)\,dx \\ &= \frac{1}{2}\Big[(h_1+h_2)a + (1-h_1)b + (1-h_2)c\Big] \end{aligned}$$

Hence the theorem.

Corollary 5.1. The expected value of the generalized zigzag uncertain variable $\sum_{j=1}^{n} y_j\beta_j$, where $\beta_j \sim \mathcal{GZ}(a_j, b_j, c_j; h_1, h_2)$ and $y_j \geq 0$ is

$$E[\sum_{j=1}^{n} y_j\beta_j] = \frac{1}{2}\Big[(h_1+h_2)\sum_{j=1}^{n} y_j a_j + (1-h_1)\sum_{j=1}^{n} y_j b_j + (1-h_2)\sum_{j=1}^{n} y_j c_j\Big]. \tag{10}$$

The above result (10) can further be expressed as

$$E[\boldsymbol{y}'\boldsymbol{\beta}] = \frac{1}{2}\Big[(h_1+h_2)\boldsymbol{y}'\boldsymbol{a} + (1-h_1)\boldsymbol{y}'\boldsymbol{b} + (1-h_2)\boldsymbol{y}'\boldsymbol{c}\Big], \tag{11}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},\ \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \text{ are the column matrices.}$$

Similarly, if $\alpha_i,\ i = 1, 2, ..., m$ are random variables and $x_1, x_2, ..., x_n$ are constants then $\sum_{i=1}^{m} x_i\alpha_i$ is also a random variable and $E[\sum_{i=1}^{m} x_i\alpha_i] = \sum_{i=1}^{m} x_i E[\alpha_i]$.

Thus in matrix notation,

$$E[\boldsymbol{x}'\boldsymbol{\alpha}] = \boldsymbol{x}'E[\boldsymbol{\alpha}] = \boldsymbol{x}'\boldsymbol{\kappa} \tag{12}$$

Therefore the total return of the uncertain random variable $r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is

$$E[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = \boldsymbol{x}'\boldsymbol{\kappa} + \frac{1}{2}\Big[(h_1+h_2)\boldsymbol{y}'\boldsymbol{a} + (1-h_1)\boldsymbol{y}'\boldsymbol{b} + (1-h_2)\boldsymbol{y}'\boldsymbol{c}\Big]. \tag{13}$$

Now we are in a position to find variance of $r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Theorem 5.2. If $\beta_j \sim \mathcal{GZ}(a_j, b_j, c_j; h_1, h_2)$ are generalized zigzag uncertain variables, for $j = 1, 2, ..., n$ and α_i's are random variables, for $i = 1, 2, ..., m$, then the variance of the chance distribution of the uncertain random variable $\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}$ is given by

$$\begin{aligned} &V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = V[\boldsymbol{x}'\boldsymbol{\alpha} + \boldsymbol{y}'\boldsymbol{\beta}] \\ &= \sigma^2(\boldsymbol{x}) + \frac{1}{12}\Big[4(h_2-h_1)(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})^2 + 4(1-h_2)(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})^2 \\ &+ 3\Big((1+h_1)\boldsymbol{y}'\boldsymbol{b} - (h_1+h_2)\boldsymbol{y}'\boldsymbol{a} - (1-h_2)\boldsymbol{y}'\boldsymbol{c}\Big) \\ &\times\Big((h_2 - 3h_1 + h_1^2 + h_1h_2)\boldsymbol{y}'\boldsymbol{a} - (1-h_1)^2\boldsymbol{y}'\boldsymbol{b} + (1-h_2)(1+h_1)\boldsymbol{y}'\boldsymbol{c}\Big)\Big] \\ &+ \frac{\sqrt{2}}{4}h_1\sigma(\boldsymbol{x})\Big[(h_1+h_2-2)\boldsymbol{y}'\boldsymbol{a} + (1-h_1)\boldsymbol{y}'\boldsymbol{b} + (1-h_2)\boldsymbol{y}'\boldsymbol{c}\Big]^2. \end{aligned} \tag{14}$$

Proof: We know that $\boldsymbol{y}'\boldsymbol{\beta}$ is a generalized zigzag uncertain variable having expected value $\frac{1}{2}[(h_1+h_2)\boldsymbol{y}'\boldsymbol{a} + (1-h_1)\boldsymbol{y}'\boldsymbol{b} + (1-h_2)\boldsymbol{y}'\boldsymbol{c}]$ (equation (11)).

We have the following uncertainty distribution of $u = \boldsymbol{y}'\boldsymbol{\beta}$, using corollary 3.1 and equation (2)

$$\zeta(u) = \left(h_1 + \frac{(h_2 - h_1)(u - \boldsymbol{y}'\boldsymbol{a})}{(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})}\right) I_{\{u:\ \boldsymbol{y}'\boldsymbol{a} \le u \le \boldsymbol{y}'\boldsymbol{b}\}} + \left(h_2 + \frac{(1 - h_2)(u - \boldsymbol{y}'\boldsymbol{b})}{(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{b}'\boldsymbol{y})}\right) I_{\{u:\ \boldsymbol{y}'\boldsymbol{b} \le u \le \boldsymbol{y}'\boldsymbol{c}\}} + I_{\{u:\ u \ge \boldsymbol{y}'\boldsymbol{c}\}}.$$

From equation(4) representing the density of the normal random variable $\boldsymbol{x}'\boldsymbol{\alpha}$ and (8) giving the distribution function of the uncertain random variable we have

$$\begin{aligned}
\tilde{\Phi}(t) &= \int_{-\infty}^{+\infty} \Bigg[\left(h_1 + \frac{(h_2 - h_1)(t - s - \boldsymbol{y}'\boldsymbol{a})}{(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})}\right) I_{\{\boldsymbol{y}'\boldsymbol{a} \le t - s \le \boldsymbol{y}'\boldsymbol{b}\}} \\
&+ \left(h_2 + \frac{(1 - h_2)(t - s - \boldsymbol{y}'\boldsymbol{b})}{(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})}\right) I_{\{\boldsymbol{y}'\boldsymbol{b} \le t - s \le \boldsymbol{y}'\boldsymbol{c}\}} + I_{\{t - s \ge \boldsymbol{y}'\boldsymbol{c}\}}\Bigg] \psi(s) ds \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{t - \boldsymbol{y}'\boldsymbol{b}}^{t - \boldsymbol{y}'\boldsymbol{a}} \left(h_1 + \frac{(h_2 - h_1)(t - s - \boldsymbol{y}'\boldsymbol{a})}{(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})}\right) \exp\left(-\frac{(s - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) ds \\
&+ \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{t - \boldsymbol{y}'\boldsymbol{c}}^{t - \boldsymbol{y}'\boldsymbol{b}} \left(h_2 + \frac{(1 - h_2)(t - s - \boldsymbol{y}'\boldsymbol{b})}{(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})}\right) \exp\left(-\frac{(s - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) ds \\
&+ \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{-\infty}^{t - \boldsymbol{y}'\boldsymbol{c}} \exp\left(-\frac{(s - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) ds.
\end{aligned}$$

Differentiating both side of the above equation with respect to t, and then using (4) and (7), the total variance is given by,

$$\begin{aligned}
&V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = \\
&\int_{-\infty}^{+\infty} \left[t - \left(\boldsymbol{\kappa}'\boldsymbol{x} + \frac{1}{2}[(h_1 + h_2)\boldsymbol{y}'\boldsymbol{a} + (1 - h_1)\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)\boldsymbol{y}'\boldsymbol{c}]\right)\right]^2 d\tilde{\Phi}(t) \\
&= \frac{h_2 - h_1}{\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})} \int_{-\infty}^{+\infty} \exp\left(-\frac{(s - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) \\
&\times \left(\int_{s + \boldsymbol{y}'\boldsymbol{a}}^{s + \boldsymbol{y}'\boldsymbol{b}} \left[t - \left(\boldsymbol{\kappa}'\boldsymbol{x} + \frac{1}{2}[(h_1 + h_2)\boldsymbol{y}'\boldsymbol{a} + (1 - h_1)\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)\boldsymbol{y}'\boldsymbol{c}]\right)\right]^2 dt\right) ds \\
&+ \frac{1 - h_2}{\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})} \int_{-\infty}^{+\infty} \exp\left(-\frac{(s - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) \\
&\times \left(\int_{s + \boldsymbol{y}'\boldsymbol{b}}^{s + \boldsymbol{y}'\boldsymbol{c}} \left[t - \left(\boldsymbol{\kappa}'\boldsymbol{x} + \frac{1}{2}[(h_1 + h_2)\boldsymbol{y}'\boldsymbol{a} + (1 - h_1)\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)\boldsymbol{y}'\boldsymbol{c}]\right)\right]^2 dt\right) ds \\
&+ \frac{h_1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{-\infty}^{+\infty} \exp\left(-\frac{(t - \boldsymbol{y}'\boldsymbol{a} - \boldsymbol{\kappa}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) \\
&\qquad \times \left[t - \left(\boldsymbol{\kappa}'\boldsymbol{x} + \frac{1}{2}[(h_1 + h_2)\boldsymbol{y}'\boldsymbol{a} + (1 - h_1)\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)\boldsymbol{y}'\boldsymbol{c}]\right)\right]^2 dt
\end{aligned}$$

Finally, changing the order of integration, we get the desired result placed in (14).

For specific values of h_1, h_2, for example, if $h_1 = 0, h_2 = 1/2$ then the expression for total variance is given by

$$V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta})] = \sigma^2(\boldsymbol{x}) + \frac{5(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})^2 + 5(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})^2 + 6(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})}{48}$$

the same was obtained by Qin [16].

5.1. The portfolio selection problem for multivariate normal and generalized zigzag uncertainty distributions

Having calculated the expected value (EV) and variance of the total uncertain random return using chance distribution in (13) and (14), respectively we are now in a position to recast our portfolio selection problem (3) as follows:

$$\begin{cases} \text{Max } z_1 = \boldsymbol{x}'\boldsymbol{\kappa} + \frac{1}{2}\Big[(h_1 + h_2)\boldsymbol{y}'\boldsymbol{a} + (1 - h_1)\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)\boldsymbol{y}'\boldsymbol{c}\Big] \\ \text{Min } z_2 = \boldsymbol{x}'\Sigma\boldsymbol{x} + \frac{1}{12}\Big[4(h_2 - h_1)(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})^2 + 4(1 - h_2)(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})^2 \\ \quad +3\Big(\big((1 + h_1)\boldsymbol{y}'\boldsymbol{b} - (h_1 + h_2)\boldsymbol{y}'\boldsymbol{a} - (1 - h_2)\boldsymbol{y}'\boldsymbol{c}\big) \\ \quad \times\big((h_2 - 3h_1 + h_1^2 + h_1h_2)\boldsymbol{y}'\boldsymbol{a} - (1 - h_1)^2\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)(1 + h_1)\boldsymbol{y}'\boldsymbol{c}\big)\Big)\Big] \\ \quad + \sqrt{2\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}\ \frac{h_1}{4}\Big[(h_1 + h_2 - 2)\boldsymbol{y}'\boldsymbol{a} + (1 - h_1)\boldsymbol{y}'\boldsymbol{b} + (1 - h_2)\boldsymbol{y}'\boldsymbol{c}\Big]^2 \\ \text{subject to, } \boldsymbol{I}_m'\boldsymbol{x} + \boldsymbol{I}_n'\boldsymbol{y} = 1 \\ \qquad \boldsymbol{L_1} \le \boldsymbol{x} \le \boldsymbol{U_1} \\ \qquad \boldsymbol{L_2} \le \boldsymbol{y} \le \boldsymbol{U_2} \\ \qquad \boldsymbol{x}, \boldsymbol{y} \ge 0. \end{cases} \tag{15}$$

This model (15) will be used to illustrate the solution procedure of the proposed portfolio selection problem in the subsequent section.

We recall here that $\boldsymbol{\kappa}' = (\kappa_1\ \kappa_2\ ...\ \kappa_m)$, where $\kappa_i = E[\alpha_i]$, $(\alpha_1\ \alpha_2\ ...\ \alpha_m) = \boldsymbol{\alpha}'$ and $\boldsymbol{\rho}' = (\rho_1\ \rho_2\ ...\ \rho_m)$, where $\rho_i = E[\beta_i]$, $(\beta_1\ \beta_2\ ...\ \beta_m) = \boldsymbol{\beta}'$.

For the solution of the model (15), we assume the following:
(i) The covariance matrix $\boldsymbol{\Sigma}$ is positive definite.
(ii) $\boldsymbol{\kappa}$ is not a multiple of $\boldsymbol{I}_m'$.
These assumptions ensures non-degeneracy of the optimal solution and it also prevents the situation where all the existing stocks are risky.

Model (15) is a multi-objective non-linear programming problem. Next our endeavor is to find efficient or non-dominating or Pareto optimal solutions of model (15), so that decision maker can choose the particular one or ones best suited for him or her. The definition of an efficient solution is recalled.

Definition 5.1. Let F be the set of all feasible solutions of the multi-objective portfolio optimization problem (15). A feasible solution $\boldsymbol{x}^* \in F$ is said to be efficient if there exists no $\boldsymbol{x} \in F$ such that $z_1(\boldsymbol{x}^*) \le z_1(\boldsymbol{x})$ and $z_2(\boldsymbol{x}^*) \ge z_2(\boldsymbol{x})$ and for at least one of the z_1, z_2 either $z_1(\boldsymbol{x}^*) < z_1(\boldsymbol{x})$ or $z_2(\boldsymbol{x}^*) > z_2(\boldsymbol{x})$ is satisfied.

In our work we have considered 4 pairs of values of (h_1, h_2) namely $(0, 1/2)$, $(0, 1/4)$, $(0, 3/4)$, and $(1/3, 2/3)$. For each of these pairs, the values of a, b, c have been estimated using an experienced expert's opinion for each of the 10 newly introduced stocks assumed to give uncertain return.

6. Illustration of the proposed method of solution of the portfolio selection problem by a real numerical example

For solving the portfolio selection problem (15), we require the values of $\boldsymbol{\kappa} = E[\boldsymbol{\alpha}]$, $\boldsymbol{\rho} = E[\boldsymbol{\beta}]$, and the covariance matrix $\boldsymbol{\Sigma}$ of random returns. For this, we consider a case study involving 30 randomly chosen stocks, of which 20 have enough historical data and the remaining 10 were recently introduced to the market. These stocks were selected from the Bombay Stock Exchange (BSE), India.

6.1. Expected values of random and uncertain returns, covariance matrix of the random returns

20 randomly chosen stocks with enough historical data were taken during the period from 1^{st} January 2014 to 31^{st} August 2023. The average returns per month of the selected stocks are shown in Table 1.

Table 1: BSE codes and the sample mean of the monthly returns of the stocks

Sl. No.(i)	1	2	3	4	5	6	7	8
Code	500112	532215	500087	500257	531335	524804	524715	500570
Mean	1.88	1.82	1.27	0.63	1.37	1.66	0.92	1.47
Sl. No.(i)	9	10	11	12	13	14	15	16
Code	532939	500520	532555	532898	533098	500470	500113	500790
Mean	1.56	1.58	1.09	1.58	1.07	2.66	1.35	1.49
Sl. No.(i)	17	18	19	20				
Name	500295	500330	500312	513599				
Mean	1.35	2.6	0.51	2.18				

Then $\boldsymbol{\kappa}' = (1.88\ 1.82\ ...\ 2.18)$.

Next we find the covariance matrix, $\boldsymbol{\Sigma} = (\sigma_{ij})_{20\times 20}$ in Table 2. The principal diagonal of $\boldsymbol{\Sigma}$ gives $\boldsymbol{\sigma}^2$, the vector of the variances of the 20 stocks.

In Table 3, we displayed the values of a, b, and c for each stock. The expected values of the stocks for the corresponding chosen return distribution are also shown. For example, for $h_1 = 0$ and $h_2 = 1/2$, $\boldsymbol{\rho} = (0.5\ 1.25\ ...\ 1.25)$.

6.2. Illustration of the proposed method of portfolio selection problem (15)

Problem (15) is a bi-objective one. For the solution of it, we first calculate the ideal values of the objectives z_1 and z_2. Let, $z_1^* = z_1(\boldsymbol{x_1}^*)$ and $z_2^* = z_2(\boldsymbol{x_2}^*)$ respectively denote the ideal values (max/min values of the respective objectives subject to the constraints). $\boldsymbol{x_1}^*$ and $\boldsymbol{x_2}^*$ are called the ideal solutions. Then using the ideal solutions, we obtained the pessimistic values of the objectives as follows: $\hat{z}_1 = \min\{z_1(\boldsymbol{x_1}^*), z_1(\boldsymbol{x_2}^*)\} = z_1(\boldsymbol{x_2}^*)$ and $\hat{z}_2 = \max\{z_2(\boldsymbol{x_1}^*), z_2(\boldsymbol{x_2}^*)\} = z_2(\boldsymbol{x_1}^*)$. These values are shown in Table 4.

Table 2: The sample covariance matrix of the monthly returns of the stocks having random return

111.8	63.4	0.2	-0.5	9.1	12.1	4.1	71.6	96.9	36.7	40.4	33.2	37.2	52.6	86.7	8.9	63.2	61.8	54.4	120.5
63.4	84.3	6.2	9.6	14	26.7	14.1	66	76	38.5	31.6	23.4	29	45.5	58.9	14.3	48.3	41.7	40	62
0.2	6.2	62	35.8	16.1	39.4	36.7	26.4	44.5	14.6	9	3.3	1.9	15.9	35.5	7.4	25.2	11.1	14.8	16.8
-0.5	9.6	35.8	83.2	11.3	45.4	45.5	36.8	58.9	20.7	1.9	-3.1	-3.4	9.8	32.3	6.6	26.4	16.6	13.2	25.6
9.1	14	16.1	11.3	63.2	25	14.8	11	10.3	11.7	6.1	7.4	9.7	6	23.9	14.3	15	-5.5	7.9	11.5
12.1	26.7	39.4	45.4	25	112.3	41.3	30.6	53.3	23.3	14.1	9	9.6	30.2	54.4	13.6	42.5	13.5	20.7	27.8
4.1	14.1	36.7	45.5	14.8	41.3	70.1	28.4	37.5	13.2	6.3	-0.1	6	26.4	45.9	7.5	28.7	13.9	17	29
71.6	66	26.4	36.8	11	30.6	28.4	208.6	115.6	67.4	36.1	24.2	39.9	65.5	82	8.9	81.5	59.5	69.2	111.3
96.9	76	44.5	58.9	10.3	53.3	37.5	115.6	603.8	65.8	70.9	51.1	73.3	60.5	108.4	28.4	108.8	114.8	67.1	156
36.7	38.5	14.6	20.7	11.7	23.3	13.2	67.4	65.8	80.3	22.8	16.9	21.6	23.8	43.6	10.8	42.4	42	33	66.6
40.4	31.6	9	1.9	6.1	14.1	6.3	36.1	70.9	22.8	61.6	29.7	35	30.9	58.3	6.2	55.9	30.8	41.7	71.3
33.2	23.4	3.3	-3.1	7.4	9	-0.1	24.2	51.1	16.9	29.7	34.4	23.4	19.9	35.3	6.1	35.2	25.9	25.4	49.9
37.2	29	1.9	-3.4	9.7	9.6	6	39.9	73.3	21.6	35	23.4	64.7	26.3	44	15	47.9	31.6	24.4	51.2
52.6	45.5	15.9	9.8	6	30.2	26.4	65.5	60.5	23.8	30.9	19.9	26.3	146.6	117.7	3.3	97.1	30.6	46.1	96.8
86.7	58.9	35.5	32.3	23.9	54.4	45.9	82	108.4	43.6	58.3	35.3	44	117.7	222.6	9.8	131.7	56	77.8	183.4
8.9	14.3	7.4	6.6	14.3	13.6	7.5	8.9	28.4	10.8	6.2	6.1	15	3.3	9.8	31.7	9.1	10.2	3.3	4.5
63.2	48.3	25.2	26.4	15	42.5	28.7	81.5	108.8	42.4	55.9	35.2	47.9	97.1	131.7	9.1	191	59.9	63.2	141.2
61.8	41.7	11.1	16.6	-5.5	13.5	13.9	59.5	114.8	42	30.8	25.9	31.6	3.6	56	10.2	59.9	155.9	45.7	74.7
54.4	40	14.8	13.2	7.9	20.7	17	69.2	67.1	33	41.7	25.4	24.4	46.1	77.8	3.3	63.2	45.7	79.4	89.1
120.5	62	16.8	25.6	11.5	27.8	29	111.3	156	66.6	71.3	49.9	51.2	96.8	183.4	4.5	141.2	74.7	89.1	370.6

Table 3: Uncertain returns and expected value(EV)s of newly listed stocks.

		$h_1 = 0,\ h_2 = 1/2$		$h_1 = 0,\ h_2 = 1/4$		$h_1 = 0,\ h_2 = 3/4$		$h_1 = 1/3,\ h_2 = 2/3$	
Sl. No. (j)	Code	(a, b, c)	EV	(a, b, c)	EV	(a, b, c)	EV	(a, b, c)	EV
1	543401	(-14,1,14)	0.5	(-21,-1,11)	1	(-10,3,26)	1	(-10,3,32)	1.33
2	543463	(-12,2,13)	1.25	(-12,1,6)	1.25	(-13,2,37)	0.75	(-12,2,25)	-1.17
3	543725	(-15,-2,13)	-1.75	(-24,-3,8)	-1.13	(-11,0,25)	-1	(-13,0,38)	-0.17
4	543712	(-16,2,18)	1.5	(-20,0,10)	1.25	(-10,1,32)	0.75	(-9,2,29)	1
5	543689	(-12,2,14)	1.5	(-19,1,8)	1.13	(-13,0,42)	0.38	(-12,4,24)	-0.67
6	543529	(-30,0,25)	-1.25	(-29,-2,12)	-0.13	(-14,0.5,41)	0.13	(-15,3.5,40)	0.33
7	543523	(-20,0,25)	1.25	(-20,0,8)	0.5	(-9,0,31)	0.5	(-11,1.5,33)	0.5
8	543532	(-17,1.5,20)	1.5	(-25,0.5,11)	1.25	(-12,3,30)	0.75	(-13,3,34)	0.17
9	543534	(-14,-1,10)	-1.5	(-24,-3,9)	-1.13	(-10,-3,33)	-1.13	(-14,2,41)	0.5
10	543533	(-18,3,17)	1.25	(-27,2,9)	1	(-12,2,37)	1.13	(-12,3,31)	0.17

Table 4: Ideal and pessimistic values of objectives for different distributions

Case	z_1^*	z_2^*	$\hat{z}_1$
$h_1 = 0,\ h_2 = 1/2$	2.636	11.5175	0.7074
$h_1 = 0,\ h_2 = 1/4$	2.636	6.2185	1.31
$h_1 = 0,\ h_2 = 3/4$	2.636	12.7558	1.0214
$h_1 = 1/3,\ h_2 = 2/3$	2.636	14.3428	1.3778

The bi-objective problem (15) is converted into a single objective one by keeping the total expected value to a given level λ and minimizing the total variance. The assigned value of λ is varied between the pessimistic value $\hat{z}_1$ and the ideal value z_1^* of the total return maximizing objective z_1 in steps of $\Delta\lambda = 0.1$. The model is then solved for each pair of values of h_1 and h_2. Observing the variation of the values of h_1 and h_2, we choose the best one. Thus, we have the following model to solve (15):

$$\begin{cases} \quad\quad \text{Min } \ z_2 \\ \text{subject to, } \ z_1 = \lambda \\ \quad\quad \lambda \in (\hat{z_1}, z_1^*) \\ \quad\quad \boldsymbol{I}_{20}^{'}\boldsymbol{x} + \boldsymbol{I}_5^{'}\boldsymbol{y} = 1 \\ \quad\quad \boldsymbol{L_1} \leq \boldsymbol{x} \leq \boldsymbol{U_1} \\ \quad\quad \boldsymbol{L_2} \leq \boldsymbol{y} \leq \boldsymbol{U_2} \\ \quad\quad \boldsymbol{x}, \boldsymbol{y} \geq 0, \end{cases} \tag{16}$$

where explicit expressions for z_1 and z_2 are given in (15).

The values of $\boldsymbol{\kappa}, \boldsymbol{\Sigma}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\rho}$ can be utilized from Table 1, 2 and 3 respectively. Further the vectors $\boldsymbol{L}_1, \boldsymbol{L}_2; \boldsymbol{U}_1, \boldsymbol{U}_2$ are respectively given by $\boldsymbol{L}_1^{'} = (0\,0\,...\,0)_{1\times 20}$, $\boldsymbol{L}_2^{'} = (0\,0\,...\,0)_{1\times 10}$; $\boldsymbol{U}_1^{'} = (0.6\,0.6\,...\,0.6)_{1\times 20}$; $\boldsymbol{U}_2^{'} = (0.6\,0.6\,...\,0.6)_{1\times 10}$.

Next we show that an optimal solution of (16) yields an efficient solution of (15).

Theorem 6.1. An optimal solutions of (16) is an efficient solution of (15).

Proof: Let, $\boldsymbol{x}^*$ be an optimal solution of model (16). We will show that $\boldsymbol{x}^*$ is also an efficient solution of (15).
If possible let, $\boldsymbol{x}^*$ is not an efficient solution of (15). Then it follows from the definition of efficient solution that, there exists $\bar{x} \in F$ such that

$$z_1(\bar{x}) \geq z_1(\boldsymbol{x}^*), \ z_2(\bar{x}) \leq z_2(\boldsymbol{x}^*). \tag{17}$$

and for at least one, one of the conditions of (17) strictly holds, i.e.,

$$\text{either, } z_1(\bar{x}) > z_1(\boldsymbol{x}^*) \text{ or, } z_2(\bar{x}) < z_2(\boldsymbol{x}^*). \tag{18}$$

Let, F be the set of all feasible solutions of (15). Clearly, the feasible solution space of (16) is a subset of F and $\boldsymbol{x}^*$ is a feasible solution of (15).

Now, Let $S = \{\boldsymbol{x} \in F : z_1(\boldsymbol{x}) > \hat{z_1}, \ z_2(\boldsymbol{x}) < \hat{z_2}\}$. Then $S \neq \phi$, the empty set, as it can be easily seen that $\boldsymbol{x}^* \in S$. Here, $\hat{z_1}$ and $\hat{z_1}$ are respectively the pessimistic values of the maximizing objectives z_1 and minimizing

objective z_2.
Next, we claim that $\bar{x} \notin F - S$. If not, then for at least one of z_1, z_2

$$z_1(\bar{x}) \leq \hat{z_1} \text{ or } z_2(\bar{x}) \geq \hat{z_2}, \tag{19}$$

will be satisfied.
Again since $\boldsymbol{x}^* \in S$, therefore

$$z_1(\boldsymbol{x}^*) > \hat{z_1},\ z_2(\boldsymbol{x}^*) < \hat{z_2}. \tag{20}$$

Now from (19) and (20) we have,

$$z_1(\bar{x}) < z_1(\boldsymbol{x}^*) \text{ or } z_2(\bar{x}) > z_1(\boldsymbol{x}^*), \tag{21}$$

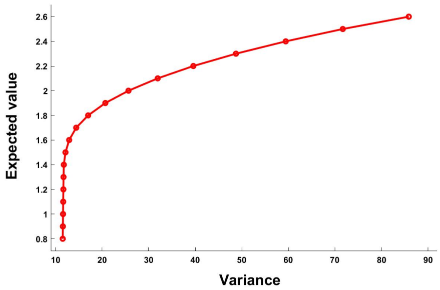
which contradicts (17). So, $\bar{x} \in S$.
Therefore, $\boldsymbol{x}^*$ and $\bar{x}$ belong to the same space S and $\bar{x}$ gives a greater value than that of $\boldsymbol{x}^*$. Which is contradiction to the fact that $\boldsymbol{x}^*$ is a optimal solution of (16). Thus, our assumption is wrong.
Hence, $\boldsymbol{x}^*$ is an efficient solution of model (15).

6.2.1. Solution of model (16)

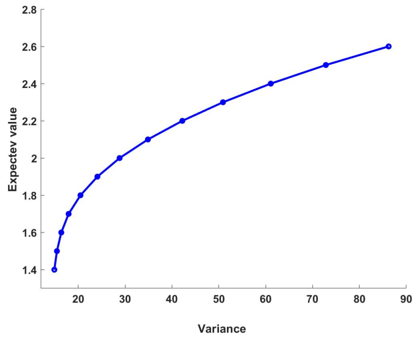
Now we apply the "fmincon" algorithm in Matlab 2018a to find the solution of (16) for different values $\lambda \in (\hat{z_1}, z_1^*)$, in steps of $\Delta\lambda = 0.1$. For $h_1 = 0$ and $h_2 = 1/4$, the solutions obtained are displayed in table 5, where columns with all zero entries are not displayed. For the other sets of values of h_1 and h_2, the solutions are obtained similarly. The graphs of total uncertain random expected value vs. variance, obtained from the solutions, are plotted in Figure 1 for 4 (four) sets of values of h_1 and h_2 : $h_1 = 0, h_2 = 1/2$; $h_1 = 1/3, h_2 = 2/3$; $h_1 = 0, h_2 = 1/4$; $h_1 = 0, h_2 = 3/4$.

Table 5: Optimal holding proportions and variances for given values of the expected returns

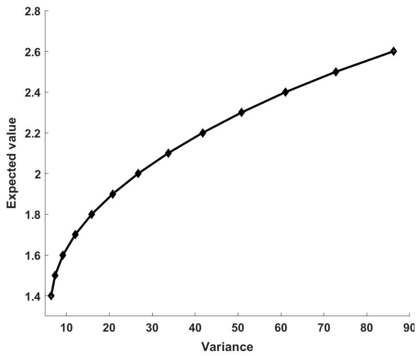
λ	x_3	x_4	x_5	x_6	x_7	x_{10}	x_{12}	x_{14}	x_{16}	x_{18}	x_{19}	y_2	Variance
1.4	0.0428	0.0127	0.0372	0	0.0028	0.002	0.1288	0.034	0.1455	0.0204	0	0.5738	**6.4547**
1.5	0.0285	0	0.0437	0.0028	0	0	01094	0.0695	0.1575	0.0527	0	0.5359	**7.3673**
1.6	0	0	0.0483	0.0104	0	0	0.09	0.1083	0.1675	0.087	0	0.4885	**9.2084**
1.7	0	0	0.0484	0.0089	0	0	0.0693	0.149	0.1749	0.1227	0	0.4268	**12.0407**
1.8	0	0	0.0484	0.0075	0	0	0.0485	0.1898	0.1824	0.1583	0	0.3651	**15.9067**
1.9	0	0	0.0485	0.0061	0	0	0.0278	0.2306	0.1898	0.194	0	0.3032	**20.8068**
2	0	0	0.0485	0.0046	0	0	0.007	0.2714	0.1973	0.2296	0	0.2416	**26.7407**
2.1	0	0	0.0466	0.003	0	0	0	0.3112	0.2022	0.2636	0	0.1734	**33.7146**
2.2	0	0	0.0438	0.0013	0	0	0	0.3505	0.2058	0.2968	0	0.1734	**41.7467**
2.3	0	0	0.0408	0	0	0	0	0.3897	0.2093	0.3299	0	0.0303	**50.8391**
2.4	0	0	0.0226	0	0	0	0	0.4325	0.1785	0.3664	0	0	**61.0759**
2.5	0	0	0	0	0	0	0	0.478	0.1159	0.4061	0	0	**72.8340**
2.6	0	0	0	0	0	0	0	0.5231	0.0283	0.4486	0	0	**86.2356**



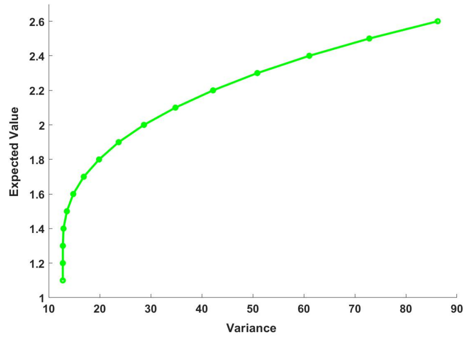
(a) $h_1 = 0,\ h_2 = 1/2$



(b) $h_1 = 1/3,\ h_2 = 2/3$



(c) $h_1 = 0,\ h_2 = 1/4$



(d) $h_1 = 0,\ h_2 = 3/4$

Figure 1: Expected value vs variance graph for different values of h_1 and h_2

6.3. Observations drawn from the figure:

From the graphs, we can see that the patterns are similar, concave in nature. Further, in each case, with the increase in expected return level, variance of the total return also increases. It is observed that for the values $h_1 = 0,\ h_2 = 1/4$ (graph (c)), rate of increase of variance, with the increase of expected return is initially the lowest. However, for the return level 1.9 onward graph (a) represents a lower rate of risk.

7. Comparative analysis

Here, we have considered four representative models depending on the values of h_1 and h_2. For each model (i.e., for fixed values of h_1 and h_2), the preassigned expected value of the total return (λ) in the constraint has been varied from its pessimistic value $\hat{z}_1$ to its ideal value z_1^*. It is seen from Table 4 that the ideal value of the objective z_1 is the same for all the models. However, its pessimistic value varies. That is why, in Table 6, the initial value of λ differs for different models. It is to be mentioned here that, for $h_1 = 0,\ h_2 = 1/2$, the model is that of Qin [16], obtained by using the zigzag uncertainty distribution. For different values of λ, the corresponding variances are listed in Table 6 for the four models.

Table 6: Comparison of different models

Return level	Variance			
	$h_1 = 0$ $h_2 = 1/2$	$h_1 = 1/3$ $h_2 = 2/3$	$h_1 = 0$ $h_2 = 1/4$	$h_1 = 0$ $h_2 = 3/4$
0.8	11.5448	—	—	—
0.9	11.5793	—	—	—
1	11.6139	—	—	12.7599
1.1	11.6486	—	—	12.7605
1.2	11.6834	—	—	12.7681
1.3	11.7183	—	—	12.7773
1.4	11.7782	14.8961	6.4547	12.8979
1.5	12.1341	15.4484	7.3673	13.5603
1.6	12.9271	16.3784	9.2084	14.813
1.7	14.4396	17.9439	12.0407	16.8722
1.8	17.007	20.4719	15.9067	19.8704
1.9	20.6972	24.0827	20.8068	23.7062
2.0	25.6689	28.8262	26.7407	28.6653
2.1	31.9382	34.8642	33.7146	34.8323
2.2	39.5925	42.2087	41.7467	42.2087
2.3	48.7494	50.8837	50.8391	50.8837
2.4	59.4315	61.0759	61.0759	61.0759
2.5	71.6883	72.834	72.834	72.834
2.6	85.8556	86.2356	86.2356	86.2356

The managerial implications and conclusions based on the comparative analysis are given below.

8. Managerial implication

The actions need to be taken by the decision maker for optimal portfolio selection problem under uncertain random environment are listed bellow:

- Instead of sticking to a fixed values of h_1 and h_2 in the uncertain return distribution, it is better to solve the problem for some representative values of the parameters. Then the solution most suited to the decision maker may be adopted.

- From the comparative analysis (table 6), it can be concluded that for a moderate investor, who prefers to take lower risk, the generalized zigzag uncertainty distribution with $h_1 = 0,\ h_2 = 1/4$ (graph (c)) is most suitable among the considered distributions. On the other hand, for an aggressive investor, who can take higher risk, can prefer the distribution with $h_1 = 0,\ h_2 = 1/2$ (graph (a)).

- To get a higher value of expected return, the investor need to invest 43.25% to 52.31% in stock 500470, 2.83% to 17.85% in stock 500790 and 36.64% to 44.86% in stock 500330 of his or her total wealth, for the model with $h_1 = 0,\ h_2 = 1/4$.

9. Conclusion and future scope

In this article, a hybrid portfolio selection problem has been studied using a generalized zigzag uncertainty distribution. As a particular case ($h_1 = 0,\ h_2 = 1/2$), one can get the results previously obtained by other authors. From the analysis of the numerical example, we observed that if the expected return is placed at a lower value, then the concomitant risk is also low, but in that case, the optimal portfolio contains a larger number of stocks. However, to get a higher return, an investor need to invest a large proportion of his or her wealth in some particular stocks, thereby resulting in high risk. From the findings of this article, we can conclude that the proposed method presented in this work is meaningful and applicable to a real-life portfolio selection problem.

As a future scope of our study, portfolio optimization problems considering adjustments to stocks and transaction costs can be considered. Also, one can use other relevant objectives of interest.

References

[1] R. Bhattacharyya, S. A. Hossain, S. Kar, M. Maiti, *Fuzzy cross-entropy, mean, variance, skewness models for portfolio selection,* J. King Saud Univ. - Comput. Inf. **26** (2014), 79–87.
[2] K-S. Chen, Y-Y. Huang, R-C. Tsaur, N-Y. Lin, *Fuzzy Portfolio Selection in the Risk Attitudes of Dimension Analysis under the Adjustable Security Proportions,* Mathematics, **11** (2023), 1143. https://doi.org/10.3390/ math11051143
[3] X. Huang, H. Ying, *Risk index based models for portfolio adjusting problem with returns subject to experts' evaluations,* Econ. Model. **30** (2013), 61–66.
[4] B. Li, R. Zhang, *A new mean-variance-entropy model for uncertain portfolio optimization with liquidity and diversification,* Chaos Solitons Fractals, **146** (2021), 110842.
[5] B. Li, R. Zhang, Y. Sun, *Multi-period portfolio selection based on uncertainty theory with bankruptcy control and liquidity,* Automatica. **147** (2023), 110751.
[6] B. Liu, *Uncertainty Theory*, 2nd edn, Springer-Verlag Berlin, 2007.
[7] B. Liu, *Some research problems in uncertainty theory* J. Uncertain Syst. **3** (2009), 3–10.
[8] B. Liu, *Uncertainty Theory: A branch of mathematics for modelling human uncertainty,* Berlin: Springer-Verlag, 2010.
[9] B. Liu, *Uncertainty Theory, 4th edn,* Springer Heidelberg New York, 2015.
[10] Y. Liu, *Uncertain random variables: A mixture of uncertainty and randomness,* Soft Comput. **17** (2013), 625–634.
[11] Y-J. Liu, W-G. Zhang, *Fuzzy multi-period portfolio selection model with time-varying loss aversion,* J Oper Res Soc. **72**(4) (2021), 935–949.
[12] H. Markowitz, *Portfolio selection,* J Finance, **7** (1952), 77–91.
[13] M.K. Mehlawat, P. Gupta, A.Z. Khan, *Portfolio optimization using higher moments in an uncertain random environment,* Inf. Sci. **576** (2021), 348–374.
[14] J. Nath, S. Chhatri, D. Bhattacharya, *A multi-objective portfolio selection problem with parameters as interval type fuzzy set* Int. J. Oper. Res. (2022), (in press) DOI: 10.1504/IJOR.2022.10051487
[15] Z. Peng, K. A. Iwamura, *sufficient and necessary condition of uncertainty distribution* J. Interdiscip. Math. **13** (2010), 277–285.
[16] Z. Qin, *Mean-variance model for portfolio optimization problem in the simultaneous presence of random and uncertain returns,* Eur. J. Oper. Res. **254** (2015), 480–488.
[17] Z. Qin, S. Kar, *Single-period inventory problem under uncertain environment*. Appl. Math. Comput. **219** (2013), 9630–9638.
[18] Z. Qin, S. Kar, X. Li, *Developments of mean-variance model for portfolio selection in uncertain environment* (2009), http://orsc.edu.cn/online/090511.pdf
[19] Z. Qin, S. Kar, H. Zheng, Uncertain portfolio adjusting model using semi-absolute deviation. Soft Comput. **20** (2016), 1–9.
[20] Y. Sheng, K. Yao, *Some formulas of variance of uncertain random variable,* Journal of uncertainty analysis and application. **2** (2014), Article 12.
[21] B. Wang, Y. Li, S. Wang, J. Watada, *A Multi-Objective Portfolio Selection Model With Fuzzy Value-at-Risk Ratio,* IEEE Trans Fuzzy Syst. **26**(6) (2018), 3673–3687.
[22] J. Zhai, M. Bai, J. Hao, *Uncertain random mean–variance–skewness models for the portfolio optimization problem,*. Optimization. **71**(13) (2022), 3941–3964.
[23] Y. Zhang, W. Liu, X. Yang, *An automatic trading system for fuzzy portfolio optimization problem with sell orders,* Expert Syst. Appl. **187** (2022), 115822